International Journal of Computer Research
Volume 22, Number 4, pp. 435–445
ISSN: 1535-6698

ASYNCHRONOUS FLOWS: THE TECHNICAL CONDITION OF PROPER OPERATION AND ITS GENERALIZATION

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Abstract

The asynchronous flows are given by Boolean functions $\Phi : \{0,1\}^n \longrightarrow \{0,1\}^n$ that iterate their coordinates $\Phi_1, ..., \Phi_n$ independently on each other. In the study of the asynchronous sequential circuits, the situation when multiple coordinates of the state can change at the same time in called a race. When the outcome of the race affects critically the run of the circuit, for example its final state, the race is called critical. To avoid the critical races that could occur, Φ is specified in general so that only one coordinate of the state can change; such a circuit is called race-free and we also say that Φ fulfills the technical condition of proper operation. We formalize in this framework the technical condition of proper operation and give its generalization, consisting in the situation when races exist, but they are not critical.

Keywords: asynchronous flow; race-free; the technical condition of proper operation

1. Introduction

The asynchronous circuits from electronics are modelled by Boolean functions $\Phi : \mathbf{B}^n \longrightarrow \mathbf{B}^n$ that iterate their coordinates $\Phi_1, ..., \Phi_n$ in arbitrary discrete time, independently on each other. As in this paper there is no bound on the duration of an iteration, we use the unbounded delay model of computation of the Boolean functions. The uncertainties related to the behavior of the circuits and their models are generated by technology and also by temperature variations and voltage supply variations.

In order to understand the dynamics of these systems we give the example of the function Φ from Table 1, whose state portrait was drawn in Figure 1 (we have adopted the terminology of state portrait, by analogy with the phase portraits of the dynamical systems theory; such drawings are usually called state transition graphs in engineering).

In Figure 1, the arrows show the increase of time. We have underlined in the tuples $(\mu_1, \mu_2, \mu_3) \in \mathbf{B}^3$ these coordinates, called unstable (or excited, or enabled), for which $\mu_i \neq \Phi_i(\mu), i \in \{1,2,3\}$; these are the coordinates that are about to switch, but the time instant and the order in which these switches happen are not known.

Table 1. An example

(μ_1, μ_2, μ_3)	$\Phi(\mu_1, \mu_2, \mu_3)$
$(0, 0, 0)$	$(0, 1, 1)$
$(0, 0, 1)$	$(0, 1, 1)$
$(0, 1, 0)$	$(0, 1, 0)$
$(0, 1, 1)$	$(1, 1, 1)$
$(1, 0, 0)$	$(0, 0, 0)$
$(1, 0, 1)$	$(1, 0, 1)$
$(1, 1, 0)$	$(1, 0, 0)$
$(1, 1, 1)$	$(1, 1, 0)$

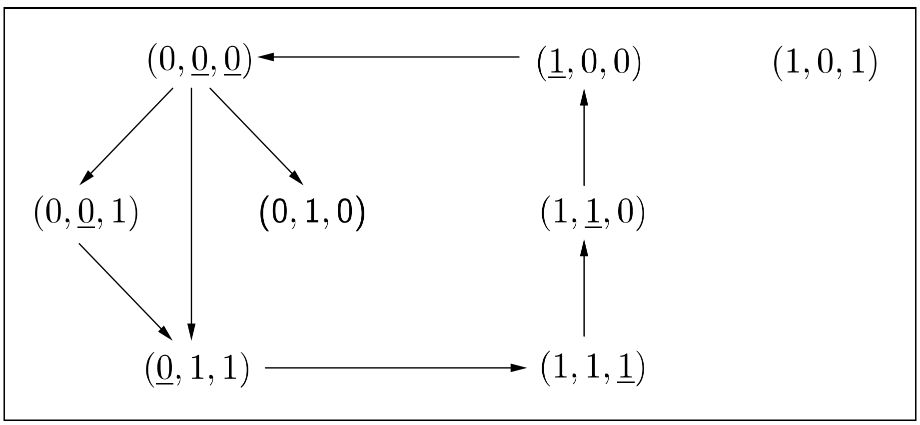

Figure 1. Dependence on the order in which Φ_1, Φ_2, Φ_3 are computed.

$(1, 0, 1)$ is an isolated fixed point (a fixed point is also called equilibrium point, or rest position, or final state), where the system stays indefinitely long. The transition $(0, 1, 1) \longrightarrow (1, 1, 1)$ consists in the computation of $\Phi_1(0, 1, 1)$; even if we do not know when it happens, we know that it happens and the system, if it is in $(0, 1, 1)$, surely gets to $(1, 1, 1)$ sometime. And the transitions $(1, 1, 1) \longrightarrow (1, 1, 0), (1, 1, 0) \longrightarrow (1, 0, 0), (1, 0, 0) \longrightarrow (0, 0, 0)$ are similar. The interesting behavior is in $(0, 0, 0)$; since if $\Phi_3(0, 0, 0)$ is computed first, or if $\Phi_2(0, 0, 0)$, $\Phi_3(0, 0, 0)$ are computed at the same time, the system gets to $(0, 1, 1)$ sometime; but if $\Phi_2(0, 0, 0)$ is computed first, then the state $(0, 1, 0)$ is reached and, as it is a fixed point of Φ, the system rests there indefinitely long.

