Topics in asynchronous systems

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Abstract

In the paper we define and characterize the asynchronous systems from the point of view of their autonomy, determinism, order, non-anticipation, time invariance, symmetry, stability and other important properties. The study is inspired by the models of the asynchronous circuits.

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Contents

1. Introduction

We mention three levels of abstraction of digital electrical engineering.

The first level is the descriptive, non-formalized one. The bricks with which this theory is built are small: logical gates, flip-flops, or bigger: handshake controls, pipelines, adders, oscillators. The analysis is made either timeless, with truth tables, or timed (discrete/real) by using different methods.

The second level was proposed by the author in some previous papers under the name of delay theory. The fundamental notion is that of delay= the mathematical model of the delay circuit, consisting in systems of ordinary and/or differential equations and/or inequalities written on $\mathbf{R} \to \{0,1\}$ functions. For example if the input u and the state x are such functions, the equation

$$x(t) = u(t - d)$$

where $t \in \mathbf{R}; d \geq 0$ is called the ideal delay, while the delays

$$\bigcap_{\xi \in [t-d_r, t)} u(\xi) \leq x(t) \leq \bigcup_{\xi \in [t-d_f, t)} u(\xi)$$

and respectively

$$\overline{x(t-0)} \cdot x(t) \cdot \bigcap_{\xi \in [t-d_r, t)} u(\xi) \cup x(t-0) \cdot \overline{x(t)} \cdot \bigcap_{\xi \in [t-d_f, t)} \overline{u(\xi)} \cup$$

$$\cup \overline{x(t-0)} \cdot \overline{x(t)} \cdot \overline{\bigcap_{\xi \in [t-d_r, t)} u(\xi)} \cup x(t-0) \cdot x(t) \cdot \overline{\bigcap_{\xi \in [t-d_f, t)} \overline{u(\xi)}} = 1$$

are inertial, i.e. non-ideal, where $d_r > 0; d_f > 0$. We interpret the last differential equation (the functions $\overline{x(t-0)} \cdot x(t); x(t-0) \cdot \overline{x(t)}$ are called the left semi-derivatives of x) in the next manner: at each time instant t, one of the next conditions is true

- x was 0 and now it is 1 and u was 1 for sufficiently long (d_r tine units)
- x was 1 and now it is 0 and u was 0 for sufficiently long
- x was 0 and now it is 0 and u was not 1 for sufficiently long
- x was 1 and now it is 1 and u was not 0 for sufficiently long

With delays and Boolean functions, any asynchronous circuit may be modeled at the most detailed logical level and this is sometimes an advantage, sometimes a disadvantage.

The third level of abstraction of digital electrical engineering is the one of the system theory that is inspired by the delay theory. In fact when the details that characterize delay theory are a (major) disadvantage, they are avoided by using asynchronous systems. An asynchronous system f (in the input-output sense) is a 'black-box', thought as a multivalued function associating to each input $u : \mathbf{R} \to \{0,1\}^m$ respectively a set of states $x : \mathbf{R} \to \{0,1\}^n; x \in f(u)$. The one-to-many association (in other words: the non-deterministic association) that f represents is motivated by the fact that the parameters that define an asynchronous circuit are not known and constant:

- they are known within the limits given by the precission of the measurement tools
- they depend on the temperature and on the power supply, thus they are variable against time in some ranges of values
- they depend on the technology that is used, but they differ even if we compare similar circuits produced in the same technology

In the paper we propose to analyze different types of asynchronous systems.

2. Preliminaries

We introduce now some notions, notations and preliminary results.

$\mathbf{R}$ is the time set. For $t, d \in \mathbf{R}$, the function $\tau^d : \mathbf{R} \to \mathbf{R}, \tau^d(t) = t - d$ is the time translation with d.

We note with $\mathbf{B}$ the set $\{0, 1\}$ and let

$$P^*(\mathbf{B}^m) = \{A | A \subset \mathbf{B}^m, A \neq \emptyset\}$$

A signal is a function $w : \mathbf{R} \to \mathbf{B}$ with the property that the real unbounded sequence $0 \le t_0 < t_1 < t_2 < \ldots$ exists so that

$$w(t) = w(t_0 - 0) \cdot \chi_{(-\infty, t_0)}(t) \oplus w(t_0) \cdot \chi_{[t_0, t_1)}(t) \oplus w(t_1) \cdot \chi_{[t_1, t_2)}(t) \oplus \ldots$$

where $\chi_{(\cdot)}$ is the characteristic function. The set of the signals is noted with S and we note furthermore:

$$S^{(m)} = \{u | u : \mathbf{R} \to \mathbf{B}^m, u_i \in S, i = \overline{1, m}\}, m \ge 1$$

$$S^{(0)} = \{0 | 0 : \mathbf{R} \to \mathbf{B}\}$$

(the one element set consisting in the null function), respectively

$$P^*(S^{(m)}) = \{U | U \subset S^{(m)}, U \neq \emptyset\}$$

For $\lambda \in \mathbf{B}^m, u \in S^{(m)}$ and $\sigma : \{1, \ldots, m\} \to \{1, \ldots, m\}$ bijective, we note

$$\overline{\lambda} = (\overline{\lambda_1}, \ldots, \overline{\lambda_m}), \quad \overline{u}(t) = (\overline{u_1(t)}, \ldots, \overline{u_m(t)})$$

$$\lambda_\sigma = (\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(m)}), \quad u_\sigma(t) = (u_{\sigma(1)}(t), \ldots, u_{\sigma(m)}(t))$$

Lemma Let $u \in S^{(m)}, m \ge 1$. The next statements are true

a) If u is not constant we note $t_0 = \min\{t | u(t - 0) \neq u(t)\}$. Then

$$\forall d \in \mathbf{R}, (u \circ \tau^d \in S^{(m)} \Longleftrightarrow t_0 + d \ge 0)$$

b) For any $d \ge 0$, we have $u \circ \tau^d \in S^{(m)}$

c) $(\forall d \in \mathbf{R}, u \circ \tau^d \in S^{(m)}) \Longleftrightarrow u$ is constant.

Proof. We suppose that the family $u^0, u^1, u^2, \ldots \in \mathbf{B}^m$ and the unbounded sequence $0 \le t_0 < t_1 < t_2 < \ldots$ are chosen so that

$$u(t) = u^0 \cdot \chi_{(-\infty, t_0)}(t) \oplus u^1 \cdot \chi_{[t_0, t_1)}(t) \oplus u^2 \cdot \chi_{[t_1, t_2)}(t) \oplus \ldots$$

and if u is not constant, then $t_0 = \min\{t | u(t - 0) \neq u(t)\}$. For any $d \in \mathbf{R}$ we can write

$$\begin{aligned} u \circ \tau^d(t) &= u(t - d) = \\ &= u^0 \cdot \chi_{(-\infty, t_0)}(t - d) \oplus u^1 \cdot \chi_{[t_0, t_1)}(t - d) \oplus u^2 \cdot \chi_{[t_1, t_2)}(t - d) \oplus \ldots \\ &= u^0 \cdot \chi_{(-\infty, t_0 + d)}(t) \oplus u^1 \cdot \chi_{[t_0 + d, t_1 + d)}(t) \oplus u^2 \cdot \chi_{[t_1 + d, t_2 + d)}(t) \oplus \ldots \end{aligned}$$

The sequence with the general term $t'_k = t_k + d;\ k \in \mathbf{N}$ is unbounded, like (t_k).

a) Obvious.

b) If u is constant, then $\forall d \in \mathbf{R}; u = u \circ \tau^d \in S^{(m)}$ and if u is not constant, then the property results from the fact that $0 \le d;\ 0 \le t_0 \le t_0 + d$ and from a) $\Longleftarrow$.

c) $\Longrightarrow$ We suppose against all reason that u is not constant. Some $d \in \mathbf{R}$ exists then so that $t_0 + d < 0$ thus, from a) we get $u \circ \tau^d \notin S^{(m)}$, contradiction. ■

3. Asynchronous systems

Definition 1 The functions $f : S^{(m)} \to P^*(S^{(n)})$ are called asynchronous systems (in the input-output sense), shortly systems. The elements $u \in S^{(m)}$, respectively $x \in f(u)$ are called inputs, respectively states (or outputs).

Remark 2 The asynchronous systems f are relations of determination between the cause u and the effect $x \in f(u)$ and our only request is that each cause has effects: $\forall u, f(u) \neq \emptyset$. When this determination consists in a system of equations and/or inequalities, f gives for any u the set $f(u)$ of the solutions of the system (writting systems of equations and/or inequalities is not the purpose of the present paper, however).

The one-to-many association $u \mapsto f(u)$ has its origin as we have already mentioned in the fact that to one cause u there correspond in general several possible effects $x \in f(u)$ depending on the variations in ambient temperature, power supply, on the technology etc.

Example 3 In the next examples we have $m = n$ at (1) and $n = 1$ at (2),...,(4):

$$f(u) = \{u \circ \tau^d\}, d \ge 0 \tag{1}$$

$$f(u) = \{x \mid \exists d \ge 0, \forall t \ge d, x(t) = u_i(t)\}, i \in \{1, ..., m\} \tag{2}$$

$$f(u) = \{x \mid \overline{x(t-0)} \cdot x(t) \le u_1(t-d) \cdot ... \cdot u_m(t-d)\} \tag{3}$$

$$f(u) = \{x \mid \overline{x(t-0)} \cdot x(t) \le \bigcap_{\xi \in [t, t+\delta_r]} x(\xi), x(t-0) \cdot \overline{x(t)} \le \bigcap_{\xi \in [t, t+\delta_f]} \overline{x(\xi)}\} \tag{4}$$

For (1), the fact that $u \in S^{(m)}$ and $d \ge 0$ implies $u \circ \tau^d \in S^{(m)}$ has been proved in the Lemma, item b). And at (4) where $\delta_r \ge 0, \delta_f \ge 0$ a system is defined that associates to each input u the set of all the (inertial) states x having the property that if they switch from 0 to 1 they remain 1 more than δ_r time units and if they switch from 1 to 0 they remain 0 more than δ_f time units.

Definition 4 Let the set $X \in P^*(S^{(n)})$ and the systems $f, g : S^{(m)} \to P^*(S^{(n)})$. They define the next systems:

- $\overline{f} : S^{(m)} \to P^*(S^{(n)})$;

$$\forall (u_1, ..., u_m) \in S^{(m)}, \overline{f}(u_1, ..., u_m) = \{\overline{x} \mid x \in f(u_1, ..., u_m)\}$$

- $f^{(m+1)} : S^{(m+1)} \to P^*(S^{(n)})$;

$$\forall (u_1, \ldots, u_{m+1}) \in S^{(m+1)}, f^{(m+1)}(u_1, \ldots, u_{m+1}) = f(u_1, \ldots, u_m)$$

- $f_{i \to j} : S^{(m)} \to P^*(S^{(n)})$ is defined for all $i, j \in \{1, \ldots, m\}, i \neq j$ by

$$\forall (u_1, \ldots, u_m) \in S^{(m)}, f_{i \to j}(u_1, \ldots, \underset{i}{u_i}, \ldots, \underset{j}{u_j}, \ldots, u_m) = f(u_1, \ldots, \underset{i}{u_i}, \ldots, \underset{j}{u_i}, \ldots, u_m)$$

- we suppose that f does not depend on $u_i, i \in \{1, \ldots, m\}$ i.e. for all $u \in S^{(m)}, f(u_1, \ldots, u_i, \ldots, u_m) = f(u_1, \ldots, \underset{i}{0}, \ldots, u_m)$. Then $f_{\hat{u}_i} : S^{(m-1)} \to P^*(S^{(n)})$ is defined in the next manner:

$$\forall (u_1, \ldots, \hat{u}_i, \ldots, u_m) \in S^{(m-1)}, f_{\hat{u}_i}(u_1, \ldots, \hat{u}_i, \ldots, u_m) = f(u_1, \ldots, \underset{i}{0}, \ldots, u_m)$$

where $\hat{u}_i$ indicates a missing coordinate

- if $\forall (u_1, \ldots, u_m) \in S^{(m)}, f(u_1, \ldots, u_m) \cap X \neq \emptyset$, respectively if $\forall (u_1, \ldots, u_m) \in S^{(m)}, f(u_1, \ldots, u_m) \cap g(u_1, \ldots, u_m) \neq \emptyset$, then the systems $f \cap X, f \cap g : S^{(m)} \to P^*(S^{(n)})$ are defined by

$$\forall (u_1, \ldots, u_m) \in S^{(m)}, (f \cap X)(u_1, \ldots, u_m) = f(u_1, \ldots, u_m) \cap X$$

$$\forall (u_1, \ldots, u_m) \in S^{(m)}, (f \cap g)(u_1, \ldots, u_m) = f(u_1, \ldots, u_m) \cap g(u_1, \ldots, u_m)$$

- $f \cup X, f \cup g : S^{(m)} \to P^*(S^{(n)})$,

$$\forall (u_1, \ldots, u_m) \in S^{(m)}, (f \cup X)(u_1, \ldots, u_m) = f(u_1, \ldots, u_m) \cup X$$

$$\forall (u_1, \ldots, u_m) \in S^{(m)}, (f \cup g)(u_1, \ldots, u_m) = f(u_1, \ldots, u_m) \cup g(u_1, \ldots, u_m)$$

4. Initial states

Definition 5 Let the system f. The function $\phi : S^{(m)} \to P^*(B^n)$,

$$\forall u, \phi(u) = \{x(0-0) | x \in f(u)\}$$

is called the initial state function of f and the set

$$\Theta_f = \bigcup_{u \in S^{(m)}} \phi(u)$$

is called the set of the initial states of f.

Definition 6 If $\Theta_f = \{x^0\}$ i.e. if

$$\forall u, \forall x \in f(u), x(0-0) = x^0$$

then we say that f is initialized and that x^0 is the initial state of f; otherwise, we say that f is not initialized and that it does not have an initial state.

Example 7 The constant function $S^{(m)} \to P^*(S^{(n)})$ equal with $\{x^0\}$ is an initialized system whose initial state is x^0.

Remark 8 Many authors prefer to work either with initialized systems, or at least with constant initial state functions. Our option is for a more general frame because we want to include in this study the trivial systems $f(u) = \{u\}$ and other similar systems.

Theorem 9 Let the systems f, g and the set of states X. The initial state functions of the systems $\overline{f}, f^{(m+1)}, f_{i \to j}, f_{\hat{u}_i}, f \cap X, f \cap g, f \cup X, f \cup g$ are the next ones:

- $\overline{\phi} : S^{(m)} \to P^*(B^n)$,

$$\forall (u_1, \ldots, u_m) \in S^{(m)}, \overline{\phi}(u_1, \ldots, u_m) = \{\overline{x^0} | x^0 \in \phi(u_1, \ldots, u_m)\}$$

- $\phi^{(m+1)} : S^{(m+1)} \to P^*(B^n)$,

$$\forall (u_1, \ldots, u_{m+1}) \in S^{(m+1)}, \phi^{(m+1)}(u_1, \ldots, u_{m+1}) = \phi(u_1, \ldots, u_m)$$

- $\phi_{i \to j} : S^{(m)} \to P^*(B^n)$ is given for all $i, j \in \{1, \ldots, m\}, i \neq j$ by

$$\forall (u_1, \ldots, u_m) \in S^{(m)}, \phi_{i \to j}(u_1, \ldots, \underset{i}{u_i}, \ldots, \underset{j}{u_j}, \ldots, u_m) = \phi(u_1, \ldots, \underset{i}{u_i}, \ldots, \underset{j}{u_i}, \ldots, u_m)$$

- if f does not depend on $u_i, i \in \{1, \ldots, m\}$, then $\phi_{\hat{u}_i} : S^{(m-1)} \to P^*(B^n)$ is given by:

$$\forall (u_1, \ldots, \hat{u}_i, \ldots, u_m) \in S^{(m-1)}, \phi_{\hat{u}_i}(u_1, \ldots, \hat{u}_i, \ldots, u_m) = \phi(u_1, \ldots, \underset{i}{0}, \ldots, u_m)$$

- if $\forall (u_1, \ldots, u_m) \in S^{(m)}, f(u_1, \ldots, u_m) \cap X \neq \emptyset$, respectively if $\forall (u_1, \ldots, u_m) \in S^{(m)}, f(u_1, \ldots, u_m) \cap g(u_1, \ldots, u_m) \neq \emptyset$, then $\phi \cap \Xi, \phi \cap \gamma : S^{(m)} \to P^*(B^n)$ are:

$$\forall (u_1, \ldots, u_m) \in S^{(m)}, (\phi \cap \Xi)(u_1, \ldots, u_m) = \phi(u_1, \ldots, u_m) \cap \Xi$$

$$\forall (u_1, \ldots, u_m) \in S^{(m)}, (\phi \cap \gamma)(u_1, \ldots, u_m) = \phi(u_1, \ldots, u_m) \cap \gamma(u_1, \ldots, u_m)$$

where $\forall (u_1, \ldots, u_m) \in S^{(m)}, \Xi(u_1, \ldots, u_m) = \{x(0 - 0) | x \in X\} \overset{not}{=} \Xi$

- $\phi \cup \Xi, \phi \cup \gamma : S^{(m)} \to P^*(B^n)$,

$$\forall (u_1, \ldots, u_m) \in S^{(m)}, (\phi \cup \Xi)(u_1, \ldots, u_m) = \phi(u_1, \ldots, u_m) \cup \Xi$$

$$\forall (u_1, \ldots, u_m) \in S^{(m)}, (\phi \cup \gamma)(u_1, \ldots, u_m) = \phi(u_1, \ldots, u_m) \cup \gamma(u_1, \ldots, u_m)$$

Proof. These result from the way that the initial state function was introduced at Definition 5. For example

$$\overline{\psi}(u) = \{x(0-0) \mid x \in \overline{f}(u)\} = \{x(0-0) \mid \overline{x} \in f(u)\} = \{\overline{x(0-0)} \mid x \in f(u)\} = \{\overline{x'} \mid x' \in \phi(u)\}$$

■

5. Parallel connection and serial connection

Remark 10 We shall identify the sets $S^{(m_1)} \times \ldots \times S^{(m_p)}$ and $S^{(m_1+\ldots+m_p)}$ for $m_1 \geq 1, \ldots, m_p \geq 1$ whose elements are of the form $(u^1, \ldots, u^p) = (u^1_1, \ldots, u^1_{m_1}, \ldots, u^p_1, \ldots, u^p_{m_p})$. By this identification we ignore the fact that the argument of $(u^1, \ldots, u^p)$ is $(t_1, \ldots, t_p) \in \mathbf{R}^p$ and the argument of $(u^1_1, \ldots, u^1_{m_1}, \ldots, u^p_1, \ldots, u^p_{m_p})$ is $t \in \mathbf{R}$ and we just keep in mind the form of the coordinates of these functions. The convention imposes furthermore the identification of $P^*(S^{(n_1)}) \times \ldots \times P^*(S^{(n_p)})$ with $P^*(S^{(n_1+\ldots+n_p)})$. See the Appendix for more details.

