WSEAS

Asynchronous Systems Theory

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Dedicated to the memory of Grigore Moisil

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Contents

CHAPTER 1

Introduction

1. Historical origins of the theory of asynchronous systems

Professor Grigore Moisil (1906-1973) is one of the computer science founders and its applications in Romania. He founded the school of the switching theory and the associated polyvalent logic. Among the members of his school we can mention the following: George Georgescu, Serban Basarab, Ioana Petrescu (married Voiculescu), Sergiu Rudeanu, Petre Ivanescu, Gheorghe Nadiu, A. Deleanu, Toma Gaspar, I. Muntean, Dragos Vaida, Gh. Ioanin, P. Constantinescu, C. Popovici, Mariana Coroi-Nedelcu.

Professor Moisil was a brilliant mind who had a profound influence on the Romanian mathematical thinking by pointing out the necessity of its orientation towards basic applications. He sensed the huge importance of automatized computations for humanity as early as the 50's of the last century but, of course, his ideas were premature, in a mathematical atmosphere dominated by Bourbaki's trend favorable rather to pure theoretical studies than to concrete applications. For instance, in 1965 [**21**] he introduced in his book a genuine collection of circuits[1] and proposed the readers to go on with their investigation. Year by year, together with his co-workers he studied various aspects of the so-called theoretical informatics, during seminars and, especially, during the communication sessions of the system theory group held at the Faculty of Mathematics, Bucharest University. Little by little, the investigations moved almost completely to the pure theoretical aspects. Even more, to our knowledge, after the years of '70, the research in the field of switching theory practically stopped, in spite of the unanimous recognition of its importance.

The discrete-time modeling of the switching phenomena introduced by Moisil proved to be a pioneering approach, but also represented a limit of his theory. Indeed, at that time, the question about the degree in which the discrete time modeling can approximate the realistic continuous time modeling was left aside.

Subsequently, under the evidence of concrete examples, the mathematical community was forced to consider more carefully the problem of the relationship between discrete and continuous-time modeling. Our investigations on switching circuits were influenced by Moisil's ideas, although not directly: we learnt about his research during university studies. By that time, the digital electrical engineering was rather descriptive than mathematically formalized. Therefore, with a view to deeply understand the phenomena in switching circuits, we became interested in their mathematical formalization. A hard work of documentation followed, hoping to find the mathematics underlying these circuits. To our big surprise, we found

[1] The words: switching circuit, asynchronous circuit, circuit, network are considered to be synonims in this context.

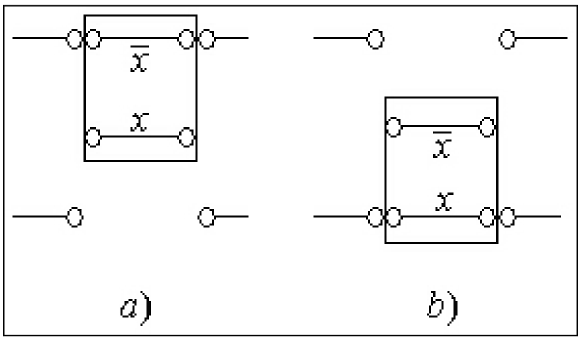

FIGURE 1. Armature with two contacts

almost nothing in our field of interest, in particular the study of the $\mathbf{R} \to \{0,1\}$ functions. In fact in the late 80's Professor Sergiu Rudeanu confirmed the lack of such a study in the mathematics of the world. As a result, by the year 2000 and even earlier we started with a systematic investigation of the asynchronous circuits and the construction of the necessary mathematical tools. These led us to two distinct categories of circuits. Roughly speaking, the first category contains delay circuits that, connected in series, keep the model and the second category contains delay circuits that, connected in series, do not keep the model. By identifying the circuit with its model, the same idea may be expressed as: *two delay circuits connected in series form a delay circuit in the first category* while in the second, *two delay circuits connected in series do not form a delay circuit*.

The splitting in two categories solved the so called paradox we noticed in 2001. Namely, we found by following the descriptive theories of our former professors two delay circuits connected in series that were not of the same kind like each of them taken separately. This was a particular case of circuits of the second category, which our professors considered to be circuits of the first category.

The solution in solving this paradox was based on the mathematical modeling by Moisil's line of thinking that has more or less imposed in Romania. Maybe this was the real influence that the great mathematician had on us.

2. Moisil legacy

Moisil presents [**21**], [**22**] the contacts and the relays in the next manner[2]. A **contact** is a device with two positions: open and closed. Figure 1 shows an armature with two contacts. The upper contact of the armature is called an **opening contact** and the lower contact is called a **closing contact**. A binary variable is associated to each contact as follows:

$x = 0$, if the closing contact is open;
$x = 1$, if the closing contact is closed;
$\overline{x} = 1$, if the opening contact is closed;
$\overline{x} = 0$, if the opening contact is open.

[2]By that time, he knew perfectly that the switching circuits may be realized with other means as well, such as the electronic tubes and the transistors. Such devices are studied in his works. We chose to present the contacts and the relays because they reproduce the best his mathematical intentions.

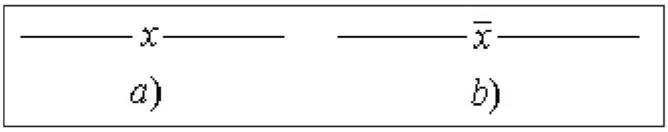

FIGURE 2. The symbols of the contacts

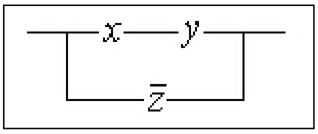

FIGURE 3. A two terminal network with contacts

In circuits, contacts are symbolized like in Figure 2. We have:
$x = 0$, if in Figure 2 a) the current may not flow through the wire;
$x = 1$, if in Figure 2 a) the current may flow through the wire;
$\overline{x} = 0$, if in Figure 2 b) the current may not flow through the wire;
$\overline{x} = 1$, if in Figure 2 b) the current may flow through the wire.

The contacts are connected in series and in parallel like in Figure 3 and they form a **two terminal network**, where **conductivity** is the variable w defined as

$w = 0$, when the two terminal network does not allow the current to flow by opening the circuit;

$w = 1$, when the two terminal network allows the current to flow by closing the circuit.

Thus w is a function of x, y, z :

$$w = F(x, y, z) = x \cdot y \cup \overline{z}.$$

We have denoted by —, $\cdot$, $\cup$ the usual Boolean laws.

At this moment $\tau > 0$ is assumed to exist so that in any interval $(n\tau, (n+1)\tau)$, any variable which appears in the network has a constant binary value, denoted by $x_n, y_n, ...$ where the discrete time n runs over $\mathbf{N} = \{0, 1, 2, ...\}$.

An (ordinary) **relay** (Figure 4) is a device consisting of an electromagnet which attracts an armature with contacts. The variable ξ, associated to the current in the relay winding, takes on, in every time interval n, one of the following two values:

$\xi_n = 0$, if during time interval n, no current passes through the relay winding;
$\xi_n = 1$, if during time interval n, a current does pass through the relay winding.

The **characteristic equation** for ordinary relays with ideal contacts is defined as

$$x_{n+1} = \xi_n,$$

meaning that the relay acts as a delay circuit. More exactly, a relay introduces in the circuit a delay of 1 time unit.

Figure 5 shows how the relays have been symbolized in a network. The circuit contains a contact and two relays X, Y and we study it by using 5 variables:
a - associated to the contact;
ξ, η - associated to the currents in the windings of X, Y;
x, y - associated to the contacts of X, Y.

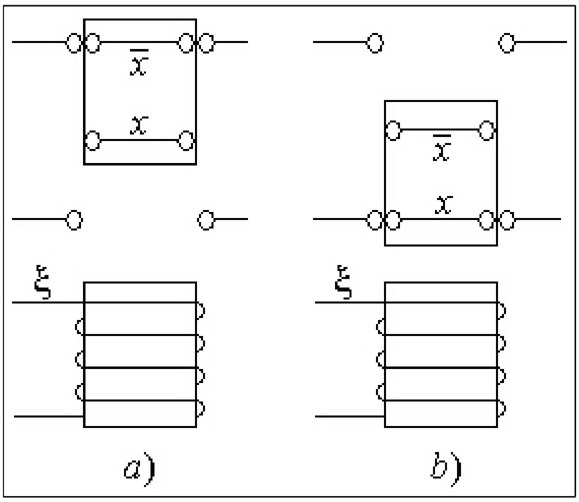

FIGURE 4. An ordinary relay

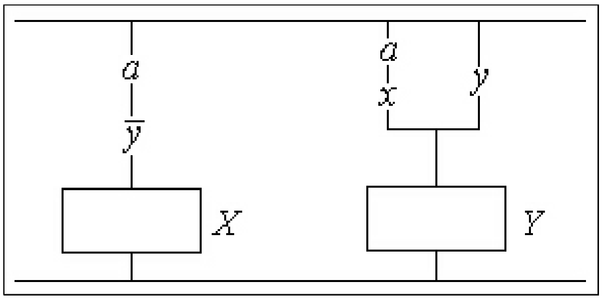

FIGURE 5. A network with two relays

The equations are:

$$\xi_n = a_n \cdot \overline{y_n},$$

$$\eta_n = a_n \cdot x_n \cup y_n,$$

$$x_{n+1} = \xi_n,$$

$$y_{n+1} = \eta_n.$$

Starting from the 'rest position'

$$a_{-1} = x_{-1} = y_{-1} = \xi_{-1} = \eta_{-1} = 0$$

where the circuit could have remained indefinitely long, the contact a is operated: $a_0 = 1$. We get the evolution from the next table.

time	a	x	y	ξ	η	
-1	0	0	0	0	0	rest position
0	1	0	0	1	0	operating a leads to operation of X
1	1	1	0	1	1	contact x is closed and Y is operated
2	1	1	1	0	1	feed-back action on X
3	1	0	1	0	1	x opens and y remains closed
4	1	0	1	0	1	stable position

([**22**], pages 102, 103).

3. About the book

From the very beginning we mention the fact that some of the concepts introduced by us bear the same name with but are different from the concepts used in the literature, since words like asynchronous, inertia, delay have many meanings. Moreover, the imprecise non-formalized concepts are associated by us with precise mathematical definitions. This is why we give a lot of definitions.

The book is intended to construct a mathematical theory of modeling the asynchronous circuits.

The asynchronous systems theory is a branch of the systems theory that has the purpose of bringing under a common framework the mathematical models of the asynchronous circuits from the digital electrical engineering. It uses:

- the general concepts of system and pseudo-system providing models for the functional blocks, where modeling is present at a synthetical level as well as

- the particular concept of delay (stable system with 1-dimensional input and 1-dimensional output), providing models for the gates and wires, where modeling is present at an analytical level.

Our first interests in asynchronous systems theory date back from 1984.

In trying to produce continuos-time models for switching circuits (some of them described by discrete time models by Moisil) we needed first to extend the calculus of the $\mathbf{R}-$valued functions to the calculus of the $\{0,1\}-$valued functions.

In this way, in the early 90's we wrote some works in mathematical analysis on $\mathbf{R} \to \{0,1\}$ functions trying to probe how far the analogies with the $\mathbf{R} \to \mathbf{R}$ functions can go. Derivatives, as well as several types of integrals can be defined on $\mathbf{R} \to \{0,1\}$ functions, together with convolution products, distributions over such test functions and, as a conclusion, a mathematical analysis written with these pseudo-Boolean functions exists, having the only disadvantage of certain trivialities due to the finiteness of $\{0,1\}$ compared with $\mathbf{R}$. In this book we limit ourselves to quote only a few results of our research strictly necessary in this topic.

In our paper [**33**] we have written the differential equations of a special case of deterministic inertial asynchronous system. These equations, briefly reproduced under the form (14.2) in Ch. 13 and describing the so-called paradox in Section 7 of Ch. 13, were inspired by the inertial principle: '*a cause produces effects if and only if it is persistent*', i.e. if the cause acts continuously $d > 0$ time units, where d is a parameter characterizing inertia. In more refined reasonings we have passed from one to two parameters, such that each of these parameters could be identified with the parameter d. Usually, these parameters are considered to be equal and they are called the *transmission delay for transitions* and respectively the *threshold for cancellation*.

Later on, in 2001, we communicated the title of our contribution to the organizers of the symposium of mathematics and its applications that the 'Politehnica' University of Timisoara had. Our paper dealt with the results of our reflections issued in our attempt to understand the principles of modeling the asynchronous circuits which brought to our attention some major theoretical shortcomings, that represented an apparent paradox. Then the title of the paper was changed, hoping that the new title will draw one's attention on the aspect that proved to be fundamental in solving all the other problems of (logical, detail level) modeling: '*how do we model the most simple asynchronous circuit, namely the delay circuit?*', i.e. the circuit that implements in digital electrical engineering the computation of the identity $1_{\{0,1\}} : \{0,1\} \to \{0,1\}$. An answer that could be given according to the existing informal literature (presented here in Section 2 of Ch. 10) was: 'we know some possibilities of modeling a delay circuit. However, as already shown in [**33**], a major difficulty appears, the serial connection of the models of two delay circuits is not the model of a delay circuit'. Something like: *a delay's delay is not a delay*, or perhaps *the inertia of inertia is not inertia*. A 'paradox' that could not be ignored.