The circuit suggests the problem of finding classes of Boolean functions Φ -identified with the asynchronous systems- where, even if we do not know the time instants and the order in which their coordinates are computed, we know that $\mu, \Phi(\mu), (\Phi \circ \Phi)(\mu), (\Phi \circ \Phi \circ \Phi)(\mu), \ldots$ are computed sometime, in this order. Thus the behavior of the systems that we are looking for reproduces in a certain way the behavior of the dynamical systems (in its discrete time, Boolean version), and this is considered to be 'nice', in a framework with many unknown parameters. The purpose of the paper is to define and characterize this situation, called the generalized technical condition of proper operation.

The technical condition of proper operation was known for many years by the theoreticians in switching circuits [6], perhaps with different names. We have also gathered useful intuition in this direction from many engineering sources such like [2], [8]. Another bibliographical direction is the one represented by the dynamical systems theory such as [1], [3], [4], [5]. An introduction in asynchronous systems may be found in [7].

We denote in the following with $\mathbf{B}$ the Boolean algebra with two elements, i.e., the set $\{0,1\}$ endowed with the complement $'\,-\,'$, the intersection $'\,\cdot\,'$, the union $'\cup'$, and the modulo 2 sum $'\oplus'$. These laws induce laws denoted with the same symbols on $\mathbf{B}^n$ where they act coordinatewise.

2. Flows

Definition 1. *For $\Phi:\mathbf{B}^n\longrightarrow\mathbf{B}^n$ and $\lambda\in\mathbf{B}^n$, we define the function $\Phi^\lambda:\mathbf{B}^n\longrightarrow\mathbf{B}^n$ by $\forall\mu\in\mathbf{B}^n,\forall i\in\{1,...,n\}$,*

$$\Phi_i^\lambda(\mu)=\begin{cases}\mu_i, if\ \lambda_i=0,\\ \Phi_i(\mu), if\ \lambda_i=1.\end{cases}$$

Definition 2. *The sequence $\alpha:\{0,1,2,...\}\longrightarrow\mathbf{B}^n$, whose terms are denoted in general with α^k (instead of $\alpha(k)$), is called **progressive** if $\forall i\in\{1,...,n\}$, the set $\{k|k\in\{0,1,2,...\},\alpha_i^k=1\}$ is infinite. The set of the progressive sequences is denoted by $\widehat{\Pi}_n$.*

Definition 3. *The **flow** $\widehat{\Phi}^\alpha(\mu,\cdot):\{-1,0,1,...\}\longrightarrow\mathbf{B}^n$ is defined by $\widehat{\Phi}^\alpha(\mu,-1)=\mu$, and $\forall k\geq -1,\widehat{\Phi}^\alpha(\mu,k+1)=\Phi^{\alpha^{k+1}}(\widehat{\Phi}^\alpha(\mu,k))$. Φ is called the **generator function**, and μ is called the **initial (value of the) state**.*

Remark 4. *Here are the explanations related with the previous definitions. Unlike Φ that is computed on all its coordinates (at the same time), Φ^λ is computed on these coordinates only where $\lambda_i=1$. $\widehat{\Phi}^\alpha(\mu,\cdot)$ represents the evolution of a state function starting from μ, that is given by the iterations of Φ_i, made independently on each other, at time instants and in an order indicated by the terms of α. The fact that α is progressive shows that any coordinate i is computed infinitely many times. And the fact that the processes that are modelled by these flows are influenced by unspecified parameters (such as technology, temperature, voltage supply) included indirectly in the model by α, is handled under the form: we are interested in special classes of functions Φ so that we can study properties of $\widehat{\Phi}^\alpha(\mu,\cdot)$ that hold for all $\mu\in\mathbf{B}^n$ and all $\alpha\in\widehat{\Pi}_n$.*

3. The Technical Condition of Proper Operation

Notation 5. *We denote by $\varepsilon^i\in\mathbf{B}^n$ the tuple $\varepsilon^i=(0,...,\underset{i}{1},...,0),i\in\{1,...,n\}$.*

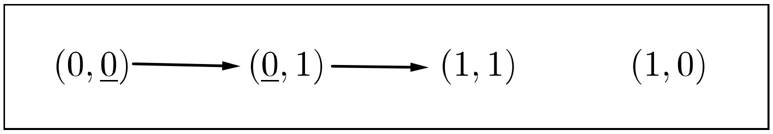

Figure 2. Function that fulfills the technical condition of proper operation.