These identifications are more meaningful than they might seem at the first sight because they allow in the next definition that p systems with p different time axes, when connected in parallel, have one time axis.

Definition 11 The parallel connection (or the direct product) of the systems $f^i : S^{(m_i)} \to P^*(S^{(n_i)}), i = \overline{1,p}$ is the system $(f^1, \ldots, f^p) : S^{(m_1+\ldots+m_p)} \to P^*(S^{(n_1+\ldots+n_p)})$ defined by

$$\forall (u^1, \ldots, u^p) \in S^{(m_1+\ldots+m_p)}, (f^1, \ldots, f^p)(u^1, \ldots, u^p) = (f^1(u^1), \ldots, f^p(u^p))$$

Definition 12 We suppose that $n_1 + \ldots + n_p = m$. The serial connection of the systems $f, f^1, \ldots, f^p$ is the system $f \circ (f^1, \ldots, f^p) : S^{(m_1+\ldots+m_p)} \to P^*(S^{(n)})$ that is defined by any of the equivalent statements:

$$f \circ (f^1, \ldots, f^p)(u^1, \ldots, u^p) = \{x \mid \exists y^1 \in f^1(u^1), \ldots, \exists y^p \in f^p(u^p), x \in f(y^1, \ldots, y^p)\}$$

$$f \circ (f^1, \ldots, f^p)(u^1, \ldots, u^p) = \bigcup_{(y^1, \ldots, y^p) \in f^1(u^1) \times \ldots \times f^p(u^p)} f(y^1, \ldots, y^p)$$

Example 13 The system $I : S \to P^*(S)$ is defined in the next way

$$I(u_i) = \{u_i\} \tag{5}$$

Then for any f and any $u = (u_1, \ldots, u_m)$ we remark that

$$(\underbrace{I, \ldots, I}_{n}) \circ f(u) = f \circ (\underbrace{I, \ldots, I}_{m})(u) = f(u)$$

More general, with the notation $I_d : S \to P^*(S), d \geq 0$

$$I_d(u_i) = \{u_i \circ \tau^d\} \tag{6}$$

we have

$$f \circ (\underbrace{I_d, \dots, I_d}_{m})(u) = f(u \circ \tau^d) \tag{7}$$

$$(\underbrace{I_d, \dots, I_d}_{n}) \circ f(u) = \{x \circ \tau^d \mid x \in f(u)\} \tag{8}$$

Let us consider for example that f represents the set of the solutions of the system

$$\overline{x(t-0)} \cdot x(t) \le \bigcap_{\xi \in [t-2,t)} (u_1(\xi) \cdot u_2(\xi)) \tag{9}$$

$$x(t-0) \cdot \overline{x(t)} = 0 \tag{10}$$

that for $(u_1, u_2) = (\chi_{[0,\infty)}, \chi_{[1,\infty)})$ is given by

$$f(u) = \{1\} \cup \{\chi_{[d',\infty)} \mid d' \ge 3\}$$

For $d = 1$ in equation (7) we have

$$f(u \circ \tau^1) = \{1\} \cup \{\chi_{[d',\infty)} \mid d' \ge 4\}$$

Theorem 14 The initial state function of the system $f \circ (f^1, \dots, f^p)$ is the function $\phi \circ (f^1, \dots, f^p) : S^{(m_1+\dots+m_p)} \to P^*(B^n)$ defined by

$$\phi \circ (f^1, \dots, f^p)(u^1, \dots, u^p) = \{x^0 \mid \exists y^1 \in f^1(u^1), \dots, \exists y^p \in f^p(u^p), x^0 \in \phi(y^1, \dots, y^p)\}$$

Proof. $\{x(0-0) \mid x \in f \circ (f^1, \dots, f^p)(u^1, \dots, u^p)\} = \{x(0-0) \mid \exists y^1 \in f^1(u^1), \dots, \exists y^p \in f^p(u^p), x \in f(y^1, \dots, y^p)\}$

$= \{x^0 \mid \exists y^1 \in f^1(u^1), \dots, \exists y^p \in f^p(u^p), x^0 \in \phi(y^1, \dots, y^p)\}$ ◾

6. Autonomy

Definition 15 The system f is autonomous (or free) if it is the constant function

$$\exists X, \forall u, f(u) = X$$

and it is non-autonomous otherwise. The usual notation for the autonomous system f is X.

Remark 16 The autonomous systems are or may be considered to be without input since the states $x \in X$ are the same for all u. Definition 15 is somehow different from other authors' point of view [1] that consider the autonomous systems be those systems where the input takes exactly one value and it belongs -in our formalization- to the one element set $S^{(0)}$. See however Theorem 18.

Example 17 The (absolute inertial) system f that was defined at Example 3 (4) is autonomous.

Theorem 18 If f is autonomous, then $\overline{f}, f^{(m+1)}, f_{i \to j}, f_{\widehat{i}}$ are autonomous, $i, j \in \{1, ..., m\}, i \neq j$ and a system $g : S^{(0)} \to P^*(S^{(n)})$ exists so that $\forall u \in S^{(m)}, \forall u' \in S^{(0)}, f(u) = g(u')$.

Proof. If $f = X$, then $\forall u, \overline{f}(u) = \{\overline{x} | x \in X\}$ etc. We take $g = f_{\widehat{1} ... \widehat{m}}$. ∎

Theorem 19 If $\forall u, X \subset f(u)$ then $f \cap X$ is autonomous and if $\forall u, f(u) \subset X$, then $f \cup X$ is autonomous.

Proof. For all u, x we have

$$x \in X \iff (x \in X \quad \text{and} \quad x \in X) \Longrightarrow (x \in f(u) \quad \text{and} \quad x \in X) \Longrightarrow x \in X$$

in other words $\forall u, X \subset f(u) \cap X \subset X$ and eventually $f \cap X = X$. ∎

Theorem 20 If f, g are autonomous, then $f \cap g$ and $f \cup g$ are autonoous.

Proof. If $\exists X, \forall u, f(u) = X$ and $\exists Y, \forall u, g(u) = Y$, then $\forall u, (f \cap g)(u) = X \cap Y$ and $\forall u, (f \cup g)(u) = X \cup Y$. ∎

Theorem 21 The initial state function Ξ of the autonomous system X is constant and the initial state functions $\overline{\Xi}, \Xi^{(m+1)}, \Xi_{i \to j}, \Xi_{\widehat{i}}$ are also constant, $i, j \in \{1, ..., m\}, i \neq j$.

Proof. The set $\forall u, \Xi(u) = \{x(0 - 0) | x \in X\}$ does not depend on u. ∎

Theorem 22 Let $f : S^{(m)} \to P^*(S^{(n)})$ and $f^i : S^{(m_i)} \to P^*(S^{(n_i)}), i = \overline{1, p}$, $n_1 + ... + n_p = m$ like before. If f is autonomous, then $f \circ (f^1, ..., f^p)$ is autonomous. If $f^1, ..., f^p$ are all autonomous, then $f \circ (f^1, ..., f^p)$ is autonomous.

Proof. If $f = X$, then $f \circ (f^1, ..., f^p) = X$ and if $f^1 = X_1, ..., f^p = X_p$, the formula

$$\forall (u^1, ..., u^p) \in S^{(m_1 + ... + m_p)}, f \circ (f^1, ..., f^p)(u^1, ..., u^p) = \bigcup_{(y^1, ..., y^p) \in X_1 \times ... \times X_p} f(y^1, ..., y^p)$$

proves the desired property. ∎

7. Finitude. Determinism

Definition 23 The system f is finite (deterministic) if it has the property that $\forall u, f(u)$ has a finite number of elements (a single element); otherwise, it is called infinite (non-deterministic).

Remark 24 In the situation when f represents the set of the solutions of a system of equations/inequalities, its determinism coincides with the uniqueness of the solution.

The deterministic systems may be identified with the $S^{(m)} \to S^{(n)}$ functions.

Finiteness is useful when, in modeling, we take in consideration the 'worst case', the 'best case', the 'most frequent' case etc.

Example 25 We have had already several examples of deterministic systems; we just remark that the Boolean functions $F : \mathbf{B}^m \to \mathbf{B}^n$ define deterministic systems by

$$\forall u, f(u) = \{F(u)\}$$

The direct product $(F^1, ..., F^p)$ of $F^i : \mathbf{B}^{m_i} \to \mathbf{B}^{n_i}, i = \overline{1,p}$ defines the deterministic system $(f^1, ..., f^p)$, where

$$\forall u^i \in S^{(m_i)}, f^i(u^i) = \{F^i(u^i)\}, i = \overline{1,p}$$

by

$$\forall (u^1, ..., u^p) \in S^{(m_1+...+m_p)}, (f^1, ..., f^p)(u^1, ..., u^p) = \{(F^1(u^1), ..., F^p(u^p))\}$$

Theorem 26 If f is finite (deterministic), then $\overline{f}, f^{(m+1)}, f_{i \to j}$ and f_{b_i} are finite (deterministic), where $i, j \in \{1, ..., m\}, i \neq j$.

Proof. We note with $|\ |$ the number of elements of a finite set and we have $\forall u, |f(u)| = |\overline{f}(u)|$ etc. ■

Theorem 27 If one of the systems f, g is finite (deterministic), then $f \cap g$ is finite (deterministic) and if both are finite, then $f \cup g$ is finite.

Proof. We suppose that f is finite (deterministic) and we infer

$$\forall u, |f(u) \cap g(u)| \leq |f(u)|$$

thus $f \cap g$ is finite (deterministic). If f, g are both finite then we have

$$\forall u, |f(u) \cup g(u)| \leq |f(u)| + |g(u)|$$

thus $f \cup g$ is finite. ■

Theorem 28 When f is deterministic, the initial state function ϕ fulfills the property: $\forall u, \phi(u)$ has a single element and the initial state functions $\overline{\phi}, \phi^{(m+1)}, \phi_{i \to j}, \phi_{b_i}$, $i, j \in \{1, ..., m\}, i \neq j$ are in the same situation.

Proof. The first assertion is obvious and the other statements take into account Theorem 26. ■

Theorem 29 If $f, f^1, ..., f^p$ are all finite (deterministic), then $f \circ (f^1, ..., f^p)$ is finite (deterministic).

Proof. For some arbitrary $(u^1, ..., u^p)$ we can write

$$f \circ (f^1, ..., f^p)(u^1, ..., u^p) = \bigcup_{(y^1, ..., y^p) \in f^1(u^1) \times ... \times f^p(u^p)} f(y^1, ..., y^p)$$

where $f^1(u^1) \times ... \times f^p(u^p)$ is finite (has one element). ■

Theorem 30 f is autonomous and finite (deterministic) if and only if $\exists X \subset S^{(n)}$ finite (consisting in a single element) so that $\forall u, f(u) = X$.

Proof. Obvious. ■

8. Order

Definition 31 The next inclusion $f \subset g$ is defined between the systems f, g:

$$\forall u, f(u) \subset g(u)$$

Remark 32 $\subset$ is a partial order without first element, but with the last element represented by the autonomous system $S^{(n)} : S^{(m)} \to P^*(S^{(n)})$,

$$\forall u \in S^{(m)}, S^{(n)}(u) = S^{(n)}$$

The sense of the inclusion $f \subset g$ is that the model offered by f is more precise, it has more information on the modeled circuit than the model offered by g, in particular the deterministic systems give the maximal information and the autonomous system $S^{(n)}$ gives the minimal information.

Example 33 We consider the next $S^{(m)} \to P^*(S)$ systems

$$f_1(u) = \{u_i\}$$

$$f_2(u) = \{x | \forall t \geq 0, x(t) = u_i(t)\}$$

$$f_3(u) = \{x | \exists t', \forall t \geq t', x(t) = u_i(t)\}$$

where $i \in \{1, ..., m\}$. We have $f_1 \subset f_2 \subset f_3$.

Theorem 34 If $f \subset g$, then $\bar{f} \subset \bar{g}$; $f^{(m+1)} \subset g^{(m+1)}$; $f_{i \to j} \subset g_{i \to j}$; $f_{\hat{i}} \subset g_{\hat{i}}$ are true, $i, j \in \{1, ..., m\}, i \neq j$.

Proof. For example

$$\forall (u_1, ..., u_m), f_{i \to j}(u_1, ..., \underset{i}{u_i}, ..., \underset{j}{u_j}, ..., u_m) = f(u_1, ..., \underset{i}{u_i}, ..., \underset{j}{u_i}, ..., u_m) \subset$$
$$\subset g(u_1, ..., \underset{i}{u_i}, ..., \underset{j}{u_i}, ..., u_m) = g_{i \to j}(u_1, ..., \underset{i}{u_i}, ..., \underset{j}{u_j}, ..., u_m)$$ ■

Theorem 35 For $X \subset S^{(n)}$ and the systems f, g, the next inclusions take place:

$$f \cap X \subset f \subset f \cup X$$

$$f \cap g \subset f \subset f \cup g$$

Proof. $\forall u, \forall x, x \in (f \cap g)(u) \iff x \in f(u) \cap g(u) \iff (x \in f(u)$ and $x \in g(u)) \Longrightarrow x \in f(u) \Longrightarrow (x \in f(u)$ or $x \in g(u)) \iff x \in f(u) \cup g(u) \iff x \in (f \cup g)(u)$ ■

Theorem 36 If $f \subset g$, then $\forall u, \varphi(u) \subset \gamma(u)$.

Proof. For any u we have $\varphi(u) = \{x(0 - 0) | x \in f(u)\} \subset \{x(0 - 0) | x \in g(u)\} = \gamma(u)$ ∎

Theorem 37 Let the systems $f, g : S^{(m)} \to P^*(S^{(n)}); f^i, g^i : S^{(m_i)} \to P^*(S^{(n_i)});$ $i = \overline{1,p}$ so that $n_1 + ... + n_p = m$. The next implications are true:

$$f \subset g \Longrightarrow f \circ (f^1, ..., f^p) \subset g \circ (f^1, ..., f^p)$$

$$f^1 \subset g^1, ..., f^p \subset g^p \Longrightarrow f \circ (f^1, ..., f^p) \subset f \circ (g^1, ..., g^p)$$

Proof. Let $(u^1, ..., u^p)$ and $x \in f \circ (f^1, ..., f^p)(u^1, ..., u^p)$, meaning that $y^1 \in f^1(u^1), ..., y^p \in f^p(u^p)$ exist so that $x \in f(y^1, ..., y^p)$; because $x \in g(y^1, ..., y^p)$, we obtain $x \in g \circ (f^1, ..., f^p)(u^1, ..., u^p)$.