Solutions were given in our papers [**36**] and mostly in [**40**]. We introduced there the concept of delay (condition) = the mathematical model of the delay circuit and also the concept of pure = ideal = fixed delay, representing a delay without inertia. All the delays that are not pure are, by definition, inertial. The 'paradox' was solved under the form: *the inertial delays connected in series are an inertial delay*, but there are two possibilities:

- by serial connection, the type of inertia is conserved. This is the example of the bounded delays (two bounded delays connected in series represent a bounded delay);

- by serial connection, the type of inertia is not conserved. This is the example of the relatively inertial delays (two relatively inertial delays connected in series do not represent a relatively inertial delay).

It just happened that in the paper from Timisoara we were situated in the second case.

Our systemic (logical, detail level, real time) mathematical modeling of the asynchronous circuits was initiated to some extent at that moment. We need to choose between several types of delays, insert them, where necessary, before/after the logical gates and in the wires, then write and solve equations and inequalities where Boolean functions that are instantaneously (without delays) computed occur, too. The technique of analysis is rather complicated, but works for small circuits and if the circuits are not small, then computers should be used in modeling.

The concepts of asynchronous system and asynchronous pseudo-system were introduced in [**37**], [**39**]. We have defined them in the sense that can be referred to in literature as 'the input-output behavior of a non-initialized, non-deterministic system', i.e. as multi-valued functions.

Roughly speaking, the n-dimensional signals are the 'nice' $\mathbf{R} \to \{0,1\}^n$ functions while an (asynchronous) system is a multi-valued function that associates with an m-dimensional signal, called an (admissible) input, a non-empty set of n-dimensional signals, called the (possible) states. The input and the states are required to have a limit as $t \to -\infty$ (an initial value). More general than that, an (asynchronous) pseudo-system possesses:

- signals without limit as $t \to -\infty$ (without initial values);

- the possibility that to an input $u : \mathbf{R} \to \{0,1\}^m$ there may correspond an empty set of states, i.e. non-admissible inputs exist.

The naturalness of our concept of pseudo-system consists in the fact that:

- it highlights the duality between the initial values and the final values of the states. The dual properties of initialization and stability can be defined in this context, that includes a duality between the initial time and the final time too;

- we must take into account the fact that very simple circuits like the RS latch, for example, have non-admissible inputs ($R \cdot S = 1$ is such an input).

A subsidiary aim of the book is to propose open problems, such as: the characterization of the Huffman systems, what is the role of injectivity and surjectivity, what non-anticipation is - things that seem to be very familiar.

The mathematical facts we presented may also be useful in studying the general topics of the systems theory. Here are some of them:

- synthesis, model checking (for detecting errors in hardware designs);
- stability;
- feedback, control;
- optimization, optimal control (for example minimal time);
- controllability, accessibility;
- structural decomposability.

It is interesting as well to establish the connections between this theory and other theories: Petri nets, temporal logic, timed automata.

The book is organized in three parts: the first is dedicated to the general systems theory, the second to the delay theory and the third to applications. Each part contains several chapters and the chapters are structured in sections. The important equations and logical properties are numbered. Thus (4.3) refers to the third outlined equation or logical property of the fourth section of the current chapter; when we refer to equation (4.3) from the current chapter we do not need to indicate the chapter, while when we refer to the same equation from another chapter, we need to indicate the chapter because it does not follow from this number. The end of the book has three appendices: one showing some intersections with temporal logic, an index and a list of notations.

Chapter 2 contains the mathematical framework necessary to model the switching circuits: spaces of $\{0,1\}-$valued functions and operations on them. In Chapter 3 we introduce a large class of models, namely the pseudo-systems and a few concepts related to them (initial and final states, initial and final time and initial and final state functions). The most important type of the set of pseudo-systems consists of the so-called systems. They represent that particular case of non-empty pseudo-systems characterized by the existence of the initial values of the inputs and of the states. The systems are treated in Chapter 4 together with some new notions of further interest. In Chapter 5 particular cases of systems are introduced and their properties are largely investigated and commented. The next chapter deals with the accesses and the transfers of the systems. The surjectivity, controllability and accessibility are the matters of Chapter 7, where comparisons with other variants existing in the literature are made. Three types of stability of systems are the subject of Chapter 8. A few examples are included. In Chapter 9, after a brief presentation of the known non-formalized definitions of the fundamental mode, the fundamental transfers are introduced and analyzed. Then, the fundamental mode is introduced, by using these transfers and its properties are investigated. The relations between the fundamental mode and accessibility are studied. Part 2 is dedicated to the delay theory. It starts with the introduction and study of delays, in Chapter 10. Several types of delays are included. The special class of bounded delays is investigated in Chapter 11, while in Chapters 12 and 13 the absolutely inertial delays and the relatively inertial delays are treated. All these three chapters differ from the previous ones in the sense that they contain a larger comparison of our notions and the traditional ones. The meaning and the interest for applications of the presented notions and properties are carefully commented. Part 3 of the book is dedicated to applications.

We tried to itemize the exposure carefully in definitions, theorems, lemmas, notations, examples and remarks since this gives good possibilities of understanding and that of referencing. Their too rich number can be boring to the reader but we are convinced that it is only in this way that a rigorous treatment was possible.

We have written in full details many dual results. Generally, the proofs are elementary and some of them have been omitted, others were included for the reason of making the exposure as readable as possible. The dual proofs have been omitted.

The book addresses to researchers in computer science, mathematicians and electrical engineers interested in modeling asynchronous circuits. Its applications are useful to the electrical engineers.

I am grateful to Professor Adelina Georgescu for accepting to read carefully my book. Many statements have been rephrased on her suggestions.

Part 1

Asynchronous systems theory

CHAPTER 2

Calculus in $\mathbf{B}^n$

Some important concepts and notations on the Boolean and pseudo-Boolean functions used throughout the book are introduced. Then the calculus for $\mathbf{B}$ is extended to $\mathbf{B}^n$, where $\mathbf{B}$ is the binary Boole algebra. The basic concepts in asynchronous circuit theory are defined with the help of the introduced functions.

1. The binary Boole algebra B

DEFINITION 1. *The set* $\mathbf{B} = \{0, 1\}$ *endowed with the discrete topology, with the order* $0 < 1$ *and with the laws* $^{-}, \cup, \cdot, \oplus$ *defined as in Figure 1 is called the* ***binary Boole*** *(or* ***Boolean****)* ***algebra****.*

DEFINITION 2. *Let* J *be an arbitrary set and consider the generalized binary sequence* $a_j \in \mathbf{B}, j \in J$. *We define the intersections and the unions*

$$\bigcap_{j\in J} a_j = \begin{cases} 0, if\ \exists j \in J, a_j = 0 \\ 1, otherwise \end{cases},$$

$$\bigcap_{j\in\emptyset} a_j = 1,$$

$$\bigcup_{j\in J} a_j = \begin{cases} 1, if\ \exists j \in J, a_j = 1 \\ 0, otherwise \end{cases},$$

$$\bigcup_{j\in\emptyset} a_j = 0.$$

NOTATION 1. *For any* $\lambda \in \mathbf{B}^m$, *we use the following notation for the complement* $\overline{\lambda}$ *of* λ

$$\overline{\lambda} = (\overline{\lambda_1}, ..., \overline{\lambda_m}).$$

NOTATION 2. *Let* A *be an arbitrary non-empty set. Then*

$$P(A) = \{A' | A' \subset A\}$$

is the set of the subsets of A *and*

$$P^*(A) = \{A' | A' \subset A, A' \neq \emptyset\}$$

$^{-}$	0	1
	1	0

$\cup$	0	1
0	0	1
1	1	1

$\cdot$	0	1
0	0	0
1	0	1

$\oplus$	0	1
0	0	1
1	1	0

FIGURE 1. The laws of $\mathbf{B}$

is the set of the non-empty subsets of A.

In the following A is any of $\mathbf{R}$, $\mathbf{B}^n$ *and some subspaces of* $\mathbf{R} \to \mathbf{B}^n$ *functions.*

DEFINITION 3. *The functions* $F : \mathbf{B}^m \to \mathbf{B}^n$,

$$\mathbf{B}^m \ni (\lambda_1, ..., \lambda_m) \longmapsto (F_1(\lambda_1, ..., \lambda_m), ..., F_n(\lambda_1, ..., \lambda_m)) \in \mathbf{B}^n$$

are called ***Boolean functions****.*

DEFINITION 4. *The* ***dual function*** $F^* : \mathbf{B}^m \to \mathbf{B}^n$ *of* F *is defined by*

$$\forall \lambda \in \mathbf{B}^m, F^*(\lambda) = \overline{F(\overline{\lambda})}.$$

REMARK 1. *The set* $\mathbf{B}$ *is a Boole algebra relative to* $\overline{}, \cup, \cdot$ *and a field relative to* $\oplus, \cdot$. *It is not an ordered field, since* $1 \geq 0$ *but* $1 \oplus 1 \leq 1 \oplus 0$.

The way that the complement $\overline{}$ *of* $\mathbf{B}$ *induced a law on* $\mathbf{B}^m$, *we could write similar notations for* $\cup, \cdot, \oplus$. *This is not going to be useful for our needs, however.*

The dual of $\overline{}$ *is* $\overline{}$ *itself; the dual of* $\cup$ *is* $\cdot$ *and vice versa. The dual of* $\oplus$ *is the coincidence* $\odot$:

$$\forall \lambda_1 \in \mathbf{B}, \forall \lambda_2 \in \mathbf{B}, \lambda_1 \odot \lambda_2 = \overline{\lambda_1 \oplus \lambda_2}.$$

2. $\mathbf{R} \to \mathbf{B}$ functions

NOTATION 3. *Let* $A \subset \mathbf{R}$ *be some set. We denote by* $\chi_A : \mathbf{R} \to \mathbf{B}$ *the* ***characteristic function*** *of the set A, defined as usual*

$$\forall t \in \mathbf{R}, \chi_A(t) = \begin{cases} 1, if\ t \in A \\ 0, else \end{cases}.$$

DEFINITION 5. *Consider the function* $x : \mathbf{R} \to \mathbf{B}$. *Its* ***support set*** *is*

$$supp\ x = \{t | t \in \mathbf{R}, x(t) = 1\}.$$

REMARK 2. *The extreme situations represented by* $\chi_\emptyset(t) = 0$, $\chi_\mathbf{R}(t) = 1$ *are constant functions and* $supp\ 0 = \emptyset$, $supp\ 1 = \mathbf{R}$. *On the other hand, any function* $x : \mathbf{R} \to \mathbf{B}$ *may be written under the form*

$$\forall t \in \mathbf{R}, x(t) = \chi_{supp\ x}(t)$$

and the laws of $\mathbf{B}$ *induce laws denoted by the same symbols in the set of the* $\mathbf{R} \to \mathbf{B}$ *functions:* $\overline{x}(t) = \overline{x(t)}$, $(x \cup y)(t) = x(t) \cup y(t)$ *etc. There is a bijective function from the set of the* $\mathbf{R} \to \mathbf{B}$ *functions to the set of the subsets of* $\mathbf{R}$, *associating with* x *the set* $supp\ x$*; via this bijection,* $\overline{x}$ *corresponds to* $\mathbf{R} \setminus supp\ x$, $x \cup y$ *corresponds to* $supp\ x \cup\ supp\ y$, $x \cdot y$ *corresponds to* $supp\ x \cap\ supp\ y$ *and* $x \oplus y$ *corresponds to* $supp\ x \Delta\ supp\ y$.

We use the same notation $0, 1$ *for the binary numbers* $0, 1 \in \mathbf{B}$ *and for the constant functions* $0, 1 : \mathbf{R} \to \mathbf{B}$. *Similarly,* $\overline{}, \cup, \cdot, \oplus$ *are used for two or three different laws each of them. These abusive notations will not create confusion.*

NOTATION 4. *Let* $d \in \mathbf{R}$ *be a real number. By* $\tau^d : \mathbf{R} \to \mathbf{R}$ *we denote the translation* $\forall t \in \mathbf{R}, \tau^d(t) = t - d$.

DEFINITION 6. *The translation of* $x : \mathbf{R} \to \mathbf{B}$ *with* $d \in \mathbf{R}$ *is the composed function* $x \circ \tau^d : \mathbf{R} \to \mathbf{B}, \forall t \in \mathbf{R}, (x \circ \tau^d)(t) = x(t - d)$.