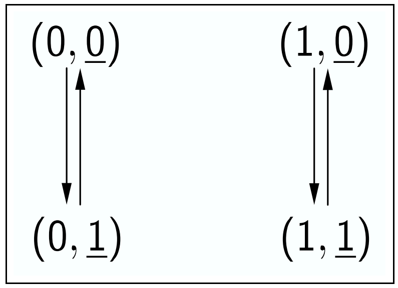

Figure 3. Function that fulfills the technical condition of proper operation.

Remark 6. *$\mathbf{B}^n$ is a linear space over the field $\mathbf{B}$; the sum of the vectors is made coordinatewise $\forall \mu \in \mathbf{B}^n, \forall \mu' \in \mathbf{B}^n$,*

$$(\mu_1, ..., \mu_n) \oplus (\mu'_1, ..., \mu'_n) = (\mu_1 \oplus \mu'_1, ..., \mu_n \oplus \mu'_n)$$

and the multiplication with scalars from $\mathbf{B}$ is made coordinatewise too. ε^i are the vectors of the canonical basis of $\mathbf{B}^n$. Note that the sum $\mu \oplus \mu'$ shows which are the coordinates of μ, μ' that differ ($\mu_i \oplus \mu'_i = 1$) and which are the coordinates of μ, μ' that are equal ($\mu_i \oplus \mu'_i = 0$).

Definition 7. *The function Φ is said to fulfill the **technical condition of proper operation** if $\forall \mu \in \mathbf{B}^n$, one of the following properties is true:*

$$\Phi(\mu) = \mu, \tag{1}$$

$$\exists i \in \{1, ..., n\}, \Phi(\mu) = \mu \oplus \varepsilon^i. \tag{2}$$

Example 8. *The identity $1_{\mathbf{B}^n} : \mathbf{B}^n \longrightarrow \mathbf{B}^n$ fulfills the technical condition of proper operation, since all $\mu \in \mathbf{B}^n$ are fixed points of $1_{\mathbf{B}^n}$.*

Example 9. *The function from Figure 2 fulfills the technical condition of proper operation.*

Example 10. *The function Φ from Figure 3 fulfills the technical condition of proper operation too.*

Example 11. *The function Φ from Figure 4 does not fulfill the technical condition of proper operation, since $\Phi(1,1) = (0,0)$.*

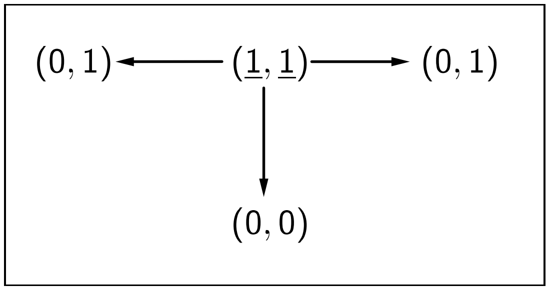

Figure 4. Function that does not fulfill the technical condition of proper operation.

4. Dynamics under the Technical Condition of Proper Operation

Theorem 12. *Let* $\alpha \in \widehat{\Pi}_n, \mu, \mu' \in \mathbf{B}^n, k_1 \in \{-1, 0, 1, ...\}$. *If* Φ *fulfills the technical condition of proper operation and* $\widehat{\Phi}^{\alpha}(\mu, k_1) = \mu'$, *then one of the following possibilities is true:*

a) $\Phi(\mu') = \mu'$ *and* $\forall k \geq k_1$,

$$\widehat{\Phi}^{\alpha}(\mu, k) = \mu' = \Phi(\mu');$$

b) $i \in \{1, ..., n\}$ *exists such that* $\Phi(\mu') = \mu' \oplus \varepsilon^i$ *and either*

$$\widehat{\Phi}^{\alpha}(\mu, k_1 + 1) = \mu' \oplus \varepsilon^i = \Phi(\mu'),$$

or $k_2 \geq k_1 + 2$ *exists with*

$$\widehat{\Phi}^{\alpha}(\mu, k_1 + 1) = ... = \widehat{\Phi}^{\alpha}(\mu, k_2 - 1) = \mu',$$

$$\widehat{\Phi}^{\alpha}(\mu, k_2) = \mu' \oplus \varepsilon^i = \Phi(\mu').$$

Proof. a) If $\Phi(\mu') = \mu'$, then for any $\lambda \in \mathbf{B}^n$ we have $\Phi^{\lambda}(\mu') = \mu'$, thus

$$\widehat{\Phi}^{\alpha}(\mu, k_1 + 1) = \Phi^{\alpha^{k_1+1}}(\widehat{\Phi}^{\alpha}(\mu, k_1)) = \Phi^{\alpha^{k_1+1}}(\mu') = \mu'$$

and the required property is proved by induction on $k \geq k_1$.