On the other hand if we suppose that $x \in f \circ (f^1, ..., f^p)(u^1, ..., u^p)$, then $y^1 \in f^1(u^1), ..., y^p \in f^p(u^p)$ exist so that $x \in f(y^1, ..., y^p)$. We get $y^1 \in g^1(u^1), ..., y^p \in g^p(u^p)$ and this implies $x \in f \circ (g^1, ..., g^p)(u^1, ..., u^p)$. ∎

Theorem 38 Let the arbitrary sets $X \subset S^{(n)}, X_i \subset S^{(n_i)}, i = \overline{1,p}$ and the systems $f, g : S^{(m)} \to P^*(S^{(n)}); f^i, g^i : S^{(m_i)} \to P^*(S^{(n_i)}); i = \overline{1,p}$ so that $n_1 + ... + n_p = m$.

a) If $\forall u, f(u) \cap X \neq \emptyset$, then $\forall (u^1, ..., u^p), (f \circ (f^1, ..., f^p))(u^1, ..., u^p) \cap X \neq \emptyset$ and
$$(f \cap X) \circ (f^1, ..., f^p) = (f \circ (f^1, ..., f^p)) \cap X$$
If $\forall (u^1, ..., u^p), f^1(u^1) \cap X_1 \neq \emptyset, ..., f^p(u^p) \cap X_p \neq \emptyset$, then $\forall (u^1, ..., u^p), (f \circ (f^1, ..., f^p))(u^1, ..., u^p) \cap (f \circ (X_1, ..., X_p)) \neq \emptyset$ and we can write
$$f \circ (f^1 \cap X_1, ..., f^p \cap X_p) \subset (f \circ (f^1, ..., f^p)) \cap (f \circ (X_1, ..., X_p))$$

b) If $\forall u, f(u) \cap g(u) \neq \emptyset$, then $\forall (u^1, ..., u^p), (f \circ (f^1, ..., f^p))(u^1, ..., u^p) \cap (g \circ (f^1, ..., f^p))(u^1, ..., u^p) \neq \emptyset$ and
$$(f \cap g) \circ (f^1, ..., f^p) \subset (f \circ (f^1, ..., f^p)) \cap (g \circ (f^1, ..., f^p))$$
If $\forall (u^1, ..., u^p), f^1(u^1) \cap g^1(u^1) \neq \emptyset, ..., f^p(u^p) \cap g^p(u^p) \neq \emptyset$, then $\forall (u^1, ..., u^p), (f \circ (f^1, ..., f^p))(u^1, ..., u^p) \cap (f \circ (g^1, ..., g^p))(u^1, ..., u^p) \neq \emptyset$ and
$$f \circ (f^1 \cap g^1, ..., f^p \cap g^p) \subset (f \circ (f^1, ..., f^p)) \cap (f \circ (g^1, ..., g^p))$$

c) We have
$$(f \cup X) \circ (f^1, ..., f^p) = (f \circ (f^1, ..., f^p)) \cup X$$
$$f \circ (f^1 \cup X_1, ..., f^p \cup X_p) \supset (f \circ (f^1, ..., f^p)) \cup (f \circ (X_1, ..., X_p))$$

d) The next properties are also true:
$$(f \cup g) \circ (f^1, ..., f^p) = (f \circ (f^1, ..., f^p)) \cup (g \circ (f^1, ..., f^p))$$
$$f \circ (f^1 \cup g^1, ..., f^p \cup g^p) \supset (f \circ (f^1, ..., f^p)) \cup (f \circ (g^1, ..., g^p))$$

Proof. We prove b) and respectively d):

$$\forall (u^1,...,u^p), ((f \cap g)\circ(f^1,...,f^p))(u^1,...,u^p) = \{x \mid \exists y^1,...,\exists y^p, y^1 \in f^1(u^1) \text{ and} ... \text{and } y^p \in f^p(u^p) \text{ and } x \in f(y^1,...,y^p) \text{ and } x \in g(y^1,...,y^p)\} \subset \{x \mid \exists y^1,...,\exists y^p, y^1 \in f^1(u^1) \text{ and} ... \text{and } y^p \in f^p(u^p) \text{ and } x \in f(y^1,...,y^p) \text{ and } \exists z^1,...,\exists z^p, z^1 \in f^1(u^1) \text{ and} ... \text{and } z^p \in f^p(u^p) \text{ and } x \in g(z^1,...,z^p)\} = ((f \circ (f^1,...,f^p)) \cap (g \circ (f^1,...,f^p)))(u^1,...,u^p)$$

$$\forall (u^1,...,u^p), (f\circ(f^1 \cap g^1,...,f^p \cap g^p))(u^1,...,u^p) = \{x \mid \exists y^1,...,\exists y^p, y^1 \in f^1(u^1) \text{ and } y^1 \in g^1(u^1) \text{ and} ... \text{and } y^p \in f^p(u^p) \text{ and } y^p \in g^p(u^p) \text{ and } x \in f(y^1,...,y^p)\} \subset \{x \mid \exists y^1,...,\exists y^p, y^1 \in f^1(u^1) \text{ and} ... \text{and } y^p \in f^p(u^p) \text{ and } \exists z^1,...,\exists z^p, z^1 \in g^1(u^1) \text{ and} ... \text{and } z^p \in g^p(u^p) \text{ and } x \in f(y^1,...,y^p) \text{ and } x \in f(z^1,...,z^p)\} = ((f \circ (f^1,...,f^p)) \cap (f \circ (g^1,...,g^p)))(u^1,...,u^p)$$

respectively

$$\forall (u^1,...,u^p), ((f \cup g)\circ(f^1,...,f^p))(u^1,...,u^p) = \{x \mid \exists y^1,...,\exists y^p, y^1 \in f^1(u^1) \text{ and} ... \text{and } y^p \in f^p(u^p) \text{ and } (x \in f(y^1,...,y^p) \text{ or } x \in g(y^1,...,y^p))\} = \{x \mid \exists y^1,...,\exists y^p, y^1 \in f^1(u^1) \text{ and} ... \text{and } y^p \in f^p(u^p) \text{ and } x \in f(y^1,...,y^p) \text{ or } y^1 \in f^1(u^1) \text{ and} ... \text{and } y^p \in f^p(u^p) \text{ and } x \in g(y^1,...,y^p)\} = ((f\circ(f^1,...,f^p)) \cup (g\circ(f^1,...,f^p)))(u^1,...,u^p)$$

$$\forall (u^1,...,u^p), (f \circ (f^1 \cup g^1,...,f^p \cup g^p))(u^1,...,u^p) = \{x \mid \exists y^1,...,\exists y^p, (y^1 \in f^1(u^1) \text{ or } y^1 \in g^1(u^1)) \text{ and} ... \text{and } (y^p \in f^p(u^p) \text{ or } y^p \in g^p(u^p)) \text{ and } x \in f(y^1,...,y^p)\} \supset \{x \mid \exists y^1,...,\exists y^p, y^1 \in f^1(u^1) \text{ and} ... \text{and } y^p \in f^p(u^p) \text{ and } x \in f(y^1,...,y^p) \text{ or } y^1 \in g^1(u^1) \text{ and} ... \text{and } y^p \in g^p(u^p) \text{ and } x \in f(y^1,...,y^p)\} = ((f \circ (f^1,...,f^p)) \cup (f \circ (g^1,...,g^p)))(u^1,...,u^p) \blacksquare$$

Remark 39 At Theorem 38, the statements from a) and b), respectively the statements from c) and d) are pairwise similar. To be remarked the asymmetry between the first statements of a) and b).

On the other hand, for the validity of the next theorem we need that the axiom of choice holds.

Theorem 40 The next properties of determinism take place:

a) Any system g includes a deterministic system f.

b) If in the inclusion $f \subset g$ the system g is deterministic, then $f = g$.

Proof. a) For any u, the axiom of choice allows choosing from the set $g(u)$ a point x and defining a selective function $f(u) = \{x\}$. f is a deterministic system and $\forall u, f(u) \subset g(u)$.

b) The formula

$$\forall u, f(u) = g(u)$$

represents the only possibility of choosing f at item a). $\blacksquare$

9. Non-anticipation, the first definition

Definition 41 f is a non-anticipatory (or causative) system if it satisfies for any $u \in S^{(m)}$ any $x \in S^{(n)}$ and any $d \in R$ one of the next equivalent conditions

a) $x \in f(u) \Rightarrow (u \circ \tau^d \in S^{(m)} \Rightarrow x \circ \tau^d \in S^{(n)})$

b) $(x \in f(u)$ and $u \circ \tau^d \in S^{(m)}) \Longrightarrow x \circ \tau^d \in S^{(n)}$

Otherwise, we say that f is anticipatory, or anti-causative.

Theorem 42 The system f is non-anticipatory if and only if $\forall u, \forall x \in f(u)$ one of the next statements is true:

a) x is constant

b) x, u are both variable and we have

$$\min\{t \mid u(t-0) \neq u(t)\} \leq \min\{t \mid x(t-0) \neq x(t)\}$$

thus the first input switch is prior to the first output switch.

Proof. If. When x is constant, $\forall d \in \mathbf{R}, x = x \circ \tau^d \in S^{(n)}$ and the conclusion of 41 b) is true. And if x, u are not constant, we note

$$t_0 = \min\{t \mid u(t-0) \neq u(t)\}$$
$$t_1 = \min\{t \mid x(t-0) \neq x(t)\}$$

In 41 a), $x \in f(u)$ is true, thus $(u \circ \tau^d \in S^{(m)} \Longrightarrow x \circ \tau^d \in S^{(n)})$ should be true when u, x, d run in $S^{(m)}, f(u)$ and $\mathbf{R}$. The next true statements are equivalent:

$(u \circ \tau^d \in S^{(m)} \Longrightarrow x \circ \tau^d \in S^{(n)}) \overset{\text{Lemma, item a)}}{\Longleftrightarrow} t_0 + d \geq 0 \Longrightarrow t_1 + d \geq 0 \Longleftrightarrow t_0 \leq t_1$

Only if. Two possibilities exist of negating the statements

Case I x is variable and u is constant

The hypothesis of 41 b) $(x \in f(u)$ and $u \circ \tau^d \in S^{(m)})$ is true for any $d \in \mathbf{R}$ thus the conclusion is true: $\forall d \in \mathbf{R}, x \circ \tau^d \in S^{(n)}$. x is constant from the Lemma, item c), contradiction

Case II x is variable, u is variable and $t_0 > t_1$

Any $d \in [-t_0, -t_1)$ gives $t_0 + d \geq 0$ and $t_1 + d < 0$, i.e. from the Lemma item a) we get $u \circ \tau^d \in S^{(m)}$ and $x \circ \tau^d \notin S^{(n)}$ contradiction with the non-anticipation of f. ∎

Corollary 43 We suppose that f is non-anticipatory and we consider the functions $u, x \in f(u)$.

a) If u is constant, then x is constant.

b) If u is not constant, then two possibilities exist: either x is constant, or x is not constant and the next condition

$$\min\{t \mid u(t-0) \neq u(t)\} \leq \min\{t \mid x(t-0) \neq x(t)\}$$

is fulfilled,

Proof. a) Special case of Theorem 42, item a), only if.

b) Special case of Theorem 42, item a), only if or coincidence with Theorem 42 item b), only if. ∎

Example 44 We have met non-anticipatory systems at Example 3 (1) and the system f_1 from Example 33 has the same property. Another case is that of the system f with $\forall u, \forall x \in f(u), x$ is the constant function. The system defined by the next equation is also non-anticipatory:

$$x(t) = \bigcap_{\xi \in (-\infty, t)} u_i(\xi)$$

where $i \in \{1, ..., m\}$, since for all u, either x is constant, or it is variable with exactly one switch from 1 to 0 and in this case we can write

$$\min\{t | x(t-0) \neq x(t)\} = \min\{t | u_i(t-0) \neq u_i(t)\} \geq \min\{t | u(t-0) \neq u(t)\}$$

see Theorem 42, if.

Theorem 45 Let f non-anticipatory and $i, j \in \{1, ..., m\}, i \neq j$. Then $\overline{f}$; $f^{(m+1)}$; $f_{i \to j}$; $f_{\hat{i}}$ are non-anticipatory.

Proof. Let $u, x \in \overline{f}(u)$ and $d \in R$ arbitrary so that $u \circ \tau^d \in S^{(m)}$. From the definition of $\overline{f}$ we have that $\overline{x} \in f(u)$ and because f is non-anticipatory $\overline{x} \circ \tau^d \in S^{(n)}$ holds and this is equivalent with any of

$$\min\{t | \overline{x}(t-d-0) \neq \overline{x}(t-d)\} \geq 0$$

$$\min\{t | x(t-d-0) \neq x(t-d)\} \geq 0$$

$$x \circ \tau^d \in S^{(n)}$$

$\overline{f}$ is non-anticipatory.

The fact that

$$(x \in f_{\hat{i}}(u) \text{ and } u \circ \tau^d \in S^{(m)}) \Longrightarrow (x \in f(u) \text{ and } u \circ \tau^d \in S^{(m)}) \Longrightarrow x \circ \tau^d \in S^{(n)}$$

proves that $f_{\hat{i}}$ is non-anticipatory. ■

Theorem 46 The next statements are equivalent for the system f:

a) f is autonomous and non-anticipatory

b) $\exists X, \forall u, f(u) = X$ and $\forall x \in X, x$ is the constant function

Proof. a) $\Longrightarrow$ b) If $\exists X, \forall u, f(u) = X$ we suppose against all reason that $\exists x \in X$ which is not constant and let $t_1 \geq 0$ with $x(t_1 - 0) \neq x(t_1)$. The existence of an u so that for some $t_0 > t_1$ we should have $\forall t < t_0, u(t) = u(0-0)$ and $u(t_0 - 0) \neq u(t_0)$ together with the hypothesis of non-anticipation of f give a contradiction, see Theorem 42 b), only if.

b) $\Longrightarrow$ a) The property is true because if $x \in X$ is constant, then $\forall d \in R, x \circ \tau^d \in S^{(n)}$. ■

Theorem 47 Let the systems f, g and $X \subset S^{(n)}$. If f is non-anticipatory, then

a) $f \cap X$ and $f \cap g$ are non-anticipatory

b) $f \cup X$ is non-anticipatory if and only if X understood as autonomous system is non-anticipatory and $f \cup g$ is non-anticipatory if and only if g is non-anticipatory.

Proof. The implication $\forall u, \forall x, \forall d \in \mathbf{R}$

$$x \in f(u) \cap g(u) \Longrightarrow x \in f(u) \Longrightarrow (u \circ \tau^d \in S^{(m)} \Longrightarrow x \circ \tau^d \in S^{(n)})$$

shows the validity of a). At b), the supposition that $f, f \cup g$ are non-anticipatory and g is anticipatory gives

$$\exists u, \exists x \in g(u) - f(u), \exists d \in \mathbf{R}, u \circ \tau^d \in S^{(m)} \quad \text{and} \quad x \circ \tau^d \notin S^{(n)}$$

contradiction. ■

Theorem 48 If $f, f^1, \ldots, f^p$ defined like previously are non-anticipatory, then $f \circ (f^1, \ldots, f^p)$ is non-anticipatory.

Proof. We suppose that $x \in f \circ (f^1, \ldots, f^p)(u^1, \ldots, u^p)$ and $(u^1 \circ \tau^d, \ldots, u^p \circ \tau^d) \in S^{(m_1 + \ldots + m_p)}$ resulting the existence of $y^1 \in f^1(u^1), \ldots, y^p \in f^p(u^p)$ so that $x \in f(y^1, \ldots, y^p)$. Because $f^1, \ldots, f^p$ are non-anticipatory, we get $y^1 \circ \tau^d \in S^{(n_1)}, \ldots, y^p \circ \tau^d \in S^{(n_p)}$ and from the fact that f is non-anticipatory, we have $x \circ \tau^d \in S^{(n)}$ so that $f \circ (f^1, \ldots, f^p)$ has resulted to be non-anticipatory. ■

Theorem 49 If f is a non-anticipatory system, then any system $g \subset f$ is non-anticipatory.

Proof. $(x \in g(u)$ and $u \circ \tau^d \in S^{(m)}) \Longrightarrow (x \in f(u)$ and $u \circ \tau^d \in S^{(m)}) \Longrightarrow x \circ \tau^d \in S^{(n)}$ ■

10. Non-anticipation, the second definition

Definition 50 The system f is non-anticipatory, or causative if

$$\forall t_1, \forall u, \forall v, (u|_{(-\infty, t_1)} = v|_{(-\infty, t_1)}) \Longrightarrow (\forall x \in f(u), \exists y \in f(v), x|_{(-\infty, t_1)} = y|_{(-\infty, t_1)})$$

and anticipatory, or anti-causative otherwise.

Remark 51 This is another perspective on non-anticipation than the previous one and the two notions are independent logically. The definition states that for any t_1 any u and any $x \in f(u)$, the restriction $x|_{(-\infty, t_1)}$ depends only on the restriction $u|_{(-\infty, t_1)}$ and is independent on the values of $u(t), t \geq t_1$.

A variant of Definition 50 exists, resulted by the replacement of the interval $(-\infty, t_1)$ with $(-\infty, t_1]$.

Example 52 Let's consider the next systems

$$f(u) = \{\chi_{[0,1)} \oplus u_1 \cdot \chi_{[2,\infty)}\}$$

$$g(u) = \begin{cases} \{1\}, & \text{if } u_1 = \chi_{[0,\infty)} \\ \{u_1\}, & \text{otherwise} \end{cases}$$

$f(u)$ is non-anticipatory in the sense of Definition 50, but it is anticipatory in the sense of Definition 41 because for $u_1(t) = \chi_{[2,\infty)}(t)$ the contradiction $u_1 \circ \tau^{-2} = \chi_{[0,\infty)} \in S, x \circ \tau^{-2} = \chi_{[-2,-1)} \oplus \chi_{[0,\infty)} \notin S$ is obtained. $g(u)$ is anticipatory in the sense of Definition 50, because for $t_1 = 1, u_1 = \chi_{[0,\infty)}, v_1 = \chi_{[0,2)}$ the contradiction $1|_{(-\infty,1)} \neq \chi_{[0,2)}|_{(-\infty,1)}$ is obtained; it is non-anticipatory in the sense of Definition 41 however.