3. Monotonous functions

DEFINITION 7. *The function $x : \mathbf{R} \to \mathbf{B}$ is called **(monotonous) increasing** if*

$$\forall t \in \mathbf{R}, \forall t' \in \mathbf{R}, t \leq t' \Longrightarrow x(t) \leq x(t')$$

*and **(monotonous) decreasing** if*

$$\forall t \in \mathbf{R}, \forall t' \in \mathbf{R}, t \leq t' \Longrightarrow x(t) \geq x(t').$$

*The property of x of being either monotonous increasing, or monotonous decreasing is expressed shortly by saying that x is **monotonous**.*

EXAMPLE 1. *The constant functions are monotonous increasing and decreasing at the same time. They are the only $\mathbf{R} \to \mathbf{B}$ functions with this property.*

EXAMPLE 2. *The non-constant increasing functions are of the form $\chi_{[d,\infty)}$, $\chi_{(d,\infty)}$ and the non-constant decreasing functions are of the form $\chi_{(-\infty,d)}$, $\chi_{(-\infty,d]}$, where $d \in \mathbf{R}$.*

REMARK 3. *The laws $\cup, \cdot$ conserve the type of monotony. For example, if x, y are monotonous increasing, then $x \cup y$ and $x \cdot y$ are monotonous increasing. The function $\overline{}$ changes the type of monotony of the non-constant functions (for example $\overline{\chi_{[d,\infty)}} = \chi_{(-\infty,d)}$).*

If x is monotonous and $d' \in \mathbf{R}$ is an arbitrary number, then $x \circ \tau^{d'}$ is monotonous and of the same type as x (example: $\chi_{[d,\infty)} \circ \tau^{d'} = \chi_{[d+d',\infty)}$).

4. Consistent sequences of real numbers. Differentiability

DEFINITION 8. *We use the notation*

$$\widetilde{Seq} = \{\{t_z | t_z \in \mathbf{R}, z \in \mathbf{Z}\}|$$

$$... < t_{-1} < t_0 < t_1 < ... \text{ is unbounded from below and from above}\}.$$

The elements of $\widetilde{Seq}$ are denoted by $t_z, z \in \mathbf{Z}$, $(t_z)_{z \in \mathbf{Z}}$ or simply by (t_z) and in the last case the fact that z runs over $\mathbf{Z}$ is understood.

THEOREM 1. *For any numbers $t' < t''$ and any sequence $(t_z) \in \widetilde{Seq}$, the set $\{z | z \in \mathbf{Z}, t_z \in [t', t'']\}$ is finite.*

PROOF. The set may have 0 elements, 1 element or more than 1 element. In the last case, the indices $z' < z''$ exist such that $t_{z'-1} < t' \leq t_{z'} < t_{z'+1} < ... < t_{z''-1} < t_{z''} \leq t'' < t_{z''+1}$. Thus $\{z | z \in \mathbf{Z}, t_z \in [t', t'']\}$ is a finite set with $z'' - z' + 1$ elements. □

DEFINITION 9. *The sequence $(t_z) \in \widetilde{Seq}$ is called **consistent** with the function $x : \mathbf{R} \to \mathbf{B}$ if*

$$\forall z \in \mathbf{Z}, \forall \xi \in (t_z, t_{z+1}), x(\xi) = x(\frac{t_z + t_{z+1}}{2}).$$

DEFINITION 10. *The function $x : \mathbf{R} \to \mathbf{B}$ is called **differentiable** if there is a sequence consistent with it, i.e. the sequence $(t_z) \in \widetilde{Seq}$ exists such that*

$$x(t) = ... \oplus x(t_{-1}) \cdot \chi_{\{t_{-1}\}}(t) \oplus x(\frac{t_{-1} + t_0}{2}) \cdot \chi_{(t_{-1},t_0)}(t) \oplus \oplus x(t_0) \cdot \chi_{\{t_0\}}(t) \oplus x(\frac{t_0 + t_1}{2}) \cdot \chi_{(t_0,t_1)}(t) \oplus ... \tag{4.1}$$

NOTATION 5. *The set of the differentiable functions is denoted by $Diff$.*

EXAMPLE 3. *The constant functions are differentiable and any* $(t_z) \in \widetilde{Seq}$ *is consistent with them.*

EXAMPLE 4. *The monotonous function* $\chi_{[0,\infty)}$ *is differentiable and the sequences* $(t_z) \in \widetilde{Seq}$ *consistent with it are those for which there is* z *such that* $t_z = 0$.

EXAMPLE 5. $\chi_{[0,1)\cup[2,3)\cup...}$ *is a differentiable function and the sequences* $(t_z) \in \widetilde{Seq}$ *consistent with it are those that include* $\mathbf{N}$ *as a subsequence.*

EXAMPLE 6. *The function* $\chi_{\{-\frac{1}{k}|k\geq 1\}}$ *is not differentiable because for any* $\varepsilon > 0$ *the statement*

$$\forall \xi \in (-\varepsilon, 0), \chi_{\{-\frac{1}{k}|k\geq 1\}}(\xi) = \chi_{\{-\frac{1}{k}|k\geq 1\}}(-\frac{\varepsilon}{2})$$

is false.

THEOREM 2. *If* $x \in Diff$ *and* (t_z) *is consistent with* x, *then any* $(t'_z) \in \widetilde{Seq}$ *containing* (t_z) *as a subsequence is consistent with* x.

PROOF. Let $x \in Diff$, $(t_z) \in \widetilde{Seq}$ consistent with x and $(t'_z) \in \widetilde{Seq}$ with $(t_z) \subset (t'_z)$ be arbitrary. Let $z \in \mathbf{Z}$ be arbitrary too. Then $z_1 \in \mathbf{Z}$ and $k \geq 1$ exist such that $t_z = t'_{z_1}$, $t_{z+1} = t'_{z_1+k}$. In addition, we can use the fact that

$$\forall i \in \{0, ..., k-1\}, \forall \xi \in (t'_{z_1+i}, t'_{z_1+i+1}), x(\xi) = x(\frac{t'_{z_1+i} + t'_{z_1+i+1}}{2}) = x(\frac{t_z + t_{z+1}}{2})$$

because the intervals $(t'_{z_1}, t'_{z_1+1}), ..., (t'_{z_1+k-1}, t'_{z_1+k})$ and the points $\frac{t'_{z_1}+t'_{z_1+1}}{2}, ...,$ $\frac{t'_{z_1+k-1}+t'_{z_1+k}}{2}$ are contained in the interval (t_z, t_{z+1}). □

THEOREM 3. *Let* $x, y \in Diff$ *and the sequences* $(t_z)_{z\in\mathbf{Z}}$ *consistent with* x, *respectively* $(t'_z)_{z\in\mathbf{Z}}$ *consistent with* y. *Then the sequence* $(t''_z)_{z\in\mathbf{Z}}$ *obtained as the union of the sets* $(t_z)_{z\in\mathbf{Z}}$, $(t'_z)_{z\in\mathbf{Z}}$ *possibly followed by a reindexing, is consistent with both* x *and* y.

PROOF. This is a consequence of Theorem 2. □

THEOREM 4. *If* x, y *are differentiable, then* $\overline{x}, x \cup y, x \cdot y, x \oplus y$ *are differentiable too.*

PROOF. We prove the statement relative to the intersection. Let be $x, y \in Diff$ and (t_z) a sequence consistent with x and y (obtained, for example, by the union of two sequences, the first consistent with x and the second consistent with y, followed by a possible reindexing). We have

$$(x \cdot y)(t) = x(t) \cdot y(t) =$$

$$\begin{aligned} = \quad & ... \oplus x(t_{-1}) \cdot y(t_{-1}) \cdot \chi_{\{t_{-1}\}}(t) \oplus x(\frac{t_{-1}+t_0}{2}) \cdot y(\frac{t_{-1}+t_0}{2}) \cdot \chi_{(t_{-1},t_0)}(t) \oplus \\ & \oplus x(t_0) \cdot y(t_0) \cdot \chi_{\{t_0\}}(t) \oplus x(\frac{t_0+t_1}{2}) \cdot y(\frac{t_0+t_1}{2}) \cdot \chi_{(t_0,t_1)}(t) \oplus ... \end{aligned}$$

meaning that (t_k) is consistent with $x \cdot y$; $x \cdot y$ is differentiable. □

THEOREM 5. *If* x *is differentiable and* $d \in \mathbf{R}$, *then* $x \circ \tau^d$ *is differentiable too.*

PROOF. Let be $x \in Diff$, $d \in \mathbf{R}$ and $(t_z) \in \widetilde{Seq}$ consistent with x. We have

$$(x \circ \tau^d)(t) = x(t-d) =$$

$$= \quad ... \oplus x(t_{-1}) \cdot \chi_{\{t_{-1}\}}(t-d) \oplus x(\frac{t_{-1}+t_0}{2}) \cdot \chi_{(t_{-1},t_0)}(t-d) \oplus$$

$$\oplus x(t_0) \cdot \chi_{\{t_0\}}(t-d) \oplus x(\frac{t_0+t_1}{2}) \cdot \chi_{(t_0,t_1)}(t-d) \oplus ... =$$

$$= ... \oplus x(t_{-1}+d-d) \cdot \chi_{\{t_{-1}+d\}}(t) \oplus$$

$$\oplus x(\frac{t_{-1}+d+t_0+d}{2} - d) \cdot \chi_{(t_{-1}+d,t_0+d)}(t) \oplus$$

$$\oplus x(t_0+d-d) \cdot \chi_{\{t_0+d\}}(t) \oplus x(\frac{t_0+d+t_1+d}{2} - d) \cdot \chi_{(t_0+d,t_1+d)}(t) \oplus ... =$$

$$= \quad ... \oplus (x \circ \tau^d)(t'_{-1}) \cdot \chi_{\{t'_{-1}\}}(t) \oplus (x \circ \tau^d)(\frac{t'_{-1}+t'_0}{2}) \cdot \chi_{(t'_{-1},t'_0)}(t) \oplus$$

$$\oplus (x \circ \tau^d)(t'_0) \cdot \chi_{\{t'_0\}}(t) \oplus (x \circ \tau^d)(\frac{t'_0+t'_1}{2}) \cdot \chi_{(t'_0,t'_1)}(t) \oplus ...$$

where the sequence with the general term $t'_z = t_z + d, z \in \mathbf{Z}$ is strictly increasing, unbounded from below and from above thus it belongs to $\widetilde{Seq}$. We have proved that it is also consistent with $x \circ \tau^d$, thus $x \circ \tau^d$ is differentiable. □

5. Left limit and right limit

DEFINITION 11. *Let be the function $x \in Diff$ and the sequence (t_z) consistent with it. Thus equation (4.1) holds. The functions*

$$x(t-0) = ... \oplus x(\frac{t_{-1}+t_0}{2}) \cdot \chi_{(t_{-1},t_0]}(t) \oplus x(\frac{t_0+t_1}{2}) \cdot \chi_{(t_0,t_1]}(t) \oplus ... \tag{5.1}$$

$$x(t+0) = ... \oplus x(\frac{t_{-1}+t_0}{2}) \cdot \chi_{[t_{-1},t_0)}(t) \oplus x(\frac{t_0+t_1}{2}) \cdot \chi_{[t_0,t_1)}(t) \oplus ... \tag{5.2}$$

are called ***the left limit*** *and* ***the right limit functions*** *of x.*

THEOREM 6. *If x is differentiable, then its left and right limit functions are differentiable. Moreover any sequence (t_z) consistent with x is consistent with its limits too.*

PROOF. These statements follow by comparing (4.1) with (5.1) and (5.2). □

REMARK 4. *From the Definition of $x(t-0)$ and $x(t+0)$ we get:*

$$x((t-0)-0) = x(t-0),$$

$$x((t-0)+0) = x(t+0),$$

$$x((t+0)-0) = x(t-0),$$

$$x((t+0)+0) = x(t+0).$$

The consequence of this remark is the following. Because with $x(t-0)$ and $x(t+0)$ we shall define the semi-derivatives and the derivatives of x, the higher order semi-derivatives are equal to the first order semi-derivatives and the higher order derivatives are equal to the first order derivatives.

THEOREM 7. *If $x \in Diff$, then $x(t-0), x(t+0)$ satisfy*

$$\forall t \in \mathbf{R}, \exists \varepsilon > 0, \forall \xi \in (t-\varepsilon, t), x(\xi) = x(t-0), \tag{5.3}$$

$$\forall t \in \mathbf{R}, \exists \varepsilon > 0, \forall \xi \in (t, t+\varepsilon), x(\xi) = x(t+0). \tag{5.4}$$

PROOF. Let (t_z) be a sequence consistent with x and consider an arbitrary $t \in \mathbf{R}$. Then a rank $z' \in \mathbf{Z}$ of (t_z) exists such that $t \in (t_{z'}, t_{z'+1}]$. Any $\varepsilon \in (0, t - t_{z'})$ satisfies (5.3), where $x(t-0) = x(\frac{t_{z'}+t_{z'+1}}{2})$. If $t < t_{z'+1}$, then any $\varepsilon \in (0, t_{z'+1} - t)$ satisfies (5.4), where $x(t+0) = x(\frac{t_{z'}+t_{z'+1}}{2})$ while if $t = t_{z'+1}$, then any $\varepsilon \in (0, t_{z'+2} - t_{z'+1})$ satisfies (5.4), with $x(t+0) = x(\frac{t_{z'+1}+t_{z'+2}}{2})$. □

THEOREM 8. *Let be $x : \mathbf{R} \to \mathbf{B}$. If two $\mathbf{R} \to \mathbf{B}$ functions denoted by $y(t), y'(t)$ exist such that*

$$\forall t \in \mathbf{R}, \exists \varepsilon > 0, \forall \xi \in (t-\varepsilon, t), x(\xi) = y(t), \tag{5.5}$$

$$\forall t \in \mathbf{R}, \exists \varepsilon > 0, \forall \xi \in (t, t+\varepsilon), x(\xi) = y'(t) \tag{5.6}$$

hold, then $x \in Diff$. Moreover, $y(t), y'(t)$ like previously are unique and they coincide with $x(t-0), x(t+0)$.