b) For any $\lambda \in \mathbf{B}^n, j \in \{1, ..., n\}$ we have

$$\Phi_j^{\lambda}(\mu') = \begin{cases} \mu'_j, \ if \ \lambda_j = 0, \\ \mu'_j, \ if \ \lambda_j = 1, j \neq i, \\ \mu'_j \oplus 1, \ if \ \lambda_j = 1, j = i, \end{cases}$$

thus $\Phi^{\lambda}(\mu') = \mu' \oplus \lambda_i \cdot \varepsilon^i$. As $\alpha \in \widehat{\Pi}_n$ implies $\{k | k \geq k_1 + 1, \alpha_i^k = 1\} \neq \varnothing$, we denote

$$k_2 = \min\{k | k \geq k_1 + 1, \alpha_i^k = 1\}$$

and we have the following possibilities.

Case $k_2 = k_1 + 1$, when

$$\widehat{\Phi}^{\alpha}(\mu, k_1 + 1) = \Phi^{\alpha^{k_1+1}}(\widehat{\Phi}^{\alpha}(\mu, k_1)) = \Phi^{\alpha^{k_1+1}}(\mu') = \mu' \oplus \alpha_i^{k_1+1} \cdot \varepsilon^i = \mu' \oplus \varepsilon^i.$$

Case $k_2 \geq k_1 + 2$, when

$$\widehat{\Phi}^{\alpha}(\mu, k_1 + 1) = \Phi^{\alpha^{k_1+1}}(\widehat{\Phi}^{\alpha}(\mu, k_1)) = \Phi^{\alpha^{k_1+1}}(\mu') = \mu' \oplus \alpha_i^{k_1+1} \cdot \varepsilon^i = \mu',$$

$$...$$

$$\widehat{\Phi}^{\alpha}(\mu, k_2 - 1) = \Phi^{\alpha^{k_2-1}}(\widehat{\Phi}^{\alpha}(\mu, k_2 - 2)) = \Phi^{\alpha^{k_2-1}}(\mu') = \mu' \oplus \alpha_i^{k_2-1} \cdot \varepsilon^i = \mu',$$

$$\widehat{\Phi}^{\alpha}(\mu, k_2) = \Phi^{\alpha^{k_2}}(\widehat{\Phi}^{\alpha}(\mu, k_2 - 1)) = \Phi^{\alpha^{k_2}}(\mu') = \mu' \oplus \alpha_i^{k_2} \cdot \varepsilon^i = \mu' \oplus \varepsilon^i.$$

■

Remark 13. *The Theorem gives the meaning of the technical condition of proper operation. In the situation when we do not know the time instant and the order in which the coordinate functions $\Phi_1, ..., \Phi_n$ are computed, what we surely know is that if $\widehat{\Phi}^{\alpha}(\mu, k_1) = \mu'$, then, independently on the values $\alpha^k \in \mathbf{B}^n$, $k \geq k_1 + 1$, some $k_2 \geq k_1 + 1$ exists such that $\widehat{\Phi}^{\alpha}(\mu, k_2) = \Phi(\mu')$.*

5. The Generalized Technical Condition of Proper Operation

Definition 14. *We say that Φ fulfills the **generalized technical condition of proper operation** if $\forall \mu \in \mathbf{B}^n$,*

$$\begin{aligned} &\exists p \geq 2, \exists i_1 \in \{1, ..., n\}, ..., \exists i_p \in \{1, ..., n\},\ \Phi(\mu) = \mu \oplus \varepsilon^{i_1} \oplus ... \oplus \varepsilon^{i_p} \Longrightarrow \\ &\Longrightarrow \forall \lambda \in \mathbf{B}^p \smallsetminus \{(1, ..., 1)\}, \Phi(\mu) = \Phi(\mu \oplus \lambda_1 \cdot \varepsilon^{i_1} \oplus ... \oplus \lambda_p \cdot \varepsilon^{i_p}). \end{aligned} \tag{3}$$

Remark 15. *For any μ, the generalized technical condition of proper operation refers to the situation when μ and $\Phi(\mu)$ differ on $p \geq 2$ coordinates, $i_1, ..., i_p$; then the value $\Phi(\mu)$ is asked to be equal with the value of Φ in any intermediate state $\mu \oplus \lambda_1 \cdot \varepsilon^{i_1} \oplus ... \oplus \lambda_p \cdot \varepsilon^{i_p}$, $\lambda \neq (1, ..., 1)$ that might result by the computation of $\leq p - 1$ unstable coordinate functions $\Phi_i, i \in \{i_1, ..., i_p\}$.*