Theorem 53 Let f a non-anticipatory system (Definition 50). Then $\overline{f}, f^{(m+1)}, f_{i \to j}, f_{\hat{i}}$ are non-anticipatory, with $i, j \in \{1, ..., m\}, i \neq j$.

Proof. Let t_1, u, v and $\overline{x} \in f(u)$ arbitrary so that $u|_{(-\infty,t_1)} = v|_{(-\infty,t_1)}$; the hypothesis that f is non-anticipatory gives the existence of $\overline{y} \in f(v)$ so that $\overline{x}|_{(-\infty,t_1)} = \overline{y}|_{(-\infty,t_1)}$ i.e. $x|_{(-\infty,t_1)} = y|_{(-\infty,t_1)}$. These show that $\overline{f}$ is non-anticipatory.

We consider $t_1, u_1, ..., u_{m+1}, v_1, ..., v_{m+1}$ and $x \in f^{(m+1)}(u_1, ..., u_{m+1}) = f(u_1, ..., u_m)$ arbitrary, so that $(u_1, ..., u_{m+1})|_{(-\infty,t_1)} = (v_1, ..., v_{m+1})|_{(-\infty,t_1)}$. From the fact that f is non-anticipatory we have the existence of $y \in f(v_1, ..., v_m) = f^{(m+1)}(v_1, ..., v_{m+1})$ so that $x|_{(-\infty,t_1)} = y|_{(-\infty,t_1)}$ i.e. $f^{(m+1)}$ is non-anticipatory.

The part of the proof corresponding to $f_{i \to j}$ and $f_{\hat{i}}$ is similar. ■

Theorem 54 If f, g are non-anticipatory systems, then $f \cup g$ is non-anticipatory.

Proof. Let t_1, u, v and $x \in f(u) \cup g(u)$ arbitrary so that $u|_{(-\infty,t_1)} = v|_{(-\infty,t_1)}$. If for example $x \in f(u)$, then the fact that f is non-anticipatory shows the existence of $y \in f(v)$ so that $x|_{(-\infty,t_1)} = y|_{(-\infty,t_1)}$; we conclude that $y \in f(u) \cup g(u)$ exists with $x|_{(-\infty,t_1)} = y|_{(-\infty,t_1)}$. ■

Theorem 55 If f is non-anticipatory, then its initial state function ϕ satisfies

$$\forall u, \forall v, u(0-0) = v(0-0) \Longrightarrow \phi(u) = \phi(v)$$

Proof. Let u, v arbitrary so that $u(0-0) = v(0-0)$, thus some t_1 exists with $u|_{(-\infty,t_1)} = v|_{(-\infty,t_1)}$. From the non-anticipation of f we get $\forall x \in f(u), \exists y \in f(v), x|_{(-\infty,t_1)} = y|_{(-\infty,t_1)}$ thus $\forall x' \in \phi(u)$ we have that $x' \in \phi(v)$. ■

Theorem 56 If $f, f^1, ..., f^p$ are non-anticipatory systems, then $f \circ (f^1, ..., f^p)$ is non-anticipatory.

Proof. Let $u^1, ..., u^p, v^1, ..., v^p$ and t_1 arbitrary with

$$u^1|_{(-\infty,t_1)} = v^1|_{(-\infty,t_1)}, ..., u^p|_{(-\infty,t_1)} = v^p|_{(-\infty,t_1)}$$

and $x \in (f \circ (f^1, \ldots, f^p))(u^1, \ldots, u^p)$ arbitrary also, thus $y^1 \in f^1(u^1), \ldots, y^p \in f^p(u^p)$ exist so that $x \in f(y^1, \ldots, y^p)$. Because $f^1, \ldots, f^p$ are non-anticipatory, $z^1 \in f^1(v^1), \ldots, z^p \in f^p(v^p)$ exist so that

$$y^1|_{(-\infty, t_1)} = z^1|_{(-\infty, t_1)}, \ldots, y^p|_{(-\infty, t_1)} = z^p|_{(-\infty, t_1)}$$

and because f is non-anticipatory we get the existence of $x' \in f(z^1, \ldots, z^p)$ with $x|_{(-\infty, t_1)} = x'|_{(-\infty, t_1)}$. $f \circ (f^1, \ldots, f^p)$ is non-anticipatory. ∎

Theorem 57 Any autonomous system $X \subset S^{(n)}$ is non-anticipatory.

Proof. For any t_1 we have

$$\forall x \in X, \exists y \in X, x|_{(-\infty, t_1)} = y|_{(-\infty, t_1)}$$

thus the conclusion of Definition 50 is true. ∎

Corollary 58 If f is non-anticipatory and $X \subset S^{(n)}$, the system $f \cup X$ is non-anticipatory.

Proof. The result follows from Theorem 54 and Theorem 57. ∎

Theorem 59 If f is a deterministic system (understood as $S^{(m)} \to S^{(n)}$ function), then the next statements are equivalent:

a) f is non-anticipatory

b) $\forall t_1, \forall u, \forall v, (u|_{(-\infty, t_1)} = v|_{(-\infty, t_1)}) \Longrightarrow (f(u)|_{(-\infty, t_1)} = f(v)|_{(-\infty, t_1)})$

Proof. Obvious. ∎

Remark 60 Proving that f, g non-anticipatory implies that $f \cap g$ is non-anticipatory was unsuccesful. This leaves open the problem of finding two non-anticipatory systems f, g so that $\forall u, f(u) \cap g(u) \neq \emptyset$ and $f \cap g$ is anticipatory.

11. Time invariance

Definition 61 The system f is time invariant if $\forall u \in S^{(m)}, \forall x \in S^{(n)}, \forall d \in \mathbf{R}$, one of the next equivalent statements is fulfilled:

a) $(u \circ \tau^d \in S^{(m)}$ and $x \in f(u)) \Longrightarrow (x \circ \tau^d \in S^{(n)}$ and $x \circ \tau^d \in f(u \circ \tau^d))$

b) $((u \circ \tau^d \in S^{(m)}$ and $x \in f(u)) \Longrightarrow x \circ \tau^d \in S^{(n)})$ and $((u \circ \tau^d \in S^{(m)}$ and $x \in f(u)) \Longrightarrow x \circ \tau^d \in f(u \circ \tau^d))$

If the previous property is not true, then f is called time variable.

Remark 62 If the signals would have been defined by replacing the request of existence of an initial time instant $t_0 \geq 0$ with the existence of an arbitrary initial time instant t_0, then time invariance would have simply been defined by $\forall u, \forall x, \forall d, (x \in f(u) \Longrightarrow x \circ \tau^d \in f(u \circ \tau^d))$. The way that S was defined however, it is tightly related with the first definition of non-anticipation: time invariance is the property of the non-anticipatory systems (Definition 41) of satisfying $x \circ \tau^d \in f(u \circ \tau^d)$ whenever $u \circ \tau^d \in S^{(m)}$ and $x \in f(u)$ hold.

Example 63 We analize two deterministic systems.

a) We show that $f(u) = \{u_i \circ \tau^{d'}\}$ is time invariant, where $i \in \{1, ..., m\}$ and $d' \geq 0$. The hypothesis $u \circ \tau^d \in S^{(m)}$ states that $u_i \circ \tau^d \in S$, from where $(u_i \circ \tau^d) \circ \tau^{d'} \in S$ and we have

- $(u_i \circ \tau^d) \circ \tau^{d'} = u_i \circ \tau^{d+d'} = (u_i \circ \tau^{d'}) \circ \tau^d \in S$ $\quad (x \circ \tau^d \in S)$
- $(u_i \circ \tau^{d'}) \circ \tau^d = (u_i \circ \tau^d) \circ \tau^{d'}$ $\quad (x \circ \tau^d \in f(u \circ \tau^d))$

b) Let the system defined by the equation

$$x(t) = \lim_{\xi \to \infty} \bigcup_{\omega \in (\xi, \infty)} (u_1(\omega) \cdot ... \cdot u_m(\omega)) \tag{11}$$

(the function in $\xi : \bigcup_{\omega \in (\xi, \infty)} (u_1(\omega) \cdot ... \cdot u_m(\omega))$ switches at most once from 1 to 0 for all u, thus the limit $\lim_{\xi \to \infty} \bigcup_{\omega \in (\xi, \infty)} (u_1(\omega) \cdot ... \cdot u_m(\omega))$ always exists and (11) defines a system indeed). Because x is the constant function, $x \circ \tau^d \in S$ is true for any d, thus the system is non-anticipatory in the sense of Definition 41. By observing that for any $d \in \mathbf{R}$,

$$\lim_{\xi \to \infty} \bigcup_{\omega \in (\xi, \infty)} (u_1(\omega - d) \cdot ... \cdot u_m(\omega - d)) = \lim_{\xi \to \infty} \bigcup_{\omega \in (\xi, \infty)} (u_1(\omega) \cdot ... \cdot u_m(\omega)) = x(t) = x(t - d)$$

the second statement from Definition 61 b) results. The system is time invariant.

Theorem 64 Let f time invariant. The next equivalence holds:

$$\forall u, \forall x, \forall d \geq 0, x \in f(u) \Longleftrightarrow x \circ \tau^d \in f(u \circ \tau^d)$$

Proof. $\Longrightarrow$ The statements $u \circ \tau^d \in S^{(m)}$ and $x \in f(u)$ are both true. We apply the time invariance of f.

$\Longleftarrow$ $(u \circ \tau^d) \circ \tau^{-d} \in S^{(m)}$ and $x \circ \tau^d \in f(u \circ \tau^d)$ are true. We apply the time invariance of f again and we get $(x \circ \tau^d) \circ \tau^{-d} \in f((u \circ \tau^d) \circ \tau^{-d})$. ∎

Theorem 65 Let f time invariant and $i, j \in \{1, ..., m\}, i \neq j$: $\overline{f}$; $f^{(m+1)}$; $f_{i \to j}$; $f_{\hat{i}}$ are time invariant.

Proof. $\overline{f}$; $f^{(m+1)}$; $f_{i \to j}$; $f_{\hat{i}}$ are non-anticipatory (Definition 41) from Theorem 45. From the truth of the implication

$$(u \circ \tau^d \in S^{(m)} \text{ and } x \in f(u)) \Longrightarrow x \circ \tau^d \in f(u \circ \tau^d)$$

for all u, x and d we get the truth of

$$(u \circ \tau^d \in S^{(m)} \text{ and } \overline{x} \in f(u)) \Longrightarrow \overline{x} \circ \tau^d \in f(u \circ \tau^d)$$

thus $\overline{f}$ is time invariant.

This part of the proof brings nothing new in the other three cases. ∎

Theorem 66 If f, g are time invariant, then $f \cap g, f \cup g$ are time invariant.

Proof. $f \cap g, f \cup g$ are non-anticipatory (Definition 41) from Theorem 47. From the truth for all u, x, d of

$$(u \circ \tau^d \in S^{(m)} \text{ and } x \in f(u)) \Longrightarrow x \circ \tau^d \in f(u \circ \tau^d)$$

$$(u \circ \tau^d \in S^{(m)} \text{ and } x \in g(u)) \Longrightarrow x \circ \tau^d \in g(u \circ \tau^d)$$

we infer with simple computations that

$$(u \circ \tau^d \in S^{(m)} \text{ and } x \in (f \cap g)(u)) \Longrightarrow x \circ \tau^d \in (f \cap g)(u \circ \tau^d)$$

$$(u \circ \tau^d \in S^{(m)} \text{ and } x \in (f \cup g)(u)) \Longrightarrow x \circ \tau^d \in (f \cup g)(u \circ \tau^d)$$

are fulfilled. ∎

Theorem 67 We suppose that $f, f^1, \ldots, f^p$ are time invariant. Then $f \circ (f^1, \ldots, f^p)$ is time invariant.

Proof. $f \circ (f^1, \ldots, f^p)$ is non-anticipatory (Definition 41), as resulting from Theorem 48. Let now $(u^1, \ldots, u^p), x \in f \circ (f^1, \ldots, f^p)(u^1, \ldots, u^p)$ arbitrary and $y^1 \in f^1(u^1), \ldots, y^p \in f^p(u^p)$ so that $x \in f(y^1, \ldots, y^p)$. The hypothesis states that

$$u^1 \circ \tau^d \in S^{(m_1)}, \ldots, u^p \circ \tau^d \in S^{(m_p)}$$

are true and from the time invariance of $f^1, \ldots, f^p$ we get that

$$y^1 \circ \tau^d \in f^1(u^1 \circ \tau^d), \ldots, y^p \circ \tau^d \in f^p(u^p \circ \tau^d)$$

are true. But f is time invariant itself thus $x \circ \tau^d \in f(y^1 \circ \tau^d, \ldots, y^p \circ \tau^d)$. $f \circ (f^1, \ldots, f^p)$ is time invariant. ∎

Theorem 68 The next statements are equivalent:

a) f is autonomous and time invariant

b) $\exists X, f = X$ and $\forall x \in X, x$ is the constant function.

Proof. a) $\Longrightarrow$ b) f is autonomous and non-anticipatory (Definition 41) thus b) is true from Theorem 46:

b) $\Longrightarrow$ a) f is autonomous and non-anticipatory from Theorem 46. Furthermore the truth of

$$(u \circ \tau^d \in S^{(m)} \quad \text{and} \quad x \in X) \Longrightarrow x \circ \tau^d \in X$$

(because $x = x \circ \tau^d$ when x is constant) shows the validity of a). ∎

Corollary 69 If f is time invariant and X satisfies $\forall x \in X, x$ is the constant function, then $f \cap X, f \cup X$ are time invariant.

Proof. This results from Theorem 66 and Theorem 68. ∎

12. Symmetry, the first definition

Definition 70 The Boolean function $F : \mathbf{B}^m \to \mathbf{B}^n$ is called (coordinatewise) symmetrical if for any bijection $\sigma : \{1, ..., m\} \to \{1, ..., m\}$ we have

$$\forall \lambda \in \mathbf{B}^m, F(\lambda) = F(\lambda_\sigma)$$

and asymmetrical otherwise.

Definition 71 The system f is (coordinatewise) symmetrical if for any bijection σ we have

$$\forall u \in S^{(m)}, f(u) = f(u_\sigma)$$

and it is asymmetrical otherwise.

Example 72 All the systems with $m = 1$ are trivially symmetrical and the systems from Example 3 (3), (4), respectively from Example 63 b) are also symmetrical. If $F : \mathbf{B}^m \to \mathbf{B}^n$ is a symmetrical function, then the deterministic system induced by F (Example 25) is symmetrical. The system

$$f(u) = \{x \mid x(t) \geq u_1(t) \cdot ... \cdot u_m(t)\}$$

is symmetrical too.

Theorem 73 f is symmetrical implies that $\overline{f}$ is symmetrical.

Proof. $\overline{f}(u) = \{\overline{x} \mid x \in f(u)\} = \{\overline{x} \mid x \in f(u_\sigma)\} = \overline{f}(u_\sigma)$ are true for all σ and u. ∎

Theorem 74 Let f, g symmetrical systems. Then $f \cap g, f \cup g$ are symmetrical systems.

Proof. We can write for σ, u and x arbitrary:
$x \in (f \cap g)(u) \iff x \in f(u)$ and $x \in g(u) \iff x \in f(u_\sigma)$ and $x \in g(u_\sigma) \iff x \in (f \cap g)(u_\sigma)$
The proof for the reunion is similar. ∎

Theorem 75 If f is symmetrical, then ϕ is symmetrical.

Proof. For any σ and u we have $\phi(u) = \{x(0 - 0) \mid x \in f(u)\} = \{x(0 - 0) \mid x \in f(u_\sigma)\} = \phi(u_\sigma)$. ∎

Remark 76 If $f^1, \ldots, f^p$ are symmetrical systems, then the next symmetry relation holds

$$f\,\delta(f^1, \ldots, f^p)(u^1, \ldots, u^p) = f\,\delta(f^{\sigma'(1)}, \ldots, f^{\sigma'(p)})(u^{\sigma'(1)}_{\sigma_{\sigma'(1)}}, \ldots, u^{\sigma'(p)}_{\sigma_{\sigma'(p)}})$$

where $\sigma_i : \{1, \ldots, m_i\} \to \{1, \ldots, m_i\}, i = \overline{1,p}$ and $\sigma' : \{1, \ldots, p\} \to \{1, \ldots, p\}$ are bijections. We observe that $f\,\delta(f^1, \ldots, f^p)$ is not a symmetrical system in general.

Theorem 77 If f is autonomous, then it is symmetrical.

Proof. $\exists X, \forall u, f(u) = X$ implies for any bijection $\sigma : \{1, \ldots, m\} \to \{1, \ldots, m\}$ that $f(u_\sigma) = X$ ∎

Corollary 78 If f is symmetrical, then $f \cap X, f \cup X$ are symmetrical.

Proof. This fact results from Theorem 74 and Theorem 77. ∎

13. Symmetry, the second definition

Definition 79 The function $F : \mathbf{B}^m \to \mathbf{B}^n$ is called symmetrical (in the rising-falling sense) if

$$\forall \lambda \in \mathbf{B}^m, F(\lambda) = \overline{F(\overline{\lambda})}$$

and asymmetrical otherwise.