PROOF. We choose an arbitrary $t_0 \in \mathbf{R}$.

Case 1. If $\forall \xi < t_0, x(\xi) = y(t_0)$, then we can choose arbitrarily the unbounded from below sequence $... < t_{-2} < t_{-1} < t_0$.

Case 2. $\exists t_{-1} < t_0, \forall \xi \in (t_{-1}, t_0), x(\xi) = y(t_0)$ and

$$x(t_{-1}) \neq y(t_0) \text{ or } \exists t' < t_{-1}, \forall \xi \in (t', t_{-1}), x(\xi) = y(t_{-1}) \neq y(t_0)$$

with the following sub-cases.

Case 2.1. If $\forall \xi < t_{-1}, x(\xi) = y(t_{-1})$, then we can choose the unbounded from below sequence $... < t_{-3} < t_{-2} < t_{-1}$ in an arbitrary manner.

Case 2.2. $\exists t_{-2} < t_{-1}, \forall \xi \in (t_{-2}, t_{-1}), x(\xi) = y(t_{-1})$ and

$$x(t_{-2}) \neq y(t_{-1}) \text{ or } \exists t' < t_{-2}, \forall \xi \in (t', t_{-2}), x(\xi) = y(t_{-2}) \neq y(t_{-1})$$

...

In all these steps, the existence of the decreasing sequence $... < t_{-2} < t_{-1} < t_0$ like above is assured by the property (5.5) and the question is whether this sequence might be bounded from below and then necessarily convergent towards some t':

$$\exists t' \in \mathbf{R}, \forall \varepsilon > 0, \exists z \in \mathbf{N}, 0 < t_{-z} - t' < \varepsilon.$$

This would contradict the property (5.6) stated at the point t'

$$\exists \varepsilon' > 0, \forall \xi \in (t', t'+\varepsilon'), x(\xi) = y'(t')$$

because in the interval $(t', t'+\varepsilon')$ we would have infinitely many terms of the sequence $(t_{-z})_{z\in\mathbf{N}}$ and also both values $0, 1$ taken by x. The sequence $... < t_{-2} < t_{-1} < t_0$ is unbounded from below and

$$\forall z \in \mathbf{N}, \forall \xi \in (t_{-z-1}, t_{-z}), x(\xi) = y(t_{-z}) = x(\frac{t_{-z-1}+t_{-z}}{2}).$$

In a dual manner, the fact that some unbounded increasing sequence $t_0 < t_1 < t_2 < ...$ exists with

$$\forall z \in \mathbf{N}, \forall \xi \in (t_z, t_{z+1}), x(\xi) = y'(t_z) = x(\frac{t_z+t_{z+1}}{2})$$

is shown. The sequence $(t_z)_{z\in\mathbf{Z}}$ is consistent with x, so that x is differentiable.

The last statements of the theorem are obvious. □

6. Pulses

DEFINITION 12. *We say that $x \in Diff$ has a 0-**pulse of length** $\delta > 0$ at the point t' if*

$$\forall \xi \in (t', t'+\delta), x(\xi) = 0,$$
$$x(t'-0) = x(t'+\delta+0) = 1.$$

*We say that $x \in Diff$ has a 1-**pulse of length** $\delta > 0$ at the point t' if*

$$\forall \xi \in (t', t'+\delta), x(\xi) = 1,$$
$$x(t'-0) = x(t'+\delta+0) = 0.$$

REMARK 5. *The values $x(t'), x(t'+\delta)$ do not occur in Definition 12 and they will be specified later under certain conditions of continuity.*

7. Continuity

THEOREM 9. *Let be the differentiable function $x : \mathbf{R} \to \mathbf{B}$. The following statements are equivalent:*

a) $x(t) = x(t-0)$;

b) if the sequence $(t_z) \in \widetilde{Seq}$ is consistent with x, then we have

$$x(t) = ... \oplus x(t_0) \cdot \chi_{(t_{-1},t_0]}(t) \oplus x(t_1) \cdot \chi_{(t_0,t_1]}(t) \oplus ... \tag{7.1}$$

The following statements are also equivalent:

a') $x(t) = x(t+0)$;

b') for any sequence $(t_z) \in \widetilde{Seq}$ consistent with x, the equation

$$x(t) = ... \oplus x(t_0) \cdot \chi_{[t_0,t_1)}(t) \oplus x(t_1) \cdot \chi_{[t_1,t_2)}(t) \oplus ... \tag{7.2}$$

holds.

PROOF. *a)* $\Longrightarrow$ *b)* By comparing (4.1) with (5.1) we get

$$..., x(t_0) = x(\frac{t_{-1}+t_0}{2}), x(t_1) = x(\frac{t_0+t_1}{2}), ... \tag{7.3}$$

and after introducing these equalities in (4.1) we get (7.1).

b) $\Longrightarrow$ *a)* If x is differentiable and (7.1) is true for a sequence (t_z) that is consistent with it, then (7.3) is true because the points $..., \frac{t_{-1}+t_0}{2}, \frac{t_0+t_1}{2}, ...$ are contained in the intervals $..., (t_{-1}, t_0), (t_0, t_1), ...$ We have

$$x(t) \overset{(7.1)}{=} ... \oplus x(t_0) \cdot \chi_{(t_{-1},t_0]}(t) \oplus x(t_1) \cdot \chi_{(t_0,t_1]}(t) \oplus ...$$
$$\overset{(7.3)}{=} ... \oplus x(\frac{t_{-1}+t_0}{2}) \cdot \chi_{(t_{-1},t_0]}(t) \oplus x(\frac{t_0+t_1}{2}) \cdot \chi_{(t_0,t_1]}(t) \oplus ... \overset{(5.1)}{=} x(t-0)$$

Second statement can be proved similarly. □

DEFINITION 13. *If $x \in Diff$ satisfies one of the conditions a), b) from Theorem 9, then it is called **differentiable left continuous**. If it satisfies one of the conditions a'), b') from Theorem 9, then is called **differentiable right continuous**.*

NOTATION 6. *The set of the differentiable left continuous functions is denoted by $\widetilde{S}^*$ and the set of the differentiable right continuous functions is denoted by $\widetilde{S}$.*

THEOREM 10. *If $x, y \in \widetilde{S}^*$, then $\overline{x}, x \cup y, x \cdot y, x \oplus y \in \widetilde{S}^*$ and if $x, y \in \widetilde{S}$, then $\overline{x}, x \cup y, x \cdot y, x \oplus y \in \widetilde{S}$.*

PROOF. Suppose, for example, that $x, y \in \widetilde{S}$ and that $(t_z) \in \widetilde{Seq}$ is a sequence consistent with x and y. We apply (7.2) and we have

$$\begin{aligned}
(x \cup y)(t) &= x(t) \cup y(t) \\
&= (... \oplus x(t_0) \cdot \chi_{[t_0,t_1)}(t) \oplus x(t_1) \cdot \chi_{[t_1,t_2)}(t) \oplus ...) \cup \\
&\quad \cup(... \oplus y(t_0) \cdot \chi_{[t_0,t_1)}(t) \oplus y(t_1) \cdot \chi_{[t_1,t_2)}(t) \oplus ...) \\
&= ... \oplus (x(t_0) \cup y(t_0)) \cdot \chi_{[t_0,t_1)}(t) \oplus (x(t_1) \cup y(t_1)) \cdot \chi_{[t_1,t_2)}(t) \oplus ... \\
&= (x \cup y)(t + 0).
\end{aligned}$$

This shows that $x \cup y \in \widetilde{S}$. $\square$

THEOREM 11. *Let be $d \in \mathbf{R}$. If $x \in \widetilde{S}^*$, then $x \circ \tau^d \in \widetilde{S}^*$ and for any $x \in \widetilde{S}$ we have $x \circ \tau^d \in \widetilde{S}$.*

PROOF. Taking into account (7.1), the relation $x \in \widetilde{S}^*$ implies for some sequence (t_z) consistent with x that

$$(x \circ \tau^d)(t) = ... \oplus (x \circ \tau^d)(t'_0) \cdot \chi_{(t'_{-1},t'_0]}(t) \oplus (x \circ \tau^d)(t'_1) \cdot \chi_{(t'_0,t'_1]}(t) \oplus ... = (x \circ \tau^d)(t - 0)$$

takes place, where the sequence (t'_z) defined by $t'_z = t_z + d, z \in \mathbf{Z}$ belongs to $\widetilde{Seq}$. Thus $x \circ \tau^d \in \widetilde{S}^*$. Similarly for the second case. $\square$

8. Initial value and final value. Signals and co-signals

DEFINITION 14. *Let be the function $x : \mathbf{R} \to \mathbf{B}$. Its **initial value** $\lim\limits_{t \to -\infty} x(t) \in \mathbf{B}$ and **final value** $\lim\limits_{t \to \infty} x(t) \in \mathbf{B}$ are defined by*

$$\exists t_0 \in \mathbf{R}, \forall \xi < t_0, x(\xi) = \lim_{t \to -\infty} x(t), \tag{8.1}$$

$$\exists t_f \in \mathbf{R}, \forall \xi \geq t_f, x(\xi) = \lim_{t \to \infty} x(t), \tag{8.2}$$

or, equivalently, by

$$\exists t_0 \in \mathbf{R}, x_{|(-\infty,t_0)} = \lim_{t \to -\infty} x(t),$$

$$\exists t_f \in \mathbf{R}, x_{|[t_f,\infty)} = \lim_{t \to \infty} x(t).$$

We have denoted by $x_{|(-\infty,t_0)}, x_{|[t_f,\infty)}$ the restrictions of x to the intervals $(-\infty, t_0)$, $[t_f, \infty)$. The last two equations show that the value of the function is constant. Other notations for $\lim\limits_{t \to -\infty} x(t), \lim\limits_{t \to \infty} x(t)$ are $x(-\infty + 0)$ and $x(\infty - 0)$.

*If for x the relation (8.1) holds ((8.2) holds), then we say that $\lim\limits_{t \to -\infty} x(t)$ **exists**, or that **the initial value of** x **exists**, or that x **has an initial value** ($\lim\limits_{t \to \infty} x(t)$ **exists**, or that **the final value of** x **exists**, or that x **has a final value**).*

REMARK 6. *If any of $\lim\limits_{t \to -\infty} x(t), \lim\limits_{t \to \infty} x(t)$ exists, then it is unique which follows from fact that x is a function.*

THEOREM 12. *Presume that the functions* $x, y : \mathbf{R} \to \mathbf{B}$ *have initial values. Then* $\overline{x}, x\cup y, x\cdot y, x\oplus y$ *have initial values equal to* $\overline{\lim_{t\to-\infty} x(t)}$, $\lim_{t\to-\infty} x(t) \cup \lim_{t\to-\infty} y(t)$, $\lim_{t\to-\infty} x(t) \cdot \lim_{t\to-\infty} y(t)$, $\lim_{t\to-\infty} x(t) \oplus \lim_{t\to-\infty} y(t)$. *Similar statements hold for the final values of* x *and* y.

PROOF. Obvious. □

THEOREM 13. *Suppose that the function* x *has an initial value (a final value) and let* $d \in \mathbf{R}$ *be arbitrary. Then* $x \circ \tau^d$ *has the same initial value (the same final value) like* x.

PROOF. Relation (8.1) implies

$$\forall \xi < t_0 + d, (x \circ \tau^d)(\xi) = x(-\infty + 0).$$

Thus the initial value $(x \circ \tau^d)(-\infty + 0)$ exists and it is equal to $x(-\infty + 0)$. □

THEOREM 14. *Let be the differentiable function* $x \in Diff$. *The following statements are equivalent:*

a) the initial value of x *exists;*

b) a sequence (t_z) *consistent with* x *exists such that*

$$x(t) = x(t_0 - 0) \cdot \chi_{(-\infty,t_0)}(t) \oplus x(t_0) \cdot \chi_{\{t_0\}}(t) \oplus x(\frac{t_0 + t_1}{2}) \cdot \chi_{(t_0,t_1)}(t) \oplus$$
$$\oplus x(t_1) \cdot \chi_{\{t_1\}}(t) \oplus x(\frac{t_1 + t_2}{2}) \cdot \chi_{(t_1,t_2)}(t) \oplus ... \tag{8.3}$$

The following statements:

a') x has a final value;

b') a sequence (t_z) exists that is consistent with x and

$$x(t) = ... \oplus x(\frac{t_{-2} + t_{-1}}{2}) \cdot \chi_{(t_{-2},t_{-1})}(t) \oplus x(t_{-1}) \cdot \chi_{\{t_{-1}\}}(t) \oplus$$
$$\oplus x(\frac{t_{-1} + t_0}{2}) \cdot \chi_{(t_{-1},t_0)}(t) \oplus x(t_0) \cdot \chi_{\{t_0\}}(t) \oplus x(t_0 + 0) \cdot \chi_{(t_0,\infty)}(t) \tag{8.4}$$

are equivalent too.