Remark 16. *The technical condition of proper operation is indeed a special case of the generalized technical condition of proper operation. This happens since, if μ and $\Phi(\mu)$ differ on 0 or 1 coordinates, then the hypothesis of (3) is false and the generalized technical condition of proper operation is fulfilled.*

Example 17. *We give in Figure 5 an example of function Φ that fulfills the generalized technical condition of proper operation. Note that in each point μ' of the trajectory where $\widehat{\Phi}^{\alpha}(\mu, k_1)$ might be, a subsequent time instant $k_2 \geq k + 1$ exists such that $\widehat{\Phi}^{\alpha}(\mu, k_2) = \Phi(\mu')$, i.e., Φ is eventually iterated, in an asynchronous way. The most interesting transfer here is from $(0, 0, 0)$ to $(0, 1, 1)$, which can take place in three different ways, as $\Phi_3(0, 0, 0)$ is computed first, $\Phi_2(0, 0, 0)$ is computed first, or $\Phi_2(0, 0, 0), \Phi_3(0, 0, 0)$ are computed at the same time. All the other transfers take place in similar conditions with the technical condition of proper operation. To be compared with the Example from Figure 1.*

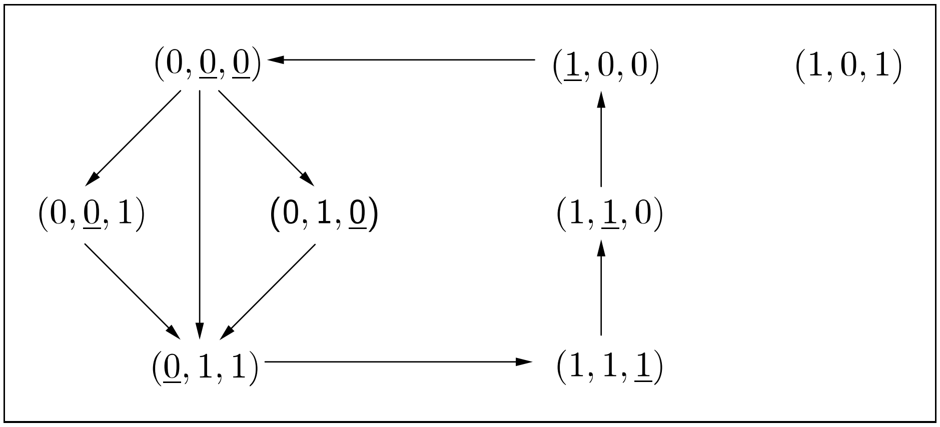

Figure 5. Function Φ that fulfills the generalized technical condition of proper operation.

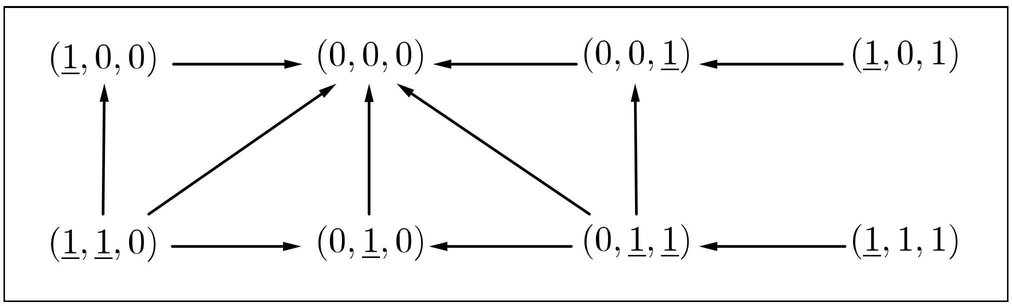

Figure 6. Function that fulfills the generalized technical condition of proper operation.

Example 18. *The function from Figure 6 fulfills also the generalized technical condition of proper operation.*

Example 19. *In Figure 7 we give the example of the function* $\Phi : \mathbf{B}^2 \longrightarrow \mathbf{B}^2, \forall \mu \in \mathbf{B}^2,$

$$\Phi(\mu_1, \mu_2) = (\overline{\mu_1}, \overline{\mu_2})$$

that does not fulfill the generalized technical condition of proper operation. This is seen from the counterexample: $\Phi(0,0) = (1,1)$, *but* $\Phi(0,1) = (1,0)$.