Definition 80 The system f is symmetrical (in the rising-falling sense) if

$$\forall u, f(u) = \overline{f}(\overline{u})$$

and respectively asymmetrical otherwise.

Remark 81 This type of symmetry of f states that the form of x under the input u coincides with the form of $\overline{x}$ under the input $\overline{u}$ and the terminology of rising-falling symmetry is due to the fact that while $x(t)$ switches at the time instant t in the rising (falling) sense, $\overline{x}(t)$ switches at the time instant t in the falling (rising) sense:

$$\forall i \in \{1, \ldots, n\}, \overline{x_i(t-0)} \cdot x_i(t) = \overline{x_i(t-0)} \cdot \overline{\overline{x_i(t)}}, \; x_i(t-0) \cdot \overline{x_i(t)} = \overline{\overline{x_i(t-0)}} \cdot \overline{x_i(t)}$$

Example 82 Some examples of symmetrical functions $F(\lambda)$ (Definition 79) are the affine functions: $\lambda_i, i = \overline{1,m}$, $\lambda_{i_1} \oplus \lambda_{i_2} \oplus \lambda_{i_3}$, $i_1, i_2, i_3 \in \{1, \ldots, m\}$, etc. The symmetrical Boolean functions define symmetrical deterministic systems, for example $F : \mathbf{B}^3 \to \mathbf{B}, F(\lambda_1, \lambda_2, \lambda_3) = \lambda_1 \oplus \lambda_2 \oplus \lambda_3$ is symmetrical and it defines the symmetrical deterministic system $f(u) = \{u_1 \oplus u_2 \oplus u_3\}$.

Let now the non-deterministic system $f(u) = \{u_1 \cdot u_2\} \cup \{u_1 \vee u_2\}$. The satisfaction of the Morgan laws

$$x(t) = u_1(t) \cdot u_2(t) \iff \overline{x(t)} = \overline{u_1(t)} \vee \overline{u_2(t)}$$
$$x(t) = u_1(t) \vee u_2(t) \iff \overline{x(t)} = \overline{u_1(t)} \cdot \overline{u_2(t)}$$

shows that it is symmetrical.

Theorem 83 If f is symmetrical, then $\overline{f}$; $f^{(m+1)}$; $f_{i \to j}$ and $f_{\mathbf{b}_i}$ are symmetrical for all $i, j \in \{1, \ldots, m\}$, $i \neq j$.

Proof. The conditions of symmetry

$$\forall u, f(u) = \overline{f}(\overline{u})$$

$$\forall u, \overline{f}(u) = f(\overline{u})$$

of f and $\overline{f}$ are equivalent, proving the first statement of the theorem.

We suppose that f is symmetrical. For any $(u_1, \ldots, u_{m+1})$ we can write

$$f^{(m+1)}(u_1, \ldots, u_{m+1}) = f(u_1, \ldots, u_m) = \overline{f}(\overline{u_1}, \ldots, \overline{u_m}) = \overline{f^{(m+1)}}(\overline{u_1}, \ldots, \overline{u_{m+1}})$$

$$f_{i \to j}(u_1, \ldots, \underset{i}{u_i}, \ldots, \underset{j}{u_j}, \ldots, u_m) = f(u_1, \ldots, \underset{i}{u_i}, \ldots, \underset{j}{u_i}, \ldots, u_m) =$$

$$= \overline{f}(\overline{u_1}, \ldots, \underset{i}{\overline{u_i}}, \ldots, \underset{j}{\overline{u_i}}, \ldots, \overline{u_m}) = \overline{f_{i \to j}}(\overline{u_1}, \ldots, \underset{i}{\overline{u_i}}, \ldots, \underset{j}{\overline{u_j}}, \ldots, \overline{u_m})$$

$$f_{\mathbf{b}_i}(u_1, \ldots, \mathbf{b}_i, \ldots, u_m) = f(u_1, \ldots, \underset{i}{0}, \ldots, u_m) = \overline{f}(\overline{u_1}, \ldots, \underset{i}{\overline{0}}, \ldots, \overline{u_m}) = \overline{f_{\mathbf{b}_i}}(\overline{u_1}, \ldots, \overline{\mathbf{b}_i}, \ldots, \overline{u_m})$$

and these prove the last three statements of the Theorem. ∎

Theorem 84 If the systems f, g are symmetrical, then the systems $f \cap g$ and $f \cup g$ are symmetrical.

Proof. $\forall u, \forall x, x \in (f \cap g)(u) \iff x \in f(u)$ and $x \in g(u) \iff \overline{x} \in f(\overline{u})$ and $\overline{x} \in g(\overline{u}) \iff \overline{x} \in (f \cap g)(\overline{u}) \iff x \in \overline{(f \cap g)}(\overline{u})$

and similarly for the second statement. ∎

Theorem 85 If f is symmetrical, then the next formula is true

$$\forall u, \varphi(u) = \overline{\varphi}(\overline{u})$$

Proof. $\forall u, \varphi(u) = \{x(0 - 0) | x \in f(u)\} = \{x(0 - 0) | x \in \overline{f}(\overline{u})\} = \overline{\varphi}(\overline{u})$ ∎

Theorem 86 If $f, f^1, \ldots, f^p$ are symmetrical systems, then $f \circ (f^1, \ldots, f^p)$ is symmetrical.

Proof. $\forall (u^1, \ldots, u^p), \forall x, x \in f \circ (f^1, \ldots, f^p)(u^1, \ldots, u^p) \iff$

$\iff \exists y^1 \in f^1(u^1), \ldots, \exists y^p \in f^p(u^p)$ s.t. $x \in f(y^1, \ldots, y^p)$

$\iff \exists \overline{y^1} \in f^1(\overline{u^1}), \ldots, \exists \overline{y^p} \in f^p(\overline{u^p})$ s.t. $\overline{x} \in f(\overline{y^1}, \ldots, \overline{y^p})$

$\iff \overline{x} \in f \circ (f^1, \ldots, f^p)(\overline{u^1}, \ldots, \overline{u^p}) \iff x \in \overline{f \circ (f^1, \ldots, f^p)}(\overline{u^1}, \ldots, \overline{u^p})$ ∎

Theorem 87 Let $f = X$ an autonomous system, with $X \subset S^{(n)}$. The next statements are equivalent:

a) f is symmetrical

b) $\forall x, x \in X \iff \overline{x} \in X$

Proof. $\forall u, f(u) = f(\overline{u}) = X$ and the equivalence between a) and b) is easily proved ∎

Corollary 88 If f is symmetrical and $X \subset S^{(n)}$ satisfies

$$\forall x, x \in X \iff \overline{x} \in X$$

then $f \cap X$ and $f \cup X$ are symmetrical.

Proof. From Theorem 84 and 87. ∎

14. Stability

Definition 89 We consider the Boolean function $F : \mathbf{B}^m \to \mathbf{B}^n$ and the next properties of the system f:

a) absolute stability

$$\forall u, \forall x \in f(u), \exists t_1, \forall t \geq t_1, x(t) = x(t_1)$$

b) relative stability

$$\forall u, \forall x \in f(u), (\exists t_1, \forall t \geq t_1, u(t) = u(t_1)) \Longrightarrow (\exists t_1, \forall t \geq t_1, x(t) = x(t_1))$$

c) stability relative to F:

$$\forall u, \forall x \in f(u), (\exists t_1, \forall t \geq t_1, F(u(t)) = F(u(t_1))) \Longrightarrow (\exists t_1, \forall t \geq t_1, x(t) = x(t_1))$$

d) delay-insensitivity relative to F:

$$\forall u, \forall x \in f(u), (\exists t_1, \forall t \geq t_1, F(u(t)) = F(u(t_1))) \Longrightarrow (\exists t_1, \forall t \geq t_1, x(t) = F(u(t_1)))$$

Remark 90 The stability problem is that of the existence of the limit $\lim\limits_{t \to \infty} x(t)$ and Definition 89 states such stability conditions true for any u and any $x \in f(u)$, the next implications being true:

$$\begin{array}{ccccc} a) & \Longrightarrow & c) & \Longleftarrow & d) \\ & & \Downarrow & & \\ & & b) & & \end{array}$$

In Definition 89, F is the 'Boolean function to be computed' and $F(u(t))$ is the cause of x. When the cause is persistent in the sense that $\lim\limits_{t \to \infty} F(u(t))$ exists and if f is delay-insensitive relative to F, we have $\lim\limits_{t \to \infty} x(t) = \lim\limits_{t \to \infty} F(u(t))$, the so called 'unbounded delay model' giving the manner in which the values of x reproduce the values of $F(u)$. The stability of f relative to F should be interpreted like this: when the cause is persistent, thus $\lim\limits_{t \to \infty} F(u(t))$ exists, we have that $\lim\limits_{t \to \infty} x(t)$ exists, thus f is stable, but the two limits are not necessarily equal; this phenomenon is called hazard when we regard the states of f as starting, but not completing (correctly) the computation of $F(u)$ and another possibility exists also that the states of $f(u)$ do not compute $F(u)$.

Example 91 The systems from Example 13 (9), (10), respectively from Example 63 (11) are absolutely stable. The system

$$f(u) = \{x | \exists t_1, \forall t \geq t_1, x(t) = u_1(t) \cdot \ldots \cdot u_m(t)\}$$

is delay-insensitive relative to $F(\lambda) = \lambda_1 \cdot \ldots \cdot \lambda_m$ and relatively stable, but it is not absolutely stable;

$$f(u) = \{x | \exists t_1, \forall t \geq t_1, x(t) = \begin{cases} 0, & \text{if } \exists \lim\limits_{\xi \to \infty} u_1(\xi) \cdot \ldots \cdot u_m(\xi) \\ u_1(t), & \text{else} \end{cases}\}$$

is relatively stable and stable relative to $F(\lambda) = \lambda_1 \cdot \ldots \cdot \lambda_m$ but it is neither absolutely stable, nor delay-insensitive relative to F and

$$f(u) = \{x | \exists t_1, \forall t \geq t_1, x(t) = \begin{cases} 0, & \text{if } \exists \lim\limits_{\xi \to \infty} u(\xi) \\ u_1(t), & \text{else} \end{cases}\}$$

is relatively stable, but it is not absolutely stable. For $F(\lambda) = \lambda_2$, by taking $(u_1, u_2, \ldots, u_m) = (\chi_{[0,1) \cup [2,3) \cup [4,5) \cup \ldots}, 0, \ldots, 0)$ we remark that f is not stable relative to F.

Theorem 92 The next statements are equivalent:

a) f is absolutely stable

b) f is stable relative to the constant function

and the next statements are also equivalent for $\mu \in B^n$:

i) $\forall u, \forall x \in f(u), \exists t_1, \forall t \geq t_1, x(t) = \mu$

ii) f is delay-insensitive relative to the constant function $F = \mu$.

Proof. a) $\Longleftrightarrow$ b) is true because a) is the conclusion of b), where b) has a hypothesis always fulfilled.

i) $\Longleftrightarrow$ ii) takes place in similar conditions with the previous equivalence. ∎

Theorem 93 If $F, G : B^m \to B^n$ are two Boolean functions with

$$\forall \lambda, \forall \lambda', F(\lambda) = F(\lambda') \Longrightarrow G(\lambda) = G(\lambda') \tag{12}$$

and if the system f is stable relative to G, then it is stable relative to F.

Proof. We suppose that f is stable relative to G:

$$\forall u, \forall x \in f(u), (\exists t_1, \forall t \geq t_1, G(u(t)) = G(u(t_1))) \Longrightarrow (\exists t_1, \forall t \geq t_1, x(t) = x(t_1))$$

and let $u, x \in f(u)$ arbitrary so that

$$\exists t_1, \forall t \geq t_1, F(u(t)) = F(u(t_1))$$

The hypothesis (12) states that

$$\exists t_1, \forall t \geq t_1, G(u(t)) = G(u(t_1))$$

from where

$$\exists t_1, \forall t \geq t_1, x(t) = x(t_1)$$

and f is stable relative to F. ∎

Theorem 94 Let the Boolean function F and the system f. If f is absolutely stable (relatively stable, stable relative to F, delay-insensitive relative to F), then the systems $\overline{f}, f^{(m+1)}, f_{i\to j}, f_{\widehat{i}}$ are absolutely stable (relatively stable, stable relative to $\overline{F}, F^{(m+1)}, F_{i\to j}, F_{\widehat{i}}$, delay-insensitive relative to $\overline{F}, F^{(m+1)}, F_{i\to j}, F_{\widehat{i}}$), where $i, j \in \{1, ..., m\}, i \neq j$ and $\overline{F}, F^{(m+1)}, F_{i\to j}, F_{\widehat{i}}$ are defined by:

$$\overline{F} : B^m \to B^n, \overline{F}(\lambda_1, ..., \lambda_m) = \overline{F(\lambda_1, ..., \lambda_m)}$$

$$F^{(m+1)} : B^{m+1} \to B^n, F^{(m+1)}(\lambda_1, ..., \lambda_{m+1}) = F(\lambda_1, ..., \lambda_m)$$

$$F_{i\to j} : B^m \to B^n, F_{i\to j}(\lambda_1, ..., \lambda_m) = F(\lambda_1, ..., \underset{i}{\lambda_i}, ..., \underset{j}{\lambda_i}, ..., \lambda_m)$$

$$F_{\widehat{i}} : B^{m-1} \to B^n, F_{\widehat{i}}(\lambda_1, ..., \widehat{\lambda_i}, ..., \lambda_m) = F(\lambda_1, ..., \underset{i}{0}, ..., \lambda_m)$$

Proof. We suppose that f is delay insensitive relative to F:

$$\forall u, \forall x \in f(u), (\exists t_1, \forall t \geq t_1, F(u(t)) = F(u(t_1))) \Longrightarrow (\exists t_1, \forall t \geq t_1, x(t) = F(u(t_1)))$$

from where

$$\forall u, \forall \overline{x} \in f(u), (\exists t_1, \forall t \geq t_1, \overline{F}(u(t)) = \overline{F}(u(t_1))) \Longrightarrow (\exists t_1, \forall t \geq t_1, \overline{x}(t) = \overline{F}(u(t_1)))$$

i.e. $\overline{f}$ is delay-insensitive relative to $\overline{F}$. Moreover, we observe that

$$\forall (u_1, ..., u_{m+1}), \forall x \in f(u_1, ..., u_m) = f^{(m+1)}(u_1, ..., u_{m+1}),$$

$$(\exists t_1, \forall t \geq t_1, F^{(m+1)}(u_1(t), ..., u_{m+1}(t)) = F^{(m+1)}(u_1(t_1), ..., u_{m+1}(t_1))) \Longleftrightarrow$$

$$\Longleftrightarrow (\exists t_1, \forall t \geq t_1, F(u_1(t), ..., u_m(t)) = F(u_1(t_1), ..., u_m(t_1))) \Longrightarrow$$

$$\Longrightarrow (\exists t_1, \forall t \geq t_1, x(t) = F(u_1(t_1), ..., u_m(t_1)) \Longleftrightarrow$$

$$\Longleftrightarrow (\exists t_1, \forall t \geq t_1, x(t) = F^{(m+1)}(u_1(t_1), ..., u_{m+1}(t_1))$$

meaning that $f^{(m+1)}$ is delay-insensitive relative to $F^{(m+1)}$ etc ∎

Remark 95 If f is stable relative to F, then it is stable relative to $\overline{F}$.

Theorem 96 Let the systen f be absolutely stable (relatively stable, stable relative to F, delay-insensitive relative to F). The next statements are true:

a) Any system $f' \subset f$ is absolutely stable (relatively stable, stable relative to F, delay-insensitive relative to F)

b) If the system g is absolutely stable (relatively stable, stable relative to F, delay-insensitive relative to F) then $f \cup g$ is absolutely stable (relatively stable, stable relative to F, delay-insensitive relative to F).

Proof. b) We suppose that f, g are delay-insensitive relative to F and let $u, x \in (f \cup g)(u)$ arbitrary, for example $x \in f(u)$. We have

$$(\exists t_1, \forall t \geq t_1, F(u(t)) = F(u(t_1))) \Longrightarrow (\exists t_1, \forall t \geq t_1, x(t) = F(u(t_1)))$$

from where we infer the delay-insensitivity of $f \cup g$ relative to F. ∎

Corollary 97 If f is absolutely stable (relatively stable, stable relative to F, delay-insensitive relative to F), then $f \cap X$ and $f \cap g$ are absolutely stable (relatively stable, stable relative to F, delay-insensitive relative to F), for any $X \subset S^{(n)}$ and any system g.

Proof. Special case of Theorem 96 a). ∎

Theorem 98 Let the functions $F : B^m \to B^n, F^i : B^{m_i} \to B^{n_i}, i = \overline{1,p}$ and the systems $f : S^{(m)} \to P^*(S^{(n)}), f^i : S^{(m_i)} \to P^*(S^{(n_i)}), i = \overline{1,p}$ so that $n_1 + ... + n_p = m$.

a) If $f, f^1, ..., f^p$ are relatively stable (stable relative to $F, F^1, ..., F^p$, delay-insensitive relative to $F, F^1, ..., F^p$), then $f \circ (f^1, ..., f^p)$ is relatively stable (stable relative to $F \circ (F^1, ..., F^p)$, delay-insensitive relative to $F \circ (F^1, ..., F^p)$)

b) If f is absolutely stable, then $f \circ (f^1, ..., f^p)$ is absolutely stable.