PROOF. *a)* $\Longrightarrow$ *b)* Let (t_z) be a sequence consistent with x. The existence of the initial value of x is related to the fact that some $z \in \mathbf{Z}$ exists with $\forall \xi < t_z, x(\xi) = x(t_z - 0)$. By a possible reindexing of the terms of the sequence (t_z) we obtain the formula (8.3).

b) $\Longrightarrow$ *a)* We have $\lim_{t\to-\infty} x(t) = x(t_0 - 0)$. □

EXAMPLE 7. *The monotonous functions have initial and final values.*

NOTATION 7. *Denote by*

$$S^* = \{x | x \in \widetilde{S}^*, \exists x(-\infty + 0)\},$$
$$S_c^* = \{x | x \in \widetilde{S}^*, \exists x(\infty - 0)\},$$
$$S = \{x | x \in \widetilde{S}, \exists x(-\infty + 0)\},$$
$$S_c = \{x | x \in \widetilde{S}, \exists x(\infty - 0)\}$$

the spaces of functions representing the set of the differentiable left continuous functions with initial values; the differentiable left continuous functions with final values;

the differentiable right continuous functions with initial values; and the differentiable right continuous functions with final values.

NOTATION 8. *We use the notations*

$$Seq = \{\{t_k | t_k \in \mathbf{R}, k \in \mathbf{N}\} | t_0 < t_1 < ... \text{ is unbounded}\},$$

$$Seq^* = \{\{t_{-k} | t_{-k} \in \mathbf{R}, k \in \mathbf{N}\} | ... < t_{-1} < t_0 \text{ is unbounded}\}.$$

REMARK 7. *If the functions $x \in Diff$ are concerned, then the sequences $(t_z)_{z \in \mathbf{Z}}$ consistent with x are those from $\widetilde{Seq}$ that make (4.1) true. If the functions $x \in Diff$ having an initial value are concerned, then the sequences consistent with x are those from Seq (with the possibility of being extended arbitrarily to sequences from $\widetilde{Seq}$), making equation (8.3) true. If the functions $x \in S$ are concerned, then the sequences that are consistent with x are those from Seq again (with the possibility of being extended arbitrarily to sequences from $\widetilde{Seq}$) making the equation*

$$x(t) = x(t_0 - 0) \cdot \chi_{(-\infty, t_0)}(t) \oplus x(t_0) \cdot \chi_{[t_0, t_1)}(t) \oplus x(t_1) \cdot \chi_{[t_1, t_2)}(t) \oplus ...$$

true, as followed by comparing (7.2) with (8.3). Similarly for the other cases, of left continuity and of existence of the final value.

DEFINITION 15. *The functions x that belong to either of $\widetilde{S}, S, S_c$ are called the **signals** and the functions x that belong to either of $\widetilde{S}^*, S^*, S_c^*$ are called the **signals*** or the **co-signals**.*

In order to avoid any confusion, we mention each time to what space of functions the signals (or signals) belong.*

THEOREM 15. *Let $X \in \{S^*, S_c^*, S, S_c\}$ and $x, y \in X$. We have $\overline{x}, x \cup y, x \cdot y, x \oplus y \in X$.*

PROOF. This fact follows from Theorem 10 and Theorem 12. □

THEOREM 16. *Let $d \in \mathbf{R}$ and $X \in \{S^*, S_c^*, S, S_c\}$. If $x \in X$, then $x \circ \tau^d \in X$.*

PROOF. The result follows from Theorem 11 and Theorem 13. □

9. Semi-derivatives and derivatives

DEFINITION 16. *Let be the differentiable function $x \in Diff$. The following functions are said: the **left semi-derivatives***

$$\begin{aligned} D_{01}x(t) &= \overline{x(t-0)} \cdot x(t), \\ D_{10}x(t) &= x(t-0) \cdot \overline{x(t)} \end{aligned}$$

*and the **right semi-derivatives***

$$\begin{aligned} D_{01}^*x(t) &= \overline{x(t)} \cdot x(t+0), \\ D_{10}^*x(t) &= x(t) \cdot \overline{x(t+0)} \end{aligned}$$

*of x, as well as the **left derivative***

$$Dx(t) = x(t-0) \oplus x(t)$$

*and the **right derivative***

$$D^*x(t) = x(t+0) \oplus x(t)$$

of x.

Remark 8. *The notation* $D_{01}x$ *puts into evidence the time instants when* x *switches at the left from 0 to 1 while* $D_{10}x$ *puts into evidence the time instants when* x *switches at the left from 1 to 0. The derivative* Dx *satisfies the relations*

$$Dx(t) = D_{01}x(t) \cup D_{10}x(t) = \overline{x(t-0)} \cdot x(t) \cup x(t-0) \cdot \overline{x(t)},$$

$$supp\ Dx = supp\ D_{01}x \cup supp\ D_{10}x$$

i.e. it puts into evidence the time instants when x *has a left discontinuity (*$x(t-0) \neq x(t)$*).*

The function $x \in Diff$ *is left continuous iff* $Dx = 0$.

Dual statements concerning the right semi-derivatives $D^*_{01}x$, $D^*_{10}x$, *the right derivative* D^*x *and the sets* $supp\ D^*_{01}x$, $supp\ D^*_{10}x$, $supp\ D^*x$ *can be made.*

Theorem 17. *For any* $x \in Diff$ *we have* $D_{01}x, D_{10}x, D^*_{01}x, D^*_{10}x, Dx, D^*x \in Diff$.

Proof. The functions $\overline{x(t)}, x(t-0), \overline{x(t-0)}, x(t+0), \overline{x(t+0)}$ are differentiable and the unions $\cup$, the products $\cdot$ and the modulo 2 sums $\oplus$ of differentiable functions are differentiable. □

Theorem 18. *Let be* $d \in \mathbf{R}$ *and* $x \in Diff$. *We have* $D_{01}(x \circ \tau^d) = (D_{01}x) \circ \tau^d$. *Similar properties hold for the other semi-derivatives and derivatives too.*

Proof. Obvious. □

Theorem 19. *Let be* $x \in Diff$.

a) If $(t_z) \in \widetilde{Seq}$ *is a sequence consistent with* x*, then the relations*

a.i) $suppD_{01}x, ..., suppD^*x \subset (t_z)$

a.ii) $suppDx \cup suppD^*x \subset (t_z)$

hold.

b) If the sequence $(t_z) \in \widetilde{Seq}$ *satisfies any of a.i), a.ii) then it is consistent with* x.

Proof. *a*) Let (t_z) be consistent with x and x be expressed under the form (4.1), for which we infer

$$D_{01}x(t) = \overline{x(t-0)} \cdot x(t) =$$

$$= (... \oplus \overline{x(\frac{t_{-1}+t_0}{2})} \cdot \chi_{(t_{-1},t_0]}(t) \oplus \overline{x(\frac{t_0+t_1}{2})} \cdot \chi_{(t_0,t_1]}(t) \oplus ...) \cdot$$

$$\cdot (... \oplus x(\frac{t_{-1}+t_0}{2}) \cdot \chi_{(t_{-1},t_0)}(t) \oplus x(t_0) \cdot \chi_{\{t_0\}}(t) \oplus$$

$$\oplus x(\frac{t_0+t_1}{2}) \cdot \chi_{(t_0,t_1)}(t) \oplus x(t_1) \cdot \chi_{\{t_1\}}(t) \oplus ...) =$$

$$= ... \oplus \overline{x(\frac{t_{-1}+t_0}{2})} \cdot x(t_0) \cdot \chi_{\{t_0\}}(t) \oplus \overline{x(\frac{t_0+t_1}{2})} \cdot x(t_1) \cdot \chi_{\{t_1\}}(t) \oplus ...$$

We have obtained that $suppD_{01}x \subset (t_z)$.

The manner of proving for the other statements from a.i) is obvious now. Moreover:

$$suppDx \cup suppD^*x =$$

$$\overset{Remark 8}{=} (suppD_{01}x \cup suppD_{10}x) \cup (suppD^*_{01}x \cup suppD^*_{10}x) \overset{a.i)}{\subset} (t_z)$$

and a.ii) follows.

b) Suppose that (t_z) is not consistent with x. This means the existence of some $z\in\mathbf{Z}$ and of some $t\in(t_z,t_{z+1})$ such that $x(t)\neq x(t-0)$ or $x(t)\neq x(t+0)$. In both cases, this contradicts one of the inclusions from a.i) and a.ii). □

THEOREM 20. *The function $x\in Diff$ has an initial value iff supp Dx and supp D^*x are both bounded from below and it has a final value iff supp Dx and supp D^*x are both bounded from above.*

PROOF. *If* Let $(t_z)\in\widetilde{Seq}$ be a sequence consistent with x which satisfies equation (4.1). From the boundedness from below of $suppDx$, $suppD^*x$ we infer the existence of the rank $z_0\in\mathbf{Z}$ with the property that $suppDx$, $suppD^*x\subset[t_{z_0},\infty)$, meaning that

$$...,x(\frac{t_{z_0-3}+t_{z_0-2}}{2})=x(t_{z_0-2}),x(\frac{t_{z_0-2}+t_{z_0-1}}{2})=x(t_{z_0-1}),$$

$$...,x(t_{z_0-2})=x(\frac{t_{z_0-2}+t_{z_0-1}}{2}),x(t_{z_0-1})=x(\frac{t_{z_0-1}+t_{z_0}}{2})$$

i.e.

$$...,x(\frac{t_{z_0-3}+t_{z_0-2}}{2})=x(t_{z_0-2})=x(\frac{t_{z_0-2}+t_{z_0-1}}{2})=x(t_{z_0-1})=x(\frac{t_{z_0-1}+t_{z_0}}{2}).$$

In other words $\forall\xi<t_{z_0},x(\xi)=x(t_{z_0}-0)$.

Only if The assumption that any of $suppDx$, $suppD^*x$ is unbounded from below gives the negation of the statement from Theorem 14 b). Thus the equivalent statement a) of this theorem is false. □

10. Lemmas on differentiable functions

THEOREM 21. *Let $x\in Diff$ and the numbers $0\leq m\leq d$. The functions*

$$\begin{aligned} y(t) &= \bigcap_{\xi\in[t-d,t-d+m]} x(\xi),\\ z(t) &= \bigcup_{\xi\in[t-d,t-d+m]} x(\xi)\end{aligned}$$

are differentiable and they satisfy the equalities

$$y(t-0)=x(t-d-0)\cdot\bigcap_{\xi\in[t-d,t-d+m)} x(\xi), \tag{10.1}$$

$$y(t+0)=\bigcap_{\xi\in(t-d,t-d+m]} x(\xi)\cdot x(t-d+m+0), \tag{10.2}$$

$$z(t-0)=x(t-d-0)\cup\bigcup_{\xi\in[t-d,t-d+m)} x(\xi), \tag{10.3}$$

$$z(t+0)=\bigcup_{\xi\in(t-d,t-d+m]} x(\xi)\cup x(t-d+m+0). \tag{10.4}$$

PROOF. If $m = 0$, then $y(t) = z(t) = x(t-d)$ is differentiable and we use Definition 2 ($\bigcap\limits_{\xi\in\emptyset} x(\xi) = 1$, $\bigcup\limits_{\xi\in\emptyset} x(\xi) = 0$).

Suppose now that $m > 0$. Let t be arbitrary and fixed. The left limit of x at $t-d$ shows the existence of $\varepsilon_1 > 0$ with

$$\forall\xi \in (t-d-\varepsilon_1, t-d), x(\xi) = x(t-d-0)$$

and the left limit of x at $t-d+m$ shows the existence of $\varepsilon_2 > 0$ such that

$$\forall\xi \in (t-d+m-\varepsilon_2, t-d+m), x(\xi) = x(t-d+m-0).$$

For any $0 < \varepsilon < \min\{\varepsilon_1, \varepsilon_2, m\}$ we infer

$$y(t-\varepsilon) = \bigcap_{\xi\in[t-d-\varepsilon,t-d+m-\varepsilon]} x(\xi) = \bigcap_{\xi\in[t-d-\varepsilon,t-d)} x(\xi) \cdot \bigcap_{\xi\in[t-d,t-d+m-\varepsilon]} x(\xi) =$$

$$= x(t-d-0)\cdot \bigcap_{\xi\in[t-d,t-d+m-\varepsilon]} x(\xi) = x(t-d-0)\cdot \bigcap_{\xi\in[t-d,t-d+m-\varepsilon]} x(\xi)\cdot x(t-d+m-0) =$$

$$= x(t-d-0) \cdot \bigcap_{\xi\in[t-d,t-d+m-\varepsilon]} x(\xi) \cdot \bigcap_{\xi\in(t-d+m-\varepsilon,t-d+m)} x(\xi) =$$

$$= x(t-d-0) \cdot \bigcap_{\xi\in[t-d,t-d+m)} x(\xi).$$

Because the value of $y(t-\varepsilon)$ does not depend on ε, we get $y(t-\varepsilon) = y(t-0)$ and because t is arbitrary, (10.1) is proved.