6. Dynamics under the Generalized Technical Condition of Proper Operation

Lemma 20. *Let* $\mu' \in \mathbf{B}^n, p \geq 2, i_1 \in \{1,...,n\},...,i_p \in \{1,...,n\}$ *and we suppose that*

$$\Phi(\mu') = \mu' \oplus \varepsilon^{i_1} \oplus ... \oplus \varepsilon^{i_p}, \tag{4}$$

$$\begin{aligned} &\forall \lambda_{i_1} \in \mathbf{B}, ..., \forall \lambda_{i_p} \in \mathbf{B}, \lambda_{i_1} \cdot ... \cdot \lambda_{i_p} = 0 \Longrightarrow \\ &\Longrightarrow \Phi(\mu') = \Phi(\mu' \oplus \lambda_{i_1} \cdot \varepsilon^{i_1} \oplus ... \oplus \lambda_{i_p} \cdot \varepsilon^{i_p}) \end{aligned} \tag{5}$$

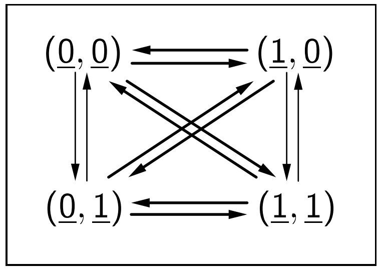

Figure 7. Function that does not fulfill the generalized technical condition of proper operation.

are true. Then $\forall \lambda \in \mathbf{B}^n, \forall \nu \in \mathbf{B}^n, \lambda_{i_1} \cdot ... \cdot \lambda_{i_p} = 0$ *implies*

$$\Phi^{\nu}(\mu' \oplus \lambda_{i_1} \cdot \varepsilon^{i_1} \oplus ... \oplus \lambda_{i_p} \cdot \varepsilon^{i_p}) = \mu' \oplus (\lambda_{i_1} \cup \nu_{i_1}) \cdot \varepsilon^{i_1} \oplus ... \oplus (\lambda_{i_p} \cup \nu_{i_p}) \cdot \varepsilon^{i_p}.$$

Proof. Let $\lambda \in \mathbf{B}^n$ arbitrary such that $\lambda_{i_1} \cdot ... \cdot \lambda_{i_p} = 0$ and we consider $\nu \in \mathbf{B}^n, j \in \{1, ..., n\}$ arbitrary also. We have:

$$\Phi_j^{\nu}(\mu' \oplus \lambda_{i_1} \cdot \varepsilon^{i_1} \oplus ... \oplus \lambda_{i_p} \cdot \varepsilon^{i_p}) = \begin{cases} \mu'_j, \ if \ \nu_j = 0, j \notin \{i_1, ..., i_p\}, \\ \mu'_j, \ if \ \nu_j = 0, j \in \{i_1, ..., i_p\}, \lambda_j = 0, \\ \mu'_j \oplus 1, \ if \ \nu_j = 0, j \in \{i_1, ..., i_p\}, \lambda_j = 1, \\ \Phi_j(\mu' \oplus \lambda_{i_1} \cdot \varepsilon^{i_1} \oplus ... \oplus \lambda_{i_p} \cdot \varepsilon^{i_p}), \ if \ \nu_j = 1 \end{cases}$$

$$\overset{(5)}{=} \begin{cases} \mu'_j, \ if \ \nu_j = 0, j \notin \{i_1, ..., i_p\}, \\ \mu'_j, \ if \ \nu_j = 0, j \in \{i_1, ..., i_p\}, \lambda_j = 0, \\ \mu'_j \oplus 1, \ if \ \nu_j = 0, j \in \{i_1, ..., i_p\}, \lambda_j = 1, \\ \Phi_j(\mu'), \ if \ \nu_j = 1 \end{cases}$$

$$\overset{(4)}{=} \begin{cases} \mu'_j, \ if \ \nu_j = 0, j \notin \{i_1, ..., i_p\}, \\ \mu'_j, \ if \ \nu_j = 0, j \in \{i_1, ..., i_p\}, \lambda_j = 0, \\ \mu'_j \oplus 1, \ if \ \nu_j = 0, j \in \{i_1, ..., i_p\}, \lambda_j = 1, \\ \mu'_j, \ if \ \nu_j = 1, j \notin \{i_1, ..., i_p\}, \\ \mu'_j \oplus 1, \ if \ \nu_j = 1, j \in \{i_1, ..., i_p\} \end{cases}$$

$$= \begin{cases} \mu'_j, \ if \ (\nu_j = 0, j \notin \{i_1, ..., i_p\}) \text{ or } (\nu_j = 0, j \in \{i_1, ..., i_p\}, \lambda_j = 0) \\ \text{or } (\nu_j = 1, j \notin \{i_1, ..., i_p\}), \\ \mu'_j \oplus 1, \ if \ (\nu_j = 0, j \in \{i_1, ..., i_p\}, \lambda_j = 1) \text{ or } (\nu_j = 1, j \in \{i_1, ..., i_p\}) \end{cases}$$

thus

$$\Phi^{\nu}(\mu' \oplus \lambda_{i_1} \cdot \varepsilon^{i_1} \oplus ... \oplus \lambda_{i_p} \cdot \varepsilon^{i_p}) = \mu' \oplus (\overline{\nu_{i_1}} \cdot \lambda_{i_1} \cup \nu_{i_1}) \cdot \varepsilon^{i_1} \oplus ... \oplus (\overline{\nu_{i_p}} \cdot \lambda_{i_p} \cup \nu_{i_p}) \cdot \varepsilon^{i_p}$$

$$= \mu' \oplus (\lambda_{i_1} \cup \nu_{i_1}) \cdot \varepsilon^{i_1} \oplus ... \oplus (\lambda_{i_p} \cup \nu_{i_p}) \cdot \varepsilon^{i_p}.$$