Proof. a) We suppose for example that $f^1, ..., f^p$ are stable relative to $F^1, ..., F^p$:

$$\forall u^1, \forall y^1 \in f^1(u^1), (\exists t_1, \forall t \geq t_1, F^1(u^1(t)) = F^1(u^1(t_1))) \Longrightarrow (\exists t_1, \forall t \geq t_1, y^1(t) = y^1(t_1))$$

$$...$$

$$\forall u^p, \forall y^p \in f^p(u^p), (\exists t_1, \forall t \geq t_1, F^p(u^p(t)) = F^p(u^p(t_1))) \Longrightarrow (\exists t_1, \forall t \geq t_1, y^p(t) = y^p(t_1))$$

or equivalently

$$\forall (u^1, ..., u^p), \forall y \in (f^1, ..., f^p)(u^1, ..., u^p),$$
$$(\exists t_1, \forall t \geq t_1, (F^1, ..., F^p)(u^1(t), ..., u^p(t)) = (F^1, ..., F^p)(u^1(t_1), ..., u^p(t_1))) \Longrightarrow$$
$$\Longrightarrow (\exists t_1, \forall t \geq t_1, y(t) = y(t_1))$$

We have noted $y = (y^1, ..., y^p)$ and we suppose from now that $(u^1, ..., u^p), y$ are arbitrary, fixed. Because f is stable relative to F, we can write

$$\forall x \in f(y), (\exists t_1, \forall t \geq t_1, F(y(t)) = F(y(t_1))) \Longrightarrow (\exists t_1, \forall t \geq t_1, x(t) = x(t_1))$$

$f \circ (f^1, ..., f^p)$ is stable relative to $F \circ (F^1, ..., F^p)$.

b) Let $u^1, ..., u^p, y^1 \in f^1(u^1), ..., y^p \in f^p(u^p)$ and $x \in f(y^1, ..., y^p)$ arbitrary. t_1 exists so that $\forall t \geq t_1, x(t) = x(t_1)$ from where the conclusion that $f \circ (f^1, ..., f^p)$ is absolutely stable follows. ∎

Theorem 99 Let the Boolean function F and the set $X \subset S^{(n)}$ that is identified with an autonomous system f. The next statements are equivalent:

a) $\forall x \in X, \exists t_1, \forall t \geq t_1, x(t) = x(t_1)$

b) f is absolutely stable

c) f is relatively stable

d) f is stable relative to F

and the next statements are also equivalent for some $\mu \in B^n$:

i) $\forall x \in X, \exists t_1, \forall t \geq t_1, x(t) = \mu$

ii) f is delay-insensitive relative to the constant function $F = \mu$.

Proof. a) and b) are obviously equivalent. We suppose that f is relatively stable and we choose u so that $\exists t_1, \forall t \geq t_1, u(t) = u(t_1)$. Then a) takes place and because the hypothesis depending on u implies a conclusion that is independent on u, we have that c) implies a). The implication a)$\Longrightarrow$ c) is obvious.

a) $\Longleftrightarrow$ d) is shown similarly with a) $\Longleftrightarrow$ c).

i) $\Longleftrightarrow$ ii) takes place because i) is the conclusion of the request of delay-insensitivity of f relative to $F = \mu$. ■

15. Fundamental mode

Definition 100 Let the Boolean function $F : \mathbf{B}^m \to \mathbf{B}^n$, the system $f : S^{(m)} \to P^*(S^{(n)})$, the input $u \in S^{(m)}$ and the state $x \in f(u)$. We suppose that an unbounded sequence $0 \leq t_0 < t_1 < t_2 < \ldots$ exists so that the next properties be stated:

$$\forall t < t_0, u(t) = u(t_0 - 0) \tag{13}$$

$$\forall k \geq 0, \forall t \in [t_k, t_{k+1}), u(t) = u(t_k) \tag{14}$$

$$\forall k \geq 0, \forall t \in [t_k, t_{k+1}), F(u(t)) = F(u(t_k)) \tag{15}$$

$$\forall k \geq 1, x \cdot \chi_{(-\infty, t_k)} \oplus x(t_k - 0) \cdot \chi_{[t_k, \infty)} \in f(u \cdot \chi_{(-\infty, t_k)} \oplus u(t_k - 0) \cdot \chi_{[t_k, \infty)}) \tag{16}$$

$$\forall k \geq 1, x(t_k - 0) = F(u(t_k - 0)) \tag{17}$$

The couple (u, x) is called

a) a pseudo-fundamental (operating) mode of f if (16) is true

b) a fundamental (operating) mode of f if (13), (14), (16) are true

c) a fundamental (operating) mode of f relative to F if (13), (15), (16) are true

d) a delay-insensitive fundamental (operating) mode of f relative to F if (13), (15), (16), (17) are true.

Remark 101 (u, x) is a pseudo-fundamental mode of f if the intervals $[t_{k-1}, t_k)$ covering $[0, \infty)$ exist (from the unboundness of $t_0, t_1, t_2, \ldots$) having the property that u is allowed to take new values in $[t_k, t_{k+1})$ possibly different from the previous ones in $[t_{k-1}, t_k)$ only if x has stabilized (at some time instant situated in the interval $[t_{k-1}, t_k)$) under the input $u \cdot \chi_{(-\infty, t_k)} \oplus u(t_k - 0) \cdot \chi_{[t_k, \infty)}$ to the value $x(t_k - 0)$. The forms of u, x do not matter, just the satisfaction of the stability condition 100 (16), that characterizes $t_1, t_2, t_3, \ldots$ as time instants when u, x are in equilibrium. We shall consider that (u, x) are in equilibrium in t_0 too.

(u, x) is a fundamental mode of f if the satisfaction of the stability condition 100 (16) takes place for u constant in $(-\infty, t_0)$ and also in each interval $[t_{k-1}, t_k)$ and (u, x) is a fundamental mode of f relative to F if the condition 100 (14) is relaxed to 100 (15). Here the role of F is that of 'Boolean function to be computed', to be compared with the delay-insensitivity of f relative to F, Definition 89 d), whose hypothesis $\exists t_1, \forall t \geq t_1, F(u(t)) = F(u(t_1))$ was replaced by 100 (15) and whose conclusion $\exists t_1, \forall t \geq t_1, x(t) = F(u(t_1))$ was replaced by 100 (16), (17).

The absence of the satisfaction of 100 (17) between the previous properties indicates either the presence of hazard: the states of the system are supposed to start the computation of $F(u)$ and this computation is unsuccessful eventually, or the fact that the state $x \in f(u)$ is not related with the computation of $F(u)$.

The definitions that are grouped in 100 include the possibility $u(t_k) = u(t_{k+1})$, respectively $F(u(t_k)) = F(u(t_{k+1}))$ or $\exists k, u(t_k) = u(t_{k+1}) = \dots$, respectively $\exists k, F(u(t_k)) = F(u(t_{k+1})) = \dots$

Theorem 102 For $F, f, u, x \in f(u)$ and $0 \le t_0 < t_1 < t_2 < \dots$ unbounded, we suppose that some of the requests 100 (13),...,(17) are satisfied. The same properties are satisfied if we replace the sequence $0 \le t_0 < t_1 < t_2 < \dots$ with $0 \le t'_0 < t'_1 < t'_2 < \dots$ where

$$t'_0 = t_0, t'_1 = t_1, \dots, t'_k = \tau, t'_{k+1} = t_k, t'_{k+2} = t_{k+1}, \dots$$

with $k \ge 1$ arbitrary and $\tau \in (t_{k-1}, t_k)$ chosen sufficiently close to t_k.

Proof. We fix $k \ge 1$ and t_k arbitrary the next properties that derive from 100 (14), (16), (17) being satisfied

$$\forall t \in [t_{k-1}, t_k), u(t) = u(t_{k-1}) \tag{18}$$

$$x \cdot \chi_{(-\infty, t_k)} \oplus x(t_k - 0) \cdot \chi_{[t_k, \infty)} \in f(u \cdot \chi_{(-\infty, t_k)} \oplus u(t_k - 0) \cdot \chi_{[t_k, \infty)}) \tag{19}$$

$$x(t_k - 0) = F(u(t_k - 0)) \tag{20}$$

We have the existence of some $\varepsilon > 0$ so that

$$\forall t \in (t_k - \varepsilon, t_k), x(t) = x(t_k - 0)$$

because x has a left limit in t_k and for any $\tau \in (t_k - \varepsilon, t_k) \cap (t_{k-1}, t_k)$ we infer the truth of

$$\forall t \in [t_{k-1}, \tau), u(t) = u(t_{k-1}) \quad \text{and} \quad \forall t \in [\tau, t_k), u(t) = u(\tau)$$

$$x \cdot \chi_{(-\infty, \tau)} \oplus x(\tau - 0) \cdot \chi_{[\tau, \infty)} \in f(u \cdot \chi_{(-\infty, \tau)} \oplus u(\tau - 0) \cdot \chi_{[\tau, \infty)}) \quad \text{and} \quad (19)$$

$$x(\tau - 0) = F(u(\tau - 0)) \quad \text{and} \quad (20)$$

thus the insertion of such a τ between the elements of (t_k) leaves the relations 100 (14), (16), (17) true. The situation is similar if we refer to 100 (15) instead of 100 (14). ∎

Definition 103 Let (u, x) a pseudo-fundamental mode of f (a fundamental mode of f, a fundamental mode of f relative to F, a delay-insensitive fundamental mode of f relative to F) and the unbounded sequence $0 \le t_0 < t_1 < t_2 < \dots$ with the property that the relations 100 (16) (the relations 100 (13), (14), (16), the relations 100 (13), (15), (16), the relations 100 (13), (15), (16), (17)) are fulfilled. Then we say that the sequence (t_k) is compatible with the mode (u, x).

Definition 104 We suppose that (u,x) is a pseudo-fundamental mode of f and let $0 \le t_0 < t_1 < t_2 < \dots$ compatible with it. The functions

$$u^{(k)} = u \cdot \chi_{(-\infty,t_k)} \oplus u(t_k - 0) \cdot \chi_{[t_k,\infty)} \tag{21}$$

$$x^{(k)} = x \cdot \chi_{(-\infty,t_k)} \oplus x(t_k - 0) \cdot \chi_{[t_k,\infty)} \tag{22}$$

$k \ge 1$ are called initial segments, or prefixes (relative to (t_k)) of u,x and the couples $(u(t_k - 0), x(t_k - 0)), k \ge 1$ are called points of equilibrium of f. By definition $(u(t_0 - 0), x(t_0 - 0))$ is a point of equilibrium of f too.

Theorem 105 $F, f, u, x \in f(u)$ and the unbounded sequence $0 \le t_0 < t_1 < t_2 < \dots$ are given.

a) Let (u,x) a pseudo-fundamental mode of f (a fundamental mode of f, a fundamental mode of f relative to F, a delay-insensitive fundamental mode of f relative to F) so that (t_k) be compatible with it. Then $(u^{(k)}, x^{(k)})$ are pseudo-fundamental modes of f (fundamental modes of f, fundamental modes of f relative to F, delay-insensitive fundamental modes of f relative to F) for all $k \ge 1$.

b) Let the couples $(u \cdot \chi_{(-\infty,t_k)} \oplus u(t_k - 0) \cdot \chi_{[t_k,\infty)}, x \cdot \chi_{(-\infty,t_k)} \oplus x(t_k - 0) \cdot \chi_{[t_k,\infty)})$ pseudo-fundamental modes of f (fundamental modes of f, fundamental modes of f relative to F, delay-insensitive fundamental modes of f relative to F) for all $k \ge 1$. Then (u,x) is a pseudo-fundamental mode of f (a fundamental mode of f, a fundamental mode of f relative to F, a delay-insensitive fundamental mode of f relative to F) and (t_k) is compatible with it.

Proof. a) We suppose for example that 100 (13), (15), (16), (17) are satisfied, we fix $k' \ge 1$ and we infer

$$\forall t < t_0, u^{(k')}(t) = u^{(k')}(t_0 - 0) = u(t_0 - 0)$$

$$\forall t \in [t_k, t_{k+1}), F(u^{(k')}(t)) = F(u^{(k')}(t_k)) = \begin{cases} F(u(t_k)), & 0 \le k < k' \\ F(u(t_{k'-1})), & k \ge k' \end{cases}$$

On the other hand, the property

$$x^{(k')} \cdot \chi_{(-\infty,t_k)} \oplus x^{(k')}(t_k - 0) \cdot \chi_{[t_k,\infty)} \in f(u^{(k')} \cdot \chi_{(-\infty,t_k)} \oplus u^{(k')}(t_k - 0) \cdot \chi_{[t_k,\infty)})$$

coincides with 100 (16) for $1 \le k \le k'$ and with

$$x^{(k')} \cdot \chi_{(-\infty,t_{k'})} \oplus x^{(k')}(t_{k'} - 0) \cdot \chi_{[t_{k'},\infty)} \in f(u^{(k')} \cdot \chi_{(-\infty,t_{k'})} \oplus u^{(k')}(t_{k'} - 0) \cdot \chi_{[t_{k'},\infty)})$$

for $k > k'$ and eventually the property

$$x^{(k')}(t_k - 0) = F(u^{(k')}(t_k - 0))$$

coincides with 100 (17) for $1 \le k \le k'$ and with

$$x^{(k')}(t_{k'} - 0) = F(u^{(k')}(t_{k'} - 0))$$

for k > k^0. $(u^{(k^0)}; x^{(k^0)})$ is a delay insensitive fundamental mode of f relative to F, the property being true for any k^0 ¸ 1.

b) Let u; x 2 f(u); 0 · t_0 < t_1 < t_2 < ::: unbounded and k^0 ¸ 1 arbitrary, ¯xed so that $u^{(k^0)}; x^{(k^0)}$ de¯ned like at 104 (21), (22) satisfy for example 100 (13), (15), (16), (17) i.e. $(u^{(k^0)}; x^{(k^0)})$ is a delay-insensitive fundamental mode of f relative to F. u; x satisfy 100 (13); 100 (15), (16), (17) are satis¯ed for 0 · k · k^0; 1 · k · k^0, 1 · k · k^0 and when k^0 is variable, we have that (u; x) is a delay-insensitive fundamental mode of f relative to F. ∎

Theorem 106 Let F; f; u and x 2 f(u). The next statements are true:

a) If (u; x) is a fundamental mode of f, then (u; x) is a fundamental mode of f relative to F

b) If F is injective and (u; x) is a fundamental mode of f relative to F, then (u; x) is a fundamental mode of f

c) If (u; x) is a fundamental mode of f (relative to F), then it is a pseudo-fundamental mode of f:

Proof. 100 (14) implies 100 (15) for any F and if F is injective, then 100 (15) implies 100 (14). ∎

Theorem 107 The next statements are equivalent:

a) (u; x) is a fundamental mode of f

b) for any function F, (u; x) is a fundamental mode of f relative to F:

Proof. b) =) a) Let F^i : B^m ! B^n; 8¸ 2 B^m; F^i(¸) = ($¸_i$; 0; :::; 0) and 0 · t^i_0 < t^i_1 < t^i_2 < ::: unbounded so that 100 (13), (15), (16) be satis¯ed for all i 2 f1; :::; mg. If 0 · t_0 < t_1 < t_2 < ::: is the sequence obtained by indexing the family (t^1_k) [::: [(t^m_k) we remark that 100 (13), (14), (16) are ful¯lled. ∎

Theorem 108 a) Let the non-anticipatory (De¯nition 50) relatively stable system f and the family of vectors u^k 2 B^m; k 2 N. The input u 2 $S^{(m)}$:

$$u(t) = u^0 ¢ Â_{[t_0;t_1)}(t) © u^1 ¢ Â_{[t_1;t_2)}(t) © :::$$

and the state x 2 f(u) exist so that (u; x) is a fundamental mode of f.

b) Let the non-anticipatory (De¯nition 50) relatively stable systems f; f^1; :::; f^p and the family of vectors z^k 2 $B^{m_1+:::+m_p}$; k 2 N. The input z 2 $S^{(m_1+:::+m_p)}$:

$$z(t) = z^0 ¢ Â_{[t_0;t_1)}(t) © z^1 ¢ Â_{[t_1;t_2)}(t) © :::$$

and the state x 2 f ± (f^1; :::; f^p)(z) exist so that (z; x) is a fundamental mode of f ± (f^1; :::; f^p).