The right limit of x at $t-d$ shows the existence of $\varepsilon_3 > 0$ such that

$$\forall\xi \in (t-d, t-d+\varepsilon_3), x(\xi) = x(t-d+0)$$

and, on the other hand, the right limit of x at $t-d+m$ shows the existence of $\varepsilon_4 > 0$ with

$$\forall\xi \in (t-d+m, t-d+m+\varepsilon_4), x(\xi) = x(t-d+m+0).$$

We take some $0 < \varepsilon' < \min\{\varepsilon_3, \varepsilon_4, m\}$ such that

$$y(t+\varepsilon') = \bigcap_{\xi\in[t-d+\varepsilon',t-d+m+\varepsilon']} x(\xi) = \bigcap_{\xi\in[t-d+\varepsilon',t-d+m]} x(\xi)\cdot \bigcap_{\xi\in(t-d+m,t-d+m+\varepsilon']} x(\xi) =$$

$$= \bigcap_{\xi\in[t-d+\varepsilon',t-d+m]} x(\xi) \cdot x(t-d+m+0) =$$

$$= x(t-d+0) \cdot \bigcap_{\xi\in[t-d+\varepsilon',t-d+m]} x(\xi) \cdot x(t-d+m+0) =$$

$$= \bigcap_{\xi\in(t-d,t-d+\varepsilon')} x(\xi) \cdot \bigcap_{\xi\in[t-d+\varepsilon',t-d+m]} x(\xi) \cdot x(t-d+m+0) =$$

$$= \bigcap_{\xi\in(t-d,t-d+m]} x(\xi) \cdot x(t-d+m+0).$$

The fact that $y(t+\varepsilon')$ does not depend on ε' shows that $y(t+\varepsilon') = y(t+0)$ and because t is arbitrary, (10.2) is proved. Therefore, by Theorem 8, y is differentiable.

The proof for z is similar. $\square$

THEOREM 22. *Under the conditions of Theorem 21 and using the previous notations, we have:*

$$\overline{y(t-0)} \cdot y(t) = \overline{x(t-d-0)} \cdot \bigcap_{\xi \in [t-d, t-d+m]} x(\xi), \tag{10.5}$$

$$y(t-0) \cdot \overline{y(t)} = x(t-d-0) \cdot \bigcap_{\xi \in [t-d, t-d+m)} x(\xi) \cdot \overline{x(t-d+m)}, \tag{10.6}$$

$$\overline{z(t-0)} \cdot z(t) = \overline{x(t-d-0)} \cdot \overline{\bigcup_{\xi \in [t-d, t-d+m)} x(\xi)} \cdot x(t-d+m), \tag{10.7}$$

$$z(t-0) \cdot \overline{z(t)} = x(t-d-0) \cdot \overline{\bigcup_{\xi \in [t-d, t-d+m]} x(\xi)}. \tag{10.8}$$

PROOF. The relations (10.5) and (10.7) are proved as follows:

$$\begin{aligned}
\overline{y(t-0)} \cdot y(t) &= \overline{x(t-d-0) \cdot \bigcap_{\xi \in [t-d, t-d+m)} x(\xi)} \cdot \bigcap_{\xi \in [t-d, t-d+m]} x(\xi) = \\
&= (\overline{x(t-d-0)} \cup \overline{\bigcap_{\xi \in [t-d, t-d+m)} x(\xi)}) \cdot \bigcap_{\xi \in [t-d, t-d+m]} x(\xi) = \\
&= \overline{x(t-d-0)} \cdot \bigcap_{\xi \in [t-d, t-d+m]} x(\xi);
\end{aligned}$$

$$\overline{z(t-0)} \cdot z(t) = \overline{x(t-d-0) \cup \bigcup_{\xi \in [t-d, t-d+m)} x(\xi)} \cdot \bigcup_{\xi \in [t-d, t-d+m]} x(\xi) =$$

$$= \overline{x(t-d-0)} \cdot \overline{\bigcup_{\xi \in [t-d, t-d+m)} x(\xi)} \cdot (\bigcup_{\xi \in [t-d, t-d+m)} x(\xi) \cup x(t-d+m)) =$$

$$= \overline{x(t-d-0)} \cdot \overline{\bigcup_{\xi \in [t-d, t-d+m)} x(\xi)} \cdot x(t-d+m).$$

□

THEOREM 23. *Let $X \in \{\widetilde{S}^*, \widetilde{S}, S^*, S_c^*, S, S_c\}$. If $x \in X$, then $y, z \in X$.*

PROOF. We choose $X = S$. If $m = 0$ and $y(t) = z(t) = x(t-d)$, then by Theorem 16, $x \circ \tau^d \in S$. From this moment on we consider that $m > 0$.

From Theorem 21 we know that y is differentiable therefore we must show that it satisfies the equality $y(t) = y(t+0)$ and $\exists \lim\limits_{t \to -\infty} y(t)$. Let t be arbitrary and fixed. The right continuity of x at $t-d$ shows that there is $\varepsilon_1 > 0$ with

$$\forall \xi \in [t-d, t-d+\varepsilon_1), x(\xi) = x(t-d)$$

and the right continuity of x at $t-d+m$ shows the existence of $\varepsilon_2 > 0$ such that

$$\forall \xi \in (t-d+m, t-d+m+\varepsilon_2), x(\xi) = x(t-d+m).$$

Let $0 < \varepsilon < \min\{\varepsilon_1, \varepsilon_2, m\}$. We conclude that

$$y(t+\varepsilon) = \bigcap_{\xi \in [t-d+\varepsilon, t-d+m+\varepsilon]} x(\xi) = \bigcap_{\xi \in [t-d+\varepsilon, t-d+m]} x(\xi) \cdot \bigcap_{\xi \in (t-d+m, t-d+m+\varepsilon]} x(\xi) =$$

$$= \bigcap_{\xi\in[t-d+\varepsilon,t-d+m]} x(\xi)\cdot x(t-d+m) = \bigcap_{\xi\in[t-d+\varepsilon,t-d+m]} x(\xi) =$$

$$= x(t-d)\cdot \bigcap_{\xi\in[t-d+\varepsilon,t-d+m]} x(\xi) =$$

$$= \bigcap_{\xi\in[t-d,t-d+\varepsilon)} x(\xi)\cdot \bigcap_{\xi\in[t-d+\varepsilon,t-d+m]} x(\xi) = \bigcap_{\xi\in[t-d,t-d+m]} x(\xi) = y(t).$$

Thus $y(t+\varepsilon) = y(t+0) = y(t)$. The function y is right continuous since t is arbitrary.

Moreover, the property of existence of the initial value is fulfilled since from

$$\forall \xi < t_0, x(\xi) = x(-\infty+0),$$

we infer

$$\forall \xi < t_0 + d - m, \bigcap_{\omega\in[\xi-d,\xi-d+m]} x(\omega) = x(-\infty+0) = y(-\infty+0).$$

We have proved that $y \in S$.

The proof for z is dual. □

THEOREM 24. *If $x \in Diff$ and $d > 0$, the functions*

$$y'(t) = \bigcap_{\xi\in[t-d,t)} x(\xi),$$
$$z'(t) = \bigcup_{\xi\in[t-d,t)} x(\xi)$$

are differentiable and satisfy the relationships

$$y'(t-0) = x(t-d-0)\cdot \bigcap_{\xi\in[t-d,t)} x(\xi),$$

$$y'(t+0) = \bigcap_{\xi\in(t-d,t]} x(\xi)\cdot x(t+0),$$

$$z'(t-0) = x(t-d-0)\cup \bigcup_{\xi\in[t-d,t)} x(\xi),$$

$$z'(t+0) = \bigcup_{\xi\in(t-d,t]} x(\xi)\cup x(t+0).$$

PROOF. Is similar to the proof of Theorem 21. □

REMARK 9. *Unlike Theorem 23, where we have proved that the right continuity of x implies the right continuity of y, in Theorem 24 the right continuity of x does not imply the right continuity of y' because we have*

$$y'(t+0) = \bigcap_{\xi\in(t-d,t]} x(\xi)\cdot x(t+0) = \bigcap_{\xi\in(t-d,t]} x(\xi) \neq y'(t)$$

and similarly for the other three situations. We conclude that a certain care must be taken when using the functions $y'(t), z'(t)$.

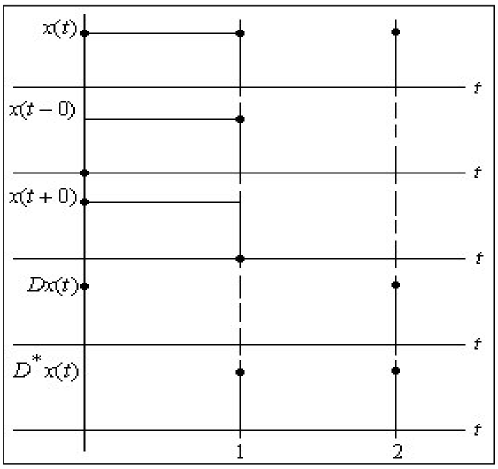

FIGURE 2. Conventions concerning the drawing of the graphs

11. Conventions about the graphs of the $\mathbf{R} \to \mathbf{B}$ functions

REMARK 10. *In order to make easier the understanding of the (in general, differentiable)* $\mathbf{R} \to \mathbf{B}$ *functions, we make the following conventions concerning the drawing of their graph:*

a) the two values 0,1 are not written on the vertical axis. They are supposed to be known, the only necessary convention is that the low value has to be associated with 0 and the high value to be associated with 1;

b) the 0 *value on the horizontal axis is not written. The convention is that* 0 *represents the intersection of the vertical and the horizontal axis;*

c) we draw vertical lines through these points where the function switches (the discontinuity points) even if the vertical lines do not belong to the graph;

d) we put bullets on the vertical lines drawn like at c), in this way underlining the points that actually belong to the graph (the value of the function at the switching point).

EXAMPLE 8. *The function* $x(t) = \chi_{[0,1]}(t) \oplus \chi_{\{2\}}(t)$ *is differentiable, that is neither left, nor right continuous. More precisely we have*

$$\begin{aligned} x(t-0) &= \chi_{(0,1]}(t), \\ x(t+0) &= \chi_{[0,1)}(t), \\ Dx(t) &= \chi_{\{0,2\}}(t), \\ D^*x(t) &= \chi_{\{1,2\}}(t). \end{aligned}$$

In Figure 2 we have drawn the graphs of these functions.

12. $\mathbf{R} \to \mathbf{B}^n$ functions

The way of obtaining from n sequences $(t_z^1), ..., (t_z^n) \in \widetilde{Seq}$ consistent with $x_1, ..., x_n \in Diff$ a sequence (t_z) consistent with all these functions was already mentioned, namely by reindexing the set $(t_z^1) \cup ... \cup (t_z^n)$. Then this sequence is used to express the validity of equation (4.1) for the function $x(t) = (x_1(t), ..., x_n(t))$. Conversely, the differentiable $x : \mathbf{R} \to \mathbf{B}^n$ functions can be defined by the existence of a sequence $(t_z) \in \widetilde{Seq}$ such that formula (4.1) holds. In this case, the coordinate functions $x_1, ..., x_n$ are differentiable themselves and the sequence (t_z) consistent with all of them is called consistent with x. The set of the differentiable $\mathbf{R} \to \mathbf{B}^n$ functions is denoted by $Diff^{(n)}$.

Let $(t_z) \in \widetilde{Seq}$ be a sequence consistent with the functions $x_1, ..., x_n \in Diff$. The equations (5.1), (5.2) that are true for $x_1, ..., x_n$ are true for the function $x(t) = (x_1(t), ..., x_n(t))$ too. Conversely, the left $x(t-0)$ and the right $x(t+0)$ limits of $x \in Diff^{(n)}$ are defined by the validity of the formulae (5.1), (5.2) wherefrom we infer that

$$x(t-0) = (x_1(t-0), ..., x_n(t-0)),$$

$$x(t+0) = (x_1(t+0), ..., x_n(t+0)).$$

The left and the right continuity of $x \in Diff^{(n)}$ consists in the equality $x(t) = x(t-0)$ and $x(t) = x(t+0)$ respectively and this is equivalent to the left and the right continuity of the coordinate functions. The formulae (7.1), (7.2) are true when written for x as well as for $x_1, ..., x_n$, where $(t_z) \in \widetilde{Seq}$ is consistent with all coordinate functions $x_1, ..., x_n$. We denote by $\widetilde{S}^{*(n)}, \widetilde{S}^{(n)} \subset Diff^{(n)}$ the sets of differentiable functions that are left continuous and right continuous respectively.