■

Theorem 21. *Let $\alpha \in \widehat{\Pi}_n, \mu, \mu' \in \mathbf{B}^n, k_1 \in \{-1, 0, 1, ...\}$. We suppose that Φ fulfills the generalized technical condition of proper operation and $\widehat{\Phi}^\alpha(\mu, k_1) = \mu'$. Then one of the following situations holds:*

a) $\Phi(\mu') = \mu'$ and $\forall k \geq k_1$,

$$\widehat{\Phi}^\alpha(\mu, k) = \mu' = \Phi(\mu');$$

b) $i \in \{1, ..., n\}$ exists such that $\Phi(\mu') = \mu' \oplus \varepsilon^i$ and either

$$\widehat{\Phi}^\alpha(\mu, k_1 + 1) = \mu' \oplus \varepsilon^i = \Phi(\mu'),$$

or $k_2 \geq k_1 + 2$ exists with

$$\widehat{\Phi}^\alpha(\mu, k_1 + 1) = ... = \widehat{\Phi}^\alpha(\mu, k_2 - 1) = \mu',$$

$$\widehat{\Phi}^\alpha(\mu, k_2) = \mu' \oplus \varepsilon^i = \Phi(\mu');$$

c) $i_1, ..., i_p \in \{1, ..., n\}$ exist, $p \geq 2$ such that $\Phi(\mu') = \mu' \oplus \varepsilon^{i_1} \oplus ... \oplus \varepsilon^{i_p}$ and either

$$\widehat{\Phi}^\alpha(\mu, k_1 + 1) = \mu' \oplus \varepsilon^{i_1} \oplus ... \oplus \varepsilon^{i_p} = \Phi(\mu'),$$

or $k_2 \geq k_1 + 2$ exists with

$$\forall k \in \{k_1 + 1, ..., k_2 - 1\}, \widehat{\Phi}^\alpha(\mu, k) \neq \Phi(\mu'),$$

$$\forall k \in \{k_1 + 1, ..., k_2\}, \forall i \in \{1, ..., n\}, \widehat{\Phi}_i^\alpha(\mu, k) \text{ are monotonous,}$$

$$\widehat{\Phi}^\alpha(\mu, k_2) = \mu' \oplus \varepsilon^{i_1} \oplus ... \oplus \varepsilon^{i_p} = \Phi(\mu').$$

Proof. The items a), b) have already been proved at Theorem 12., we prove c). For $\lambda \in \mathbf{B}^n$ we have $\forall j \in \{1, ..., n\}$,

$$\Phi_j^\lambda(\mu') = \begin{cases} \mu'_j, \ if \ \lambda_j = 0, \\ \mu'_j, \ if \ \lambda_j = 1, j \notin \{i_1, ..., i_p\}, \\ \mu'_j \oplus 1, \ if \ \lambda_j = 1, j \in \{i_1, ..., i_p\} \end{cases}$$

thus we can write

$$\Phi^\lambda(\mu') = \mu' \oplus \lambda_{i_1} \cdot \varepsilon^{i_1} \oplus ... \oplus \lambda_{i_p} \cdot \varepsilon^{i_p}.$$

We denote

$$k_2 = \begin{cases} k_1 + 1, \ if \ \alpha_{i_1}^{k_1+1} = ... = \alpha_{i_p}^{k_1+1} = 1, \\ \min\{k | k \geq k_1 + 1, \alpha_{i_1}^{k_1+1} \cup ... \cup \alpha_{i_1}^{k} = 1 \text{ and } ... \\ \text{and } \alpha_{i_p}^{k_1+1} \cup ... \cup \alpha_{i_p}^{k} = 1\}, \ else \end{cases} \tag{6}$$

and we have the following possibilities.