Proof. b) We consider the family of vectors z^k 2 $B^{m_1+:::+m_p}$; k 2 N and we ¯x t_0 ¸ 0 arbitrary. For the input

$$z^{(1)}(t) = z^0 ¢ Â_{[t_0;1)}(t)$$

from the relative stability of $f; f^1; \dots; f^p$ we infer the existence of $y^{(1)} \in (f^1; \dots; f^p)(z^{(1)})$; $x^{(1)} \in f(y^{(1)})$ and $t_1 > t_0$ so that

$$y^{(1)}(t) = y^{(1)}(t) \cdot \chi_{(-\infty; t_1)}(t) \oplus y^{(1)}(t_1 - 0) \cdot \chi_{[t_1; \infty)}(t)$$

$$x^{(1)}(t) = x^{(1)}(t) \cdot \chi_{(-\infty; t_1)}(t) \oplus x^{(1)}(t_1 - 0) \cdot \chi_{[t_1; \infty)}(t)$$

We define

$$z^{(2)}(t) = z^0 \cdot \chi_{[t_0; t_1)}(t) \oplus z^1 \cdot \chi_{[t_1; \infty)}(t)$$

From the non-anticipation and the relative stability of $f; f^1; \dots; f^p$ we infer the existence of $y^{(2)} \in (f^1; \dots; f^p)(z^{(2)})$; $x^{(2)} \in f(y^{(2)})$ and $t_2 > t_1$ so that

$$y^{(2)}(t) = y^{(1)}(t) \cdot \chi_{(-\infty; t_1)}(t) \oplus y^{(2)}(t) \cdot \chi_{[t_1; t_2)}(t) \oplus y^{(2)}(t_2 - 0) \cdot \chi_{[t_2; \infty)}(t)$$

$$x^{(2)}(t) = x^{(1)}(t) \cdot \chi_{(-\infty; t_1)}(t) \oplus x^{(2)}(t) \cdot \chi_{[t_1; t_2)}(t) \oplus x^{(2)}(t_2 - 0) \cdot \chi_{[t_2; \infty)}(t)$$

We can define in this moment

$$z^{(3)}(t) = z^0 \cdot \chi_{[t_0; t_1)}(t) \oplus z^1 \cdot \chi_{[t_1; t_2)}(t) \oplus z^2 \cdot \chi_{[t_2; \infty)}(t)$$

$$\dots$$

By using iteratively the non-anticipation and the relative stability of $f; f^1; \dots; f^p$ we obtain

$$z^{(k+1)}(t) = z^0 \cdot \chi_{[t_0; t_1)}(t) \oplus z^1 \cdot \chi_{[t_1; t_2)}(t) \oplus \dots \oplus z^k \cdot \chi_{[t_k; \infty)}(t)$$

$y^{(k+1)} \in (f^1; \dots; f^p)(z^{(k+1)})$; $x^{(k+1)} \in f(y^{(k+1)})$ and $t_{k+1} > t_k$ so that

$$y^{(k+1)}(t) = y^{(1)}(t) \cdot \chi_{(-\infty; t_1)}(t) \oplus y^{(2)}(t) \cdot \chi_{[t_1; t_2)}(t) \oplus \dots \oplus y^{(k+1)}(t_{k+1} - 0) \cdot \chi_{[t_{k+1}; \infty)}(t)$$

$$x^{(k+1)}(t) = x^{(1)}(t) \cdot \chi_{(-\infty; t_1)}(t) \oplus x^{(2)}(t) \cdot \chi_{[t_1; t_2)}(t) \oplus \dots \oplus x^{(k+1)}(t_{l+1} - 0) \cdot \chi_{[t_{k+1}; \infty)}(t)$$

The functions

$$z(t) = z^0 \cdot \chi_{[t_0; t_1)}(t) \oplus z^1 \cdot \chi_{[t_1; t_2)}(t) \oplus \dots$$

$$x(t) = x^{(1)} \cdot \chi_{(-\infty; t_1)}(t) \oplus x^{(2)} \cdot \chi_{[t_1; t_2)}(t) \oplus \dots$$

satisfy the required property. ■

Theorem 109 a) Let the Boolean function F, the family of vectors $x^k \in Range(F)$; $k \in N$ and the non-anticipatory (Definition 50) system f that is stable relative to F (that is delay-insensitive relative to F). The input $u \in S^{(m)}$:

$$u(t) = u^0 \cdot \chi_{[t_0; t_1)}(t) \oplus u^1 \cdot \chi_{[t_1; t_2)}(t) \oplus \dots$$

and the state $x \in f(u)$ exist so that

$$\forall k \in N; F(u^k) = x^k$$

and $(u; x)$ is a fundamental mode of f relative to F (a delay-insensitive fundamental mode of f relative to F).

b) Let the Boolean functions $F, F^1, ..., F^p$, the family of vectors $x^k \in Range(F \circ (F^1, ..., F^p)), k \in N$ and the non-anticipatory (Definition 50) systems $f, f^1, ..., f^p$ that are stable relative to $F, F^1, ..., F^p$ (that are delay-insensitive relative to $F, F^1, ..., F^p$). The input $z \in S^{(m_1 + ... + m_p)}$:

$$z(t) = z^0 \cdot \chi_{[t_0, t_1)}(t) \oplus z^1 \cdot \chi_{[t_1, t_2)}(t) \oplus ...$$

and the state $x \in f \circ (f^1, ..., f^p)(z)$ exist so that

$$\forall k \in N, F \circ (F^1, ..., F^p)(z^k) = x^k$$

and (z, x) is a fundamental mode of $f \circ (f^1, ..., f^p)$ relative to $F \circ (F^1, ..., F^p)$ (a delay-insensitive fundamental mode of $f \circ (f^1, ..., f^p)$ relative to $F \circ (F^1, ..., F^p)$).

Proof. b) We choose arbitrarily the family $z^k \in B^{m_1 + ... + m_p}$ so that $x^k = F \circ (F^1, ..., F^p)(z^k), k \in N$ and the proof coincides with the one from 108 b), where 'relative stability' is replaced by 'stability relative to $F, F^1, ..., F^p$'. We have in addition the condition of delay-insensitivity stating

$$y^{(k)}(t_k - 0) = (F^1, ..., F^p)(z^{(k)}(t_k - 0))$$

$$x^{(k)}(t_k - 0) = F(y^{(k)}(t_k - 0))$$

for all $k \geq 1$, from where we get

$$\forall k \geq 1, x(t_k - 0) = x^{(k)}(t_k - 0) = F(y^{(k)}(t_k - 0)) =$$

$$= F \circ (F^1, ..., F^p)(z^{(k)}(t_k - 0)) = F \circ (F^1, ..., F^p)(z(t_k - 0))$$

■

Theorem 110 We suppose that (u, x) is a pseudo-fundamental mode of f (a fundamental mode of f, a fundamental mode of f relative to F, a delay-insensitive fundamental mode of f relative to F). Then $(u, \overline{x})$ is a pseudo-fundamental mode of $\overline{f}$ (a fundamental mode of $\overline{f}$, a fundamental mode of $\overline{f}$ relative to $\overline{F}$, a delay-insensitive fundamental mode of $\overline{f}$ relative to $\overline{F}$).

Proof. Equations 100 (15),...,(17) imply

$$\forall k \geq 0, \forall t \in [t_k, t_{k+1}), \overline{F}(u(t)) = \overline{F}(u(t_k))$$

$$\forall k \geq 1, \overline{x} \cdot \chi_{(-\infty, t_k)} \oplus \overline{x}(t_k - 0) \cdot \chi_{[t_k, \infty)} \in \overline{f}(u \cdot \chi_{(-\infty, t_k)} \oplus u(t_k - 0) \cdot \chi_{[t_k, \infty)})$$

$$\forall k \geq 1, \overline{x}(t_k - 0) = \overline{F}(u(t_k - 0))$$

showing the statements of the Theorem. ■

Theorem 111 If (u, x) is a pseudo-fundamental mode of f (a fundamental mode of f, a fundamental mode of f relative to F, a delay-insensitive fundamental mode of f relative to F) and $f \subset g$, then (u, x) is a pseudo-fundamental mode of g (a fundamental mode of g, a fundamental mode of g relative to F, a delay-insensitive fundamental mode of g relative to F).

Proof. The condition

$$\forall k \geq 1, x \cdot \chi_{(-\infty,t_k)} \oplus x(t_k - 0) \cdot \chi_{[t_k,\infty)} \in g(u \cdot \chi_{(-\infty,t_k)} \oplus u(t_k - 0) \cdot \chi_{[t_k,\infty)})$$

follows from 100 (16) and from the fact that $f \subset g$. ■

Theorem 112 We suppose that (u, x) is a pseudo-fundamental mode of f (a fundamental mode of f, a fundamental mode of f relative to F, a delay-insensitive fundamental mode of f relative to F), that f is time invariant and let $d \in \mathbf{R}$ so that $u \circ \tau^d \in S^{(m)}$. Then $(u \circ \tau^d, x \circ \tau^d)$ is a pseudo-fundamental mode of f (a fundamental mode of f, a fundamental mode of f relative to F, a delay-insensitive fundamental mode of f relative to F).

Proof. We suppose that u is the constant function and from the time-invariance of f we have that $\forall x \in f(u)$, x is the constant function (Corollary 43 a)). If some of 100 (13),...,(17) are true, then by the replacement of u, x with $u \circ \tau^d = u$, $x \circ \tau^d = x$ the same statements are true.

We suppose now that u is not constant, implying the existence of

$$t' = \min\{t | u(t - 0) \neq u(t)\}$$

and the hypothesis $u \circ \tau^d \in S^{(m)}$ means that $t' + d \geq 0$. The truth of some of the statements 100 (13),...,(17) implies the validity of these statements after the replacement of $u, x, 0 \leq t_0 < t_1 < t_2 < \ldots$ with $u \circ \tau^d, x \circ \tau^d, 0 \leq t_0 + d < t_1 + d < t_2 + d < \ldots$ and we have supposed without loss that $t_0 = t'$ (if x is constant, this statement is obvious and if x is not constant, then

$$t'' = \min\{t | x(t - 0) \neq x(t)\}$$

exists and the non-anticipation -Definition 41- of f gives $t' \leq t''$, see Corollary 43 b), so that $t_0 = t'$ is possible again). ■

Theorem 113 Let the coordinatewise symmetrical Boolean function F (Definition 70) and the coordinatewise symmetrical system f (Definition 71). If (u, x) is a pseudo-fundamental mode of f (a fundamental mode of f, a fundamental mode of f relative to F, a delay-insensitive fundamental mode of f relative to F), then for all bijections $\sigma : \{1, \ldots, m\} \to \{1, \ldots, m\}$, (u_σ, x) is a pseudo-fundamental mode of f (a fundamental mode of f, a fundamental mode of f relative to F, a delay-insensitive fundamental mode of f relative to F).

Proof. From 100 (13),...,(17) and from the coordinatewise symmetry of F and f we infer that

$$\forall t < t_0, u_\sigma(t) = u_\sigma(t_0 - 0)$$

$$\forall k \geq 0, \forall t \in [t_k, t_{k+1}), u_\sigma(t) = u_\sigma(t_k)$$

$$\forall k \geq 0, \forall t \in [t_k, t_{k+1}), F(u_\sigma(t)) = F(u(t)) = F(u(t_k)) = F(u_\sigma(t_k))$$

$$\forall k \geq 1, x \cdot \chi_{(-\infty,t_k)} \oplus x(t_k - 0) \cdot \chi_{[t_k,\infty)} \in f(u \cdot \chi_{(-\infty,t_k)} \oplus u(t_k - 0) \cdot \chi_{[t_k,\infty)}) =$$
$$= f(u_\sigma \cdot \chi_{(-\infty,t_k)} \oplus u_\sigma(t_k - 0) \cdot \chi_{[t_k,\infty)})$$

$$\forall k \geq 1, x(t_k - 0) = F(u(t_k - 0)) = F(u_\sigma(t_k - 0))$$

are fulfilled. ∎

Theorem 114 Let the rising-falling symmetrical function F (Definition 79), the rising-falling symmetrical system f (Definition 80), $u \in S^{(m)}$ and $x \in f(u)$. If (u, x) is a pseudo-fundamental mode of f (a fundamental mode of f, a fundamental mode of f relative to F, a delay-insensitive fundamental mode of f relative to F), then $(\overline{u}, \overline{x})$ is a pseudo-fundamental mode of f (a fundamental mode of f, a fundamental mode of f relative to F, a delay-insensitive fundamental mode of f relative to F).

Proof. We infer from 100 (13),...,(17) and from the hypothesis of rising-falling symmetry of F, f that

$$\forall t < t_0, \overline{u}(t) = \overline{u}(t_0 - 0)$$

$$\forall k \geq 0, \forall t \in [t_k, t_{k+1}), \overline{u}(t) = \overline{u}(t_k)$$

$$\forall k \geq 0, \forall t \in [t_k, t_{k+1}), F(\overline{u}(t)) = \overline{F}(u(t)) = \overline{F}(u(t_k)) = F(\overline{u}(t_k))$$

$$\forall k \geq 1, \overline{x} \cdot \chi_{(-\infty, t_k)} \oplus \overline{x}(t_k - 0) \cdot \chi_{[t_k, \infty)} \in \overline{f}(u \cdot \chi_{(-\infty, t_k)} \oplus u(t_k - 0) \cdot \chi_{[t_k, \infty)}) =$$
$$= f(\overline{u} \cdot \chi_{(-\infty, t_k)} \oplus \overline{u}(t_k - 0) \cdot \chi_{[t_k, \infty)})$$

$$\forall k \geq 1, \overline{x}(t_k - 0) = \overline{F}(u(t_k - 0)) = F(\overline{u}(t_k - 0))$$

are true. ∎

16. Generator function

Definition 115 Let $\Phi : B^m \times B^n \to B^n, u \in S^{(m)}$ and $x \in S^{(n)}$. We say that the state x is generated by the (generator) function Φ and the input (function) u and that Φ, u generate (the state, the trajectory, the path) x if the unbounded sequence $0 \leq t_0 < t_1 < t_2 < \dots$ exists so that we have:

$$u(t) = u(t_0 - 0) \cdot \chi_{(-\infty, t_0)}(t) \oplus u(t_0) \cdot \chi_{[t_0, t_1)}(t) \oplus u(t_1) \cdot \chi_{[t_1, t_2)}(t) \oplus \dots \quad (23)$$

$$x(t) = x(t_0 - 0) \cdot \chi_{(-\infty, t_0)}(t) \oplus x(t_0) \cdot \chi_{[t_0, t_1)}(t) \oplus x(t_1) \cdot \chi_{[t_1, t_2)}(t) \oplus \dots \quad (24)$$

$$\forall k \in N, \forall i \in \{1, \dots, n\}, (x_i(t_{k+1}) = x_i(t_k) \quad \text{or} \quad x_i(t_{k+1}) = \Phi_i(u(t_k), x(t_k))) \quad (25)$$

$$\{i | i \in \{1, \dots, n\}, \exists k \in N, \exists a \in B, a = x_i(t_k) = x_i(t_{k+1}) = \dots \quad \text{and}$$
$$\text{and} \quad \overline{a} = \Phi_i(u(t_k), x(t_k)) = \Phi_i(u(t_{k+1}), x(t_{k+1})) = \dots\} = \emptyset \quad (26)$$

Remark 116 We interpret Definition 115 that formalizes in this context the unbounded delay model from the asynchronous circuits theory.

a) For any u, x an unbounded sequnce (t_k) like at (23), (24) exists. These two equations fix such a sequence, that becomes the discrete time set.

b) Equations (25), (26) represent a restatement of Definition 2.10, items b); c) from [3] see also paragrapf 7 of that paper, by following an idea of Anatoly Chebotarev. (25) states that for any (discrete) time moment t_k, the new value of the coordinate x_i is equal either with the old one, or with $\Phi_i(u(t_k), x(t_k))$ (or with both). At (26) it is stated that any computation of the 'next state' $\Phi_i(u(t_k), x(t_k))$ is eventually made.

c) The common picture of all the trajectories that are generated by Φ and u was associated [2] with the propositional branching time temporal logic: when $x_i(t_{k+1}) = x_i(t_k)$, respectively when $x_i(t_{k+1}) = \Phi_i(u(t_k), x(t_k))$, the 'proposition' x runs in two different branches of time.

d) Similarly with what happens at the fundamental mode, see Definition 103, if x is generated by Φ and u and $0 \leq t_0 < t_1 < t_2 < \ldots$ is an unbounded sequence so that (23),...,(26) be true, we can call the sequnce (t_k) compatible with (u, x). Several sequences (t_k) exist that are compatible with (u, x), see for example the proof of Theorem 118.

Definition 117 Let the state x generated by Φ and u. The coordinate $i \in \{1, \ldots, n\}$ and the coordinate function x_i are called excited or enabled at the time instant t if $x_i(t) \neq \Phi_i(u(t), x(t))$ and they are called stable, or disabled at the time instant t if $x_i(t) = \Phi_i(u(t), x(t))$.

If $x(t) = \Phi(u(t), x(t))$ i.e. if all the coordinates are stable, we say that the state x is stable at the time instant t and $(u(t), x(t))$ is called an equilibrium point of Φ.

Theorem 118 We suppose that x is generated by Φ and u. If it is stable at the time instant t', then $\forall t \geq t', x(t) = x(t')$.

Proof. We suppose that some $k \in \mathbf{N}$ exists so that $x(t_k) = \Phi(u(t_k), x(t_k))$ (if the previous property is not true, then $x(t') = \Phi(u(t'), x(t'))$ is fulfilled for $t' \notin (t_k)$; we reindex the elements of the set $t' \cup (t_k)$ and we get an unbounded sequence $0 \leq t'_0 < t'_1 < t'_2 < \ldots$ that makes (23),...,(26) from Definition 115 be fulfilled and the property true). We have $x(t_k) = x(t_{k+1}) = \ldots$ ∎

Notation 119 The set of the states x with $x(0-0) = x^0$ that are generated by Φ and u is noted with $L_\Phi(u, x^0)$.