The definition of the initial value $\lim\limits_{t\to-\infty} x(t) \in \mathbf{B}^n$ and final value $\lim\limits_{t\to\infty} x(t) \in \mathbf{B}^n$ of $x : \mathbf{R} \to \mathbf{B}^n$ is expressed by the formulae (8.1), (8.2) also while the existence of the initial/final value of x consists in the existence of the initial/final values of the coordinate functions. The formulae (8.3), (8.4) are true if $x \in Diff^{(n)}$ has an initial value, respectively a final value (i.e. if $x_1, ..., x_n \in Diff$ have initial and final values respectively). The new notations are:

$$S^{*(n)} = \{x | x \in \widetilde{S}^{*(n)}, \exists x(-\infty + 0)\},$$

$$S_c^{*(n)} = \{x | x \in \widetilde{S}^{*(n)}, \exists x(\infty - 0)\},$$

$$S^{(n)} = \{x | x \in \widetilde{S}^{(n)}, \exists x(-\infty + 0)\},$$

$$S_c^{(n)} = \{x | x \in \widetilde{S}^{(n)}, \exists x(\infty - 0)\}.$$

The functions that belong to $\widetilde{S}^{(n)}, S^{(n)}, S_c^{(n)}$ are called the n-**dimensional signals** and the functions that belong to $\widetilde{S}^{*(n)}, S^{*(n)}, S_c^{*(n)}$ are called the n-**dimensional co-signals**, or **signals***.

Even if this fact is possible, we shall not use the vector notation of the semi-derivatives and derivatives of the differentiable functions $x : \mathbf{R} \to \mathbf{B}^n$. The semi-derivatives and the derivatives of the coordinate functions $x_1, ..., x_n$ will occur occasionally.

Remark that from the definition of $\overline{x}, x \cup y, x \cdot y, x \oplus y$ for the functions $x, y : \mathbf{R} \to \mathbf{B}$, the Boolean functions $F : \mathbf{B}^m \to \mathbf{B}^n$ that define, for each $u : \mathbf{R} \to \mathbf{B}^m$, the function $F(u(\cdot)) : \mathbf{R} \to \mathbf{B}^n$, $\mathbf{R} \ni t \longmapsto F(u(t)) \in \mathbf{B}^n$ bring the space $Diff^{(m)}$ in $Diff^{(n)}$, $\widetilde{S}^{*(m)}$ in $\widetilde{S}^{*(n)}$ etc., initial values in initial values $(F(\lim\limits_{t\to\infty} u(t)) = \lim\limits_{t\to\infty} F(u(t)))$ and final values in final values.

13. Cartesian products of functions and spaces of functions

DEFINITION 17. *The* ***Cartesian product*** *(the* ***sum-dimension vector function****) of the functions* $x : \mathbf{R} \to \mathbf{B}^n$ *and* $x' : \mathbf{R} \to \mathbf{B}^{n'}$ *is the function* $x \times x' : \mathbf{R} \to \mathbf{B}^n \times \mathbf{B}^{n'}$,

$$(x \times x')(t) = (x(t), x'(t)).$$

Sometimes, instead of $x \times x'$, *we use the notation* (x, x'). *It is often convenient to identify* $\mathbf{B}^n \times \mathbf{B}^{n'}$ *with* $\mathbf{B}^{n+n'}$ *and then write*

$$(x \times x')(t) = (x_1(t), ..., x_n(t), x'_1(t), ..., x'_{n'}(t)).$$

DEFINITION 18. *Let* $X \subset (\mathbf{B}^n)^{\mathbf{R}}, X' \subset (\mathbf{B}^{n'})^{\mathbf{R}}$ *be two non-empty sets and we denote* $(\mathbf{B}^n)^{\mathbf{R}} = \{x | x : \mathbf{R} \to \mathbf{B}^n\}$. *Their* ***Cartesian product*** *is defined as*

$$X \times X' = \{x \times x' | x \in X, x' \in X'\}.$$

DEFINITION 19. *For* X, X' *like before, the* ***Cartesian product*** $P(X) \times P(X')$ *of* $P(X)$ *and* $P(X')$ *is the set*

$$P(X) \times P(X') = P(X \times X')$$

and similarly for $P^*(X) \times P^*(X')$.

REMARK 11. *In Definitions 17,18, 19 the argument* t *of the functions of the product is the same (a unique time axis and a unique present time exist). We conclude, for example, that any* $x \in Diff^{(n)}$ *is the Cartesian product of its coordinates:*

$$x(t) = (x_1(t), ..., x_n(t)) = (x_1 \times ... \times x_n)(t)$$

and moreover that

$$Diff^{(n)} \times Diff^{(n')} = Diff^{(n+n')},$$
$$P(Diff^{(n)}) \times P(Diff^{(n')}) = P(Diff^{(n+n')}).$$

Relations of the same kind are obtained if we replace $Diff^{(n)}$ *by its previously defined subspaces:* $\widetilde{S}^{*(n)}$, $S^{*(n)}$, $S_c^{*(n)}$, $\widetilde{S}^{(n)}$, $S^{(n)}$, $S_c^{(n)}$.

If in Definition 17 the functions x, x' *are constant and equal to* $\mu \in \mathbf{B}^n, \mu' \in \mathbf{B}^{n'}$ *then the two equations become*

$$\mu \times \mu' = (\mu, \mu'),$$
$$\mu \times \mu' = (\mu_1, ..., \mu_n, \mu'_1, ..., \mu'_{n'}).$$

CHAPTER 3

Pseudo-systems

The mathematical concepts occurring in the description of the pseudo-systems are defined in terms of the notions introduced in Ch. 2. The appropriate choice of the mathematical concepts which must be associated with the signals, pseudo-systems, initial and final states, initial and final time, initial and final state functions is carefully motivated.

1. Choosing the right continuity of the signals

At this moment, in principle, we have the following possibilities:

a) to work with differentiable functions $x \in Diff^{(n)}$;

b) from now on to choose the work with left-continuous functions $x \in \widetilde{S}^{*(n)}$, or x belongs to some subspace of $\widetilde{S}^{*(n)}$;

c) to make the choice of working with right-continuous functions $x \in \widetilde{S}^{(n)}$, or x belongs to some subspace of $\widetilde{S}^{(n)}$.

The possibility a) looks correct, but, maybe, too general. In the differential equations and inequalities left and right (semi-)derivatives occur such that the study of the solutions is difficult. Our belief is that the 'Dirac impulse' $\chi_{\{d\}}$ considered as the typical example of differentiable function with left and with right discontinuities does not reflect the properties of the electrical devices that are generally inertial. This is why we eliminate it from our study. It remains as a subject of reflection whether our belief is reasonable.

The b) alternative of restricting the functions involved in our theory to the left continuous ones looks good, in the sense that it corresponds to our aims of being as simple as possible. For reasons imposed by our previous works, this is not our option, but the subject of reflection that arised is: could we have studied everything in this paper based on the left-continuous functions? With what consequences? We prefer to associate these functions with the systems that run with the time axis reversed, from the future to the past.

The c) alternative, dual to b) represents our decision for the rest of the book. By using this restriction, we omit using right (semi-)derivatives, that are null for such functions, but also the Dirac impulses. The gain in simplicity seems noteworthy and the loss of generality seems minimal or null.

2. The definition of the pseudo-systems

Remark 12. *The pseudo-systems are multi-valued functions from the set of differentiable right continuous* $\mathbf{R} \to \mathbf{B}^m$ *functions called the inputs to (empty or non-empty) sets of differentiable right continuous* $\mathbf{R} \to \mathbf{B}^n$ *functions called the states. Under a very general form they initiate the problem of modeling the asynchronous circuits from the digital electrical engineering and allow us to present the* *duality between the initial states and the initial time, on one hand and the final states and final time, on the other hand.*

DEFINITION 20. *The functions $f : \widetilde{S}^{(m)} \to P(\widetilde{S}^{(n)}), m, n \geq 1$ are called the* ***asynchronous pseudo-systems in the input-output sense****, or shortly* ***pseudo-systems****. We say that f represents a* ***pseudo-system under the closed-form****. The elements $u \in \widetilde{S}^{(m)}$ are called the* ***inputs*** *(in the pseudo-system):* ***admissible*** *if $f(u) \neq \emptyset$ and* ***non-admissible*** *if $f(u) = \emptyset$, while the elements $x \in f(u)$ are called the (****possible****)* ***states****, or (****possible****)* ***outputs*** *(of the pseudo-system). The sets $\widetilde{S}^{(m)}, \widetilde{S}^{(n)}$ are called the* ***input space*** *and the* ***state*** *(the* ***output****)* ***space*** *and m, n are called the* ***dimensions of the input*** *and* ***of the output space****. The set $\mathbf{R}$ is the* ***time set****.*

DEFINITION 21. *A* ***pseudo-system under the implicit form*** *consists in one or several equations and/or inequalities where u is given, $t \in \mathbf{R}$ is the temporal variable and x is the indeterminate.*

REMARK 13. *The pseudo-systems are multi-valued functions (or relations) that associate with each input u the set of the possible states $f(u)$. The concept originates in the modeling of the asynchronous circuits.*

A non-admissible input, i.e. an input u for which $f(u) = \emptyset$, is thought of to be the cause of no effect that can be expressed by f and an admissible input u, for which $f(u) \neq \emptyset$ is considered the cause of several possible effects $x \in f(u)$. The multi-valued character of the cause-effect association is due to statistical fluctuations in the fabrication process, the variations in the ambiental temperature, the power supply etc.

Let $\lambda, \mu \in \mathbf{B}$. The inequality $\lambda \leq \mu$ is equivalent to the equality $\overline{\lambda} \cup \mu = 1$, while the equality $\lambda = \mu$ is equivalent to the inequalities $\lambda \leq \mu, \mu \leq \lambda$ (by 'equivalent' properties we mean that the sets of couples (λ, μ) satisfying the two properties are equal). This fact shows us that it is the same thing to indicate the pseudo-systems under the implicit form as equations or as inequalities.

NOTATION 9. *If $\forall u \in \widetilde{S}^{(m)}, f(u)$ has exactly one element, then for the pseudo-system f we use the same notation $f : \widetilde{S}^{(m)} \to \widetilde{S}^{(n)}$ as for the uni-valued functions.*

3. Examples

EXAMPLE 9. *The null pseudo-system is defined by $f : \widetilde{S}^{(m)} \to P(\widetilde{S}^{(n)}), \forall u \in \widetilde{S}^{(m)}, f(u) = \emptyset$, i.e. all its inputs are non-admissible. This corresponds to the situation when f models nothing.*

EXAMPLE 10. *The total pseudo-system $f : \widetilde{S}^{(m)} \to P(\widetilde{S}^{(n)})$ is defined by $\forall u \in \widetilde{S}^{(m)}, f(u) = \widetilde{S}^{(n)}$ and has all the inputs admissible. It models all the circuits with m-dimensional inputs and n-dimensional states and gives no information on these circuits.*

EXAMPLE 11. *The identical pseudo-system $I_m : \widetilde{S}^{(m)} \to \widetilde{S}^{(m)}$ is defined by: $\forall u \in \widetilde{S}^{(m)}, I_m(u) = u$. It models m wires without inertia and without delays.*

EXAMPLE 12. *The projection on the $j - th$ coordinate, $j \in \{1, ..., m\}$ is the pseudo-system $\pi_j : \widetilde{S}^{(m)} \to \widetilde{S}, \forall u \in \widetilde{S}^{(m)}, \pi_j(u) = u_j$.*

EXAMPLE 13. *The vector $\mu \in \mathbf{B}^n$ defines the constant function $\mu : \widetilde{S}^{(m)} \to \widetilde{S}^{(n)}$. This is an interesting example of pseudo-system, since it suggests us a circuit* *modeled with 'stuck-at'* μ_i *faults,* $i=\overline{1,n}$. *We have identified the constant* μ *with the constant function* $x(t)=\mu$.

EXAMPLE 14. *More general than previously, a set* $A\subset \mathbf{B}^n$ *defines the constant pseudo-system* $A:\widetilde{S}^{(m)}\rightarrow P(\widetilde{S}^{(n)})$.

EXAMPLE 15. *Let be the function* $F:\mathbf{B}^m\rightarrow \mathbf{B}^n$. *We define the pseudo-system* $f:\widetilde{S}^{(m)}\rightarrow \widetilde{S}^{(n)}$ *by* $\forall u\in \widetilde{S}^{(m)}, f(u)(t)=F(u(t))$. *This example makes use of the fact that for every* $u\in \widetilde{S}^{(m)}$, *when* t *runs over* $\mathbf{R}$, *then* $F(u(t))$ *belongs to* $\widetilde{S}^{(n)}$. *The modeled circuit represents the ideal logical gates and, more generally, the ideal combinational circuits, that work without inertia and without delays.*

EXAMPLE 16. *Let* ρ *be an equivalence relation on* $\widetilde{S}^{(m)}$ *and denote by* $[u]_\rho$ *the equivalence class of* u *relative to* ρ. *We have the pseudo-system* $f:\widetilde{S}^{(m)}\rightarrow P(\widetilde{S}^{(m)})$ *defined by* $\forall u\in \widetilde{S}^{(m)}, f(u)=[u]_\rho$. *Here are some equivalence relations on* $\widetilde{S}^{(m)}$:

- $u\rho v \Longleftrightarrow \exists d\in \mathbf{R}, u=v\circ \tau^d$;
- $u\rho v \Longleftrightarrow \exists t\in \mathbf{R}, u_{|(-\infty,t)}=v_{|(-\infty,t)}$;
- $u\rho v \Longleftrightarrow \exists t\in \mathbf{R}, u_{|[t,\infty)}=v_{|[t,\infty)}$;
- $u\rho v \Longleftrightarrow \exists \alpha>0, \forall t\in \mathbf{R}, u(t)=v(\alpha\cdot t)$ *(two signals are equivalent if they are equal irrespective of the time unit);*
- $u\rho v \Longleftrightarrow \forall j\in\{1,...,m\}, \lim\limits_{t\rightarrow -\infty}\bigcap\limits_{\xi\in(-\infty,t)} u_j(\xi)=\lim\limits_{t\rightarrow -\infty}\bigcap\limits_{\xi\in(-\infty,t)} v_j(\xi)$;
- $u\rho v \Longleftrightarrow \forall j\in\{1,...,m\}, \lim\limits_{t\rightarrow \infty}\bigcup\limits_{\xi\in[t,\infty)} u_j(\xi)=\lim\limits_{t\rightarrow \infty}\bigcup\limits_{\xi\in[t,\infty)} v_j(\xi)$.