Case $k_2 = k_1 + 1$, when

$$\widehat{\Phi}^\alpha(\mu, k_1 + 1) = \Phi^{\alpha^{k_1+1}}(\widehat{\Phi}^\alpha(\mu, k_1)) = \Phi^{\alpha^{k_1+1}}(\mu') = \mu' \oplus \alpha_{i_1}^{k_1+1} \cdot \varepsilon^{i_1} \oplus ... \oplus \alpha_{i_p}^{k_1+1} \cdot \varepsilon^{i_p}$$

$$= \mu' \oplus \varepsilon^{i_1} \oplus ... \oplus \varepsilon^{i_p};$$

Case $k_2 \geq k_1 + 2$, when

$$\widehat{\Phi}^{\alpha}(\mu, k_1 + 1) = \Phi^{\alpha^{k_1+1}}(\widehat{\Phi}^{\alpha}(\mu, k_1)) = \Phi^{\alpha^{k_1+1}}(\mu') = \mu' \oplus \alpha_{i_1}^{k_1+1} \cdot \varepsilon^{i_1} \oplus ... \oplus \alpha_{i_p}^{k_1+1} \cdot \varepsilon^{i_p},$$

$$\widehat{\Phi}^{\alpha}(\mu, k_1 + 2) = \Phi^{\alpha^{k_1+2}}(\widehat{\Phi}^{\alpha}(\mu, k_1 + 1)) = \Phi^{\alpha^{k_1+2}}(\mu' \oplus \alpha_{i_1}^{k_1+1} \cdot \varepsilon^{i_1} \oplus ... \oplus \alpha_{i_p}^{k_1+1} \cdot \varepsilon^{i_p})$$

$$\overset{Lemma\ 20.}{=} \mu' \oplus (\alpha_{i_1}^{k_1+1} \cup \alpha_{i_1}^{k_1+2}) \cdot \varepsilon^{i_1} \oplus ... \oplus (\alpha_{i_p}^{k_1+1} \cup \alpha_{i_p}^{k_1+2}) \cdot \varepsilon^{i_p},$$

$$...$$

$$\widehat{\Phi}^{\alpha}(\mu, k_2 - 1) = \Phi^{\alpha^{k_2-1}}(\widehat{\Phi}^{\alpha}(\mu, k_2 - 2))$$

$$= \Phi^{\alpha^{k_2-1}}(\mu' \oplus (\alpha_{i_1}^{k_1+1} \cup ... \cup \alpha_{i_1}^{k_2-2}) \cdot \varepsilon^{i_1} \oplus ... \oplus (\alpha_{i_p}^{k_1+1} \cup ... \cup \alpha_{i_p}^{k_2-2}) \cdot \varepsilon^{i_p})$$

$$\overset{Lemma\ 20.}{=} \mu' \oplus (\alpha_{i_1}^{k_1+1} \cup ... \cup \alpha_{i_1}^{k_2-1}) \cdot \varepsilon^{i_1} \oplus ... \oplus (\alpha_{i_p}^{k_1+1} \cup ... \cup \alpha_{i_p}^{k_2-1}) \cdot \varepsilon^{i_p},$$

$$\widehat{\Phi}^{\alpha}(\mu, k_2) = \Phi^{\alpha^{k_2}}(\widehat{\Phi}^{\alpha}(\mu, k_2 - 1))$$

$$= \Phi^{\alpha^{k_2}}(\mu' \oplus (\alpha_{i_1}^{k_1+1} \cup ... \cup \alpha_{i_1}^{k_2-1}) \cdot \varepsilon^{i_1} \oplus ... \oplus (\alpha_{i_p}^{k_1+1} \cup ... \cup \alpha_{i_p}^{k_2-1}) \cdot \varepsilon^{i_p})$$

$$\overset{Lemma\ 20.}{=} \mu' \oplus (\alpha_{i_1}^{k_1+1} \cup ... \cup \alpha_{i_1}^{k_2}) \cdot \varepsilon^{i_1} \oplus ... \oplus (\alpha_{i_p}^{k_1+1} \cup ... \cup \alpha_{i_p}^{k_2}) \cdot \varepsilon^{i_p}$$

$$\overset{(6)}{=} \mu' \oplus \varepsilon^{i_1} \oplus ... \oplus \varepsilon^{i_p}.$$

Moreover, $\forall k \in \{k_1 + 1, ..., k_2\}, \forall i \in \{1, ..., n\}$, the functions $\alpha_i^{k_1+1} \cup ... \cup \alpha_i^k$ are increasing, thus $\widehat{\Phi}_i^{\alpha}(\mu, k)$ are monotonous. ■

Remark 22. *The meaning of the generalized technical condition of proper operation as shown by the previous Theorem is the following. Let us suppose that* $\widehat{\Phi}^{\alpha}(\mu, k_1) = \mu'$*. Then, for any* $\alpha^{k_1+1}, \alpha^{k_1+2}, ... \in \mathbf{B}^n$ *(we do not know the time instants when the coordinate functions* Φ_i *are computed), some* $k_2 \geq k_1 + 1$ *exists with the property that* $\widehat{\Phi}^{\alpha}(\mu, k_2) = \Phi(\mu')$.

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