Remark 120 $L_\Phi(u, x^0)$ may be considered to be a $S^{(m)} \to P^*(S^{(n)})$ function, i.e. an asynchronous system with the initial state x^0.

On the other hand, we observe that for any u, some $x \in L_\Phi(u, x^0)$ exists so that

$$u(t) = u(t_0 - 0) \cdot \chi_{(-\infty, t_0)}(t) \oplus u(t_0) \cdot \chi_{[t_0, t_1)}(t) \oplus u(t_1) \cdot \chi_{[t_1, t_2)}(t) \oplus \ldots$$

$$x(t) = x(t_0 - 0)\cdot\chi_{(-\infty,t_0)}(t) \oplus x(t_0^0)\cdot\chi_{[t_0^0,t_0^1)}(t) \oplus \ldots \oplus x(t_0^{p_0})\cdot\chi_{[t_0^{p_0},t_0^{p_0+1})}(t) \oplus$$
$$\oplus\, x(t_1^0)\cdot\chi_{[t_1^0,t_1^1)}(t) \oplus \ldots \oplus x(t_1^{p_1})\cdot\chi_{[t_1^{p_1},t_1^{p_1+1})}(t) \oplus \ldots$$

where $x(t_0 - 0) = x(t_0) = x^0$,

$t_0 = t_0^0 < t_0^1 < \ldots < t_0^{p_0} < t_0^{p_0+1} = t_1 = t_1^0 < t_1^1 < \ldots < t_1^{p_1} < t_1^{p_1+1} = t_2 = t_2^0 < \ldots$

$p_0, p_1, p_2, \ldots \in N$ and

$$x(t_k^{j+1}) = \Phi(u(t_k), x(t_k^j)),\ j = \overline{0, p_k}, k \in N$$

It is interesting the situation when any $x \in L_\Phi(u, x^0)$ is of this form and the propositional branching time temporal logic becomes propositional linear time temporal logic.

Definition 121 We say that the system f is generated by the (generator) function Φ if

$$\forall u, f(u) = \bigcup_{x^0 \in \chi(u)} L_\Phi(u, x^0)$$

Example 122 In the next four examples $m = n = 1$, $\Phi : B \times B \to B, B \times B \ni (\lambda, \mu) \mapsto \Phi(\lambda, \mu) \in B$ and x^0 is the initial state.

a) $\Phi(\lambda, \mu) = x^1, x^1 \in B$ (the constant function)

$$L_\Phi(u, x^0) = \{x \mid \exists t_0 \geq 0, x(t) = x^0\cdot\chi_{(-\infty,t_0)}(t) \oplus x^1\cdot\chi_{[t_0,\infty)}(t)\}$$

see also Theorem 123.

b) $\Phi(\lambda, \mu) = \lambda$ (the projection on the first coordinate)

$$L_\Phi(u, x^0) = \{x \mid \text{the unbounded sequence } 0 \leq t_0 < t_1 < t_2 < \ldots \text{ exists so that}$$
$$x(t) = x^0\cdot\chi_{(-\infty,t_0)}(t) \oplus u(t_0)\cdot\chi_{[t_0,t_1)}(t) \oplus u(t_1)\cdot\chi_{[t_1,t_2)}(t) \oplus \ldots\}$$

Thus if $0 \leq t'_0 < t'_1 < t'_2 < \ldots$ is an unbounded sequence satisfying

$$u(t) = u(t_0^0 - 0)\cdot\chi_{(-\infty,t_0^0)}(t) \oplus u(t_0^0)\cdot\chi_{[t_0^0,t_1^0)}(t) \oplus u(t_1^0)\cdot\chi_{[t_1^0,t_2^0)}(t) \oplus \ldots$$

and $0 \leq t_0 < t_1 < t_2 < \ldots$ is a subsequence of (t_k^0), then the state $x \in L_\Phi(u, x^0)$ reproduces some of the successive values of u (infinitely many values). We remark that if $\lim_{t\to\infty} u(t)$ exists, then $\lim_{t\to\infty} x(t)$ exists and $\lim_{t\to\infty} u(t) = \lim_{t\to\infty} x(t)$.

c) $\Phi(\lambda, \mu) = \mu$ (the projection on the second coordinate)

$$L_\Phi(u, x^0) = \{x^0\}$$

d) $\Phi(\lambda, \mu) = \lambda\cdot\mu$

$$L_\Phi(u, x^0) = \{x \mid \text{the unbounded sequence } 0 \leq t_0 < t_1 < t_2 < \ldots \text{ exists so that}$$
$$x(t) = x^0\cdot\chi_{(-\infty,t_0)}(t) \oplus x^0\cdot u(t_0)\cdot\chi_{[t_0,t_1)}(t) \oplus x^0\cdot u(t_0)\cdot u(t_1)\cdot\chi_{[t_1,t_2)}(t) \oplus \ldots\}$$

Like at b), $u(t_0), u(t_1), u(t_2), \ldots$ are some of the successive values talen by u.

Theorem 123 Let the function Φ and the initial state x^0. If $\exists x^1, \Phi = x^1$ (the constant function) then

$$L_\Phi(u, x^0) = \{x \mid \forall i \in \{1, ..., n\}, \exists t_i \geq 0, x_i(t) = x_i^0 \cdot \chi_{(-\infty, t_i)}(t) \oplus x_i^1 \cdot \chi_{[t_i, \infty)}(t)\}$$

Proof. In Definition 115, (25) shows for any i that x_i may switch from x_i^0 to x_i^1 and (26) shows that if $x_i^0 \neq x_i^1$ then some $t_i \geq 0$ exists so that x_i switches at t_i from x_i^0 to x_i^1. ■

Corollary 124 If f is generated by $\Phi = x^1$ then

$$\forall u, \forall x \in f(u), \exists x^0 \in \varphi(u), \forall i \in \{1, ..., n\}, \exists t_i \geq 0, \; x_i(t) = x_i^0 \cdot \chi_{(-\infty, t_i)}(t) \oplus x_i^1 \cdot \chi_{[t_i, \infty)}(t)$$

Theorem 125 Let f, Φ, x^0 and we suppose that

$$\forall u, f(u) = L_\Phi(u, x^0)$$

a) If $\Gamma : \mathbf{B}^m \times \mathbf{B}^n \to \mathbf{B}^n$ satisfies $\forall (\lambda_1, ..., \lambda_m) \in \mathbf{B}^m, \forall (\mu_1, ..., \mu_n) \in \mathbf{B}^n$,

$$\Gamma(\lambda_1, ..., \lambda_m, \mu_1, ..., \mu_n) = \overline{\Phi(\lambda_1, ..., \lambda_m, \overline{\mu_1}, ..., \overline{\mu_n})}$$

then

$$\forall u, \overline{f}(u) = L_\Gamma(u, \overline{x^0})$$

b) If $\Gamma : \mathbf{B}^{m+1} \times \mathbf{B}^n \to \mathbf{B}^n$ satisfies $\forall (\lambda_1, ..., \lambda_{m+1}) \in \mathbf{B}^{m+1}, \forall (\mu_1, ..., \mu_n) \in \mathbf{B}^n$,

$$\Gamma(\lambda_1, ..., \lambda_{m+1}, \mu_1, ..., \mu_n) = \Phi(\lambda_1, ..., \lambda_m, \mu_1, ..., \mu_n)$$

then

$$\forall (u_1, ..., u_{m+1}) \in S^{(m+1)}, f^{(m+1)}(u_1, ..., u_{m+1}) = L_\Gamma(u_1, ..., u_{m+1}, x^0)$$

c) If $\Gamma : \mathbf{B}^m \times \mathbf{B}^n \to \mathbf{B}^n$ satisfies for $i, j \in \{1, ..., m\}, i \neq j$: $\forall (\lambda_1, ..., \lambda_m) \in \mathbf{B}^m, \forall (\mu_1, ..., \mu_n) \in \mathbf{B}^n$,

$$\Gamma(\lambda_1, ..., \underset{i}{\lambda_i}, ..., \underset{j}{\lambda_j}, ..., \lambda_m, \mu_1, ..., \mu_n) = \Phi(\lambda_1, ..., \underset{i}{\lambda_i}, ..., \underset{j}{\lambda_i}, ..., \lambda_m, \mu_1, ..., \mu_n)$$

then

$$\forall (u_1, ..., u_m) \in S^{(m)}, f_{i \to j}(u_1, ..., u_m) = L_\Gamma(u_1, ..., u_m, x^0)$$

d) We suppose that Φ satisfies for some $i \in \{1, ..., m\}$: $\forall (\lambda_1, ..., \lambda_m) \in \mathbf{B}^m, \forall (\mu_1, ..., \mu_n) \in \mathbf{B}^n$,

$$\Phi(\lambda_1, ..., \underset{i}{0}, ..., \lambda_m, \mu_1, ..., \mu_n) = \Phi(\lambda_1, ..., \underset{i}{1}, ..., \lambda_m, \mu_1, ..., \mu_n)$$

Then $f_{\hat{i}}$ has sense and if $\Gamma : \mathbf{B}^{m-1} \times \mathbf{B}^n \to \mathbf{B}^n$ fulfills the condition $\forall (\lambda_1, ..., \widehat{\lambda_i}, ..., \lambda_m) \in \mathbf{B}^{m-1}, \forall (\mu_1, ..., \mu_n) \in \mathbf{B}^n$,

$$\Gamma(\lambda_1, ..., \widehat{\lambda_i}, ..., \lambda_m, \mu_1, ..., \mu_n) = \Phi(\lambda_1, ..., \underset{i}{0}, ..., \lambda_m, \mu_1, ..., \mu_n)$$

we have

$$\forall (u_1, ..., \widehat{u_i}, ..., u_m) \in S^{(m-1)}, f_{\hat{i}}(u_1, ..., \widehat{u_i}, ..., u_m) = L_\Gamma(u_1, ..., \widehat{u_i}, ..., u_m, x^0)$$

Proof. At a), if the equations (23),...,(26) from Definition 115 are fulfilled by u,x,Φ then they are fulfilled by $u,\overline{x},-$ etc. ∎

Remark 126 A series of corollaries of Theorem 125 refers to the general case, when f is generated by Φ, but it is not initialized. Another series of corollaries of Theorem 125 follows from the supposition that Φ satisfies

$$\forall(\lambda_1,\ldots,\lambda_m)\in \mathbf{B}^m,\forall(\mu_1,\ldots,\mu_n)\in \mathbf{B}^n,\Phi(\lambda_1,\ldots,\lambda_m,\mu_1,\ldots,\mu_n)=\overline{\Phi(\lambda_1,\ldots,\lambda_m,\overline{\mu_1},\ldots,\overline{\mu_n})}$$

at a), (some examples of such functions for $m=1,n=2$ are given by $(\mu_2,\lambda\cdot\mu_1\oplus\lambda\cdot\mu_2\oplus\lambda\oplus\mu_1)$; $(\mu_2\oplus 1,\lambda\oplus\mu_2\oplus 1)$; respectively $(\lambda\oplus\mu_1,\lambda\cdot\mu_1\oplus\lambda\cdot\mu_2\oplus\mu_1)$) and

$$\forall(\lambda_1,\ldots,\lambda_m)\in \mathbf{B}^m,\forall(\mu_1,\ldots,\mu_n)\in \mathbf{B}^n,\Phi(\lambda_1,\ldots,\underset{i}{\lambda_i},\ldots,\underset{j}{\lambda_j},\ldots,\lambda_m,\mu_1,\ldots,\mu_n)=$$
$$\Phi(\lambda_1,\ldots,\underset{i}{\lambda_j},\ldots,\underset{j}{\lambda_i},\ldots,\lambda_m,\mu_1,\ldots,\mu_n)$$

respectively

$$\forall(\lambda_1,\ldots,\lambda_m)\in \mathbf{B}^m,\forall(\mu_1,\ldots,\mu_n)\in \mathbf{B}^n,\Phi(\lambda_1,\ldots,\underset{i}{\lambda_i},\ldots,\underset{j}{0},\ldots,\lambda_m,\mu_1,\ldots,\mu_n)=$$
$$\Phi(\lambda_1,\ldots,\underset{i}{\lambda_i},\ldots,\underset{j}{1},\ldots,\lambda_m,\mu_1,\ldots,\mu_n)$$

at c).

On the other hand systems exist that are not generated by any function, for example those from Example 3 (1),(3),(4) that are characterized by the parameters $d\geq 0,\delta_r\geq 0,\delta_f\geq 0$ are in this situation.

The problem of the generator functions leaves open a lot of questions, from the generation of the intersection and the reunion of the systems, to the connections with other topics from our work, such as the parallel connection and the serial connection, the symmetry in both variants and the stability.

Appendix. Details related with Remark 10

With $u^1\in S^{(m_1)},\ldots,u^p\in S^{(m_p)}$ we form the functions $(u^1,\ldots,u^p):\mathbf{R}^p\to \mathbf{B}^{m_1}\times\ldots\times\mathbf{B}^{m_p}$,

$$\forall(t_1,\ldots,t_p)\in\mathbf{R}^p,(u^1,\ldots,u^p)(t_1,\ldots,t_p)=(u^1(t_1),\ldots,u^p(t_p))$$

$(u^1,\ldots,u^p)\in S^{(m_1)}\times\ldots\times S^{(m_p)}$ and respectively $u^1\oplus\ldots\oplus u^p:\mathbf{R}\to\mathbf{B}^{m_1+\ldots+m_p}$,

$$\forall t\in\mathbf{R},(u^1\oplus\ldots\oplus u^p)(t)=(u^1_1(t),\ldots,u^1_{m_1}(t),\ldots,u^p_1(t),\ldots,u^p_{m_p}(t))$$

$u^1\oplus\ldots\oplus u^p\in S^{(m_1+\ldots+m_p)}$. A bijection $\pi:S^{(m_1)}\times\ldots\times S^{(m_p)}\to S^{(m_1+\ldots+m_p)}$ exists,

$$\forall(u^1,\ldots,u^p)\in S^{(m_1)}\times\ldots\times S^{(m_p)},\pi(u^1,\ldots,u^p)=u^1\oplus\ldots\oplus u^p$$

allowing us to identify $S^{(m_1)}\times\ldots\times S^{(m_p)}$ with $S^{(m_1+\ldots+m_p)}$.

We form two sets with $X_1 \in P^*(S^{(n_1)}), \ldots, X_p \in P^*(S^{(n_p)})$: $(X_1, \ldots, X_p) \in P^*(S^{(n_1)}) \times \ldots \times P^*(S^{(n_p)})$ and respectively $X_1 \| \ldots \| X_p \in P^*(S^{(n_p+\ldots+n_p)})$ that is defined this way

$$X_1 \| \ldots \| X_p = \{x^1 \| \ldots \| x^p | x^1 \in X_1, \ldots, x^p \in X_p\}$$

We have a bijection $\varphi : P^*(S^{(n_1)}) \times \ldots \times P^*(S^{(n_p)}) \to P^*(S^{(n_p+\ldots+n_p)})$,

$$\forall (X_1, \ldots, X_p) \in P^*(S^{(n_1)}) \times \ldots \times P^*(S^{(n_p)}), \varphi(X_1, \ldots, X_p) = X_1 \| \ldots \| X_p$$

that allows us to identify the sets $P^*(S^{(n_1)}) \times \ldots \times P^*(S^{(n_p)})$ and $P^*(S^{(n_p+\ldots+n_p)})$.

With the functions $f^i : S^{(m_i)} \to P^*(S^{(n_i)}), i = \overline{1,p}$ we form two functions $(f^1, \ldots, f^p) : S^{(m_1)} \times \ldots \times S^{(m_p)} \to P^*(S^{(n_1)}) \times \ldots \times P^*(S^{(n_p)})$,

$$\forall (u^1, \ldots, u^p) \in S^{(m_1)} \times \ldots \times S^{(m_p)}, (f^1, \ldots, f^p)(u^1, \ldots, u^p) = (f^1(u^1), \ldots, f^p(u^p))$$

and respectively $f^1 \| \ldots \| f^p : S^{(m_1+\ldots+m_p)} \to P^*(S^{(n_1+\ldots+n_p)})$,

$$\forall (u^1 \| \ldots \| u^p) \in S^{(m_1+\ldots+m_p)}, (f^1 \| \ldots \| f^p)(u^1 \| \ldots \| u^p) = f^1(u^1) \| \ldots \| f^p(u^p)$$

The commutativity of the diagram

$$\begin{array}{ccc} S^{(m_1)} \times \ldots \times S^{(m_p)} & \xrightarrow{(f^1, \ldots, f^p)} & P^*(S^{(n_1)}) \times \ldots \times P^*(S^{(n_p)}) \\ \pi \downarrow & & \downarrow \varphi \\ S^{(m_1+\ldots+m_p)} & \xrightarrow{f^1 \| \ldots \| f^p} & P^*(S^{(n_1+\ldots+n_p)}) \end{array}$$

makes us identify the functions $(f^1, \ldots, f^p)$ and $f^1 \| \ldots \| f^p$.

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