In writing the last two definitions we have made use of the fact that for any $u\in \widetilde{S}^{(m)}$ *and any* $j\in\{1,...,m\}$, *the functions of* t : $\bigcap\limits_{\xi\in(-\infty,t)} u_j(\xi)$, $\bigcup\limits_{\xi\in[t,\infty)} u_j(\xi)$ *switch at most once from* 0 *to* 1 *when* t *decreases, and from* 1 *to* 0 *when* t *increases respectively. Thus they are monotonous and the two limits as* $t\rightarrow -\infty$, *and* $t\rightarrow\infty$ *respectively exist.*

EXAMPLE 17. *The pseudo-system* $f:\widetilde{S}\rightarrow P(\widetilde{S})$ *is defined in the implicit form by the double inequality*

$$\bigcap_{\xi\in[t-d,t)} u(\xi)\leq x(t)\leq \bigcup_{\xi\in[t-d,t)} u(\xi), \tag{3.1}$$

where $d>0$. *When* $u,x\in\widetilde{S}$, *the functions* $\bigcap\limits_{\xi\in[t-d,t)} u(\xi)$ *and* $\bigcup\limits_{\xi\in[t-d,t)} u(\xi)$ *are just differentiable, they are not right-continuous (as discussed in Remark 9). This is the model of a delay circuit, where the delay between* u *and* x *is bounded by* d.

4. Initial states and final states

REMARK 14. *We reveal the following properties of the pseudo-system* f:

$$\forall u\in \widetilde{S}^{(m)}, \forall x\in f(u), \exists \mu\in \mathbf{B}^n, \exists t_0\in \mathbf{R}, \forall t<t_0, x(t)=\mu; \tag{4.1}$$

$$\forall u\in \widetilde{S}^{(m)}, \exists \mu\in \mathbf{B}^n, \forall x\in f(u), \exists t_0\in \mathbf{R}, \forall t<t_0, x(t)=\mu; \tag{4.2}$$

$$\exists \mu\in \mathbf{B}^n, \forall u\in \widetilde{S}^{(m)}, \forall x\in f(u), \exists t_0\in \mathbf{R}, \forall t<t_0, x(t)=\mu; \tag{4.3}$$

$$\forall u\in \widetilde{S}^{(m)}, \forall x\in f(u), \exists \mu\in \mathbf{B}^n, \exists t_f\in \mathbf{R}, \forall t\geq t_f, x(t)=\mu; \tag{4.4}$$

$$\forall u\in \widetilde{S}^{(m)}, \exists \mu\in \mathbf{B}^n, \forall x\in f(u), \exists t_f\in \mathbf{R}, \forall t\geq t_f, x(t)=\mu; \tag{4.5}$$

$$\exists \mu \in \mathbf{B}^n, \forall u \in \widetilde{S}^{(m)}, \forall x \in f(u), \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = \mu. \tag{4.6}$$

We can see that if $f(u) \neq \emptyset$, then $\exists \mu$ in (4.1),...,(4.6) should be interpreted as $\exists ! \mu$, the existence of a unique μ with that property.

If (4.1) is true with f non-null, then it defines a partial function $\widetilde{S}^{(n)} \to \mathbf{B}^n$ that associates with each $x \in \bigcup\limits_{u \in \widetilde{S}^{(m)}} f(u)$ its initial value μ. If (4.2) is true with f non-null, then it defines a partial function $\widetilde{S}^{(m)} \to \mathbf{B}^n$ that associates with each admissible input u the common initial value μ of all $x \in f(u)$. Dually, if f is non-null and (4.4), (4.5) are true, two partial functions $\widetilde{S}^{(n)} \to \mathbf{B}^n$ and $\widetilde{S}^{(m)} \to \mathbf{B}^n$ are defined.

The null pseudo-system f fulfills trivially all properties (4.1),...,(4.6) with arbitrary $\mu \in \mathbf{B}^n$ and $t_0, t_f \in \mathbf{R}$.

Remark the dualities between (4.1) and (4.4); (4.2) and (4.5); (4.3) and (4.6) and the truth of the implications

$$(4.3) \Longrightarrow (4.2) \Longrightarrow (4.1),$$

$$(4.6) \Longrightarrow (4.5) \Longrightarrow (4.4).$$

Remark also that in (4.1), ..., (4.3) $\forall t < t_0$ *and* $\forall t \leq t_0$ *are equivalent and in* (4.4), ..., (4.6) $\forall t > t_f$ *and* $\forall t \geq t_f$ *are equivalent. We have adopted* $\forall t < t_0$ *and* $\forall t \geq t_f$ *in order to underline the right continuity of the $n-$signals $x \in \widetilde{S}^{(n)}$.*

DEFINITION 22. *If f satisfies (4.1), we say that it **has initial states**. In this case the vectors μ are called (the) **initial states** (of f), or (the) **initial values of the states** (of f).*

DEFINITION 23. *Suppose that f satisfies (4.2). In this situation we say that it **has race-free initial states**. Moreover, the initial states μ are called **race-free** themselves.*

DEFINITION 24. *When f satisfies (4.3), we use to say that it **has a (constant) initial state** μ. In this case we say that f is **initialized** and that μ is its **(constant) initial state**.*

DEFINITION 25. *If f satisfies (4.4), it is called **absolutely stable** and we also say that it **has final states**. In this case the vectors μ bear the name of **final states** (of f), or **final values of the states** (of f).*

DEFINITION 26. *If f fulfills the property (4.5), it is called **absolutely race-free stable** and we say that it **has race-free final states** μ. In this case the final states μ are called **race-free**.*

DEFINITION 27. *Suppose that the pseudo-system f satisfies (4.6). Then it is called **absolutely constantly stable** or, equivalently, we say that it **has a (constant) final state** μ. In this situation the vector μ is called the (**constant**) **final state**.*

REMARK 15. *The previous terminology is related to the dualities initial-final, initialized-absolutely stable as well as with hardware engineering. In hardware engineering, 'race' means: 'which coordinate of x switches first is the winner' or perhaps 'several ways to go'. In this case 'race-free' means 'one way to go'. The interpretation of the race-freedom is (vaguely): 'for any statistical fluctuations in the fabrication process...', see Remark 13.*

5. Initial time and final time

Remark 16. *We reveal the following properties on the pseudo-system* $f : \widetilde{S}^{(m)} \to P(\widetilde{S}^{(n)})$*:*

$$\forall u \in \widetilde{S}^{(m)}, \forall x \in f(u) \cap S^{(n)}, \exists \mu \in \mathbf{B}^n, \exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = \mu; \tag{5.1}$$

$$\forall u \in \widetilde{S}^{(m)}, \exists t_0 \in \mathbf{R}, \forall x \in f(u) \cap S^{(n)}, \exists \mu \in \mathbf{B}^n, \forall t < t_0, x(t) = \mu; \tag{5.2}$$

$$\exists t_0 \in \mathbf{R}, \forall u \in \widetilde{S}^{(m)}, \forall x \in f(u) \cap S^{(n)}, \exists \mu \in \mathbf{B}^n, \forall t < t_0, x(t) = \mu; \tag{5.3}$$

$$\forall u \in \widetilde{S}^{(m)}, \forall x \in f(u) \cap S_c^{(n)}, \exists \mu \in \mathbf{B}^n, \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = \mu; \tag{5.4}$$

$$\forall u \in \widetilde{S}^{(m)}, \exists t_f \in \mathbf{R}, \forall x \in f(u) \cap S_c^{(n)}, \exists \mu \in \mathbf{B}^n, \forall t \geq t_f, x(t) = \mu; \tag{5.5}$$

$$\exists t_f \in \mathbf{R}, \forall u \in \widetilde{S}^{(m)}, \forall x \in f(u) \cap S_c^{(n)}, \exists \mu \in \mathbf{B}^n, \forall t \geq t_f, x(t) = \mu. \tag{5.6}$$

The properties (5.1) and (5.4) are fulfilled by all pseudo-systems and they are presented here for the sake of symmetry of the exposure only.

The similarity of this remark with Remark 14 is relative. Indeed, by defining a partial function $\widetilde{S}^{(n)} \to \mathbf{R}$*, for example in the case of (5.1), to associate the number* $t_0 \in \mathbf{R}$ *with each state* $x \in \bigcup_{u \in \widetilde{S}^{(m)}} f(u) \cap S^{(n)}$ *is not quite natural because* t_0 *is not unique (anyway we can make use of the axiom of choice). The reasoning is the same for the number* $t_f \in \mathbf{R}$*.*

If f *is the null pseudo-system or, more generally, if in one of (5.1),...,(5.3)* $\forall u \in \widetilde{S}^{(m)}$, $f(u) \cap S^{(n)} = \emptyset$*, or in one of (5.4),...,(5.6)* $\forall u \in \widetilde{S}^{(m)}$, $f(u) \cap S_c^{(n)} = \emptyset$*, then that property is trivially fulfilled.*

The dualities between (5.1) and (5.4); (5.2) and (5.5); (5.3) and (5.6) take place and the following implications hold:

$$(5.3) \Longrightarrow (5.2) \Longrightarrow (5.1);$$

$$(5.6) \Longrightarrow (5.5) \Longrightarrow (5.4).$$

Once again, $\forall t < t_0$ *and* $\forall t \leq t_0$ *are equivalent in* (5.1), ..., (5.3) *and* $\forall t > t_f$ *and* $\forall t \geq t_f$ *are equivalent in* (5.4), ..., (5.6).

Definition 28. *If* f *satisfies (5.1), we say that it* ***has an unbounded initial time*** *and any* t_0 *satisfying this property is called an* ***unbounded initial time (instant)****.*

Definition 29. *Let be* f *fulfilling the property (5.2). We say that it* ***has a bounded initial time*** *and any* t_0 *making this property true is called a* ***bounded initial time (instant)****.*

Definition 30. *When* f *satisfies (5.3), we use to say that it* ***has a fixed initial time*** *and any* t_0 *fulfilling (5.3) is called a* ***fixed initial time (instant)****.*

Definition 31. *Suppose that* f *satisfies (5.4). Then we say that it* ***has an unbounded final time*** *and any* t_f *satisfying this property is called an* ***unbounded final time (instant)****.*

Definition 32. *If* f *fulfills the property (5.5), we say that it* ***has a bounded final time****. Any number* t_f *satisfying (5.5) is called a* ***bounded final time (instant)****.*

Definition 33. *We suppose that the pseudo-system f satisfies the property (5.6). Then we say that it* ***has a fixed final time*** *and any number t_f satisfying (5.6) is called a* ***fixed final time (instant)****.*

Theorem 25. *If the pseudo-system f has initial states, then the following non-exclusive possibilities exist:*

a) f has initial states and an unbounded initial time iff

$$\forall u \in \widetilde{S}^{(m)}, \forall x \in f(u), \exists \mu \in \mathbf{B}^n, \exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = \mu;$$

b) f has initial states and a bounded initial time iff

$$\forall u \in \widetilde{S}^{(m)}, \exists t_0 \in \mathbf{R}, \forall x \in f(u), \exists \mu \in \mathbf{B}^n, \forall t < t_0, x(t) = \mu;$$

c) f has initial states and a fixed initial time iff

$$\exists t_0 \in \mathbf{R}, \forall u \in \widetilde{S}^{(m)}, \forall x \in f(u), \exists \mu \in \mathbf{B}^n, \forall t < t_0, x(t) = \mu;$$

d) f has race-free initial states and an unbounded initial time iff

$$\forall u \in \widetilde{S}^{(m)}, \exists \mu \in \mathbf{B}^n, \forall x \in f(u), \exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = \mu;$$

e) f has race-free initial states and a bounded initial time iff

$$\forall u \in \widetilde{S}^{(m)}, \exists \mu \in \mathbf{B}^n, \exists t_0 \in \mathbf{R}, \forall x \in f(u), \forall t < t_0, x(t) = \mu;$$

f) f has race-free initial states and a fixed initial time iff

$$\exists t_0 \in \mathbf{R}, \forall u \in \widetilde{S}^{(m)}, \exists \mu \in \mathbf{B}^n, \forall x \in f(u), \forall t < t_0, x(t) = \mu;$$

g) f has a constant initial state and an unbounded initial time iff

$$\exists \mu \in \mathbf{B}^n, \forall u \in \widetilde{S}^{(m)}, \forall x \in f(u), \exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = \mu;$$

h) f has a constant initial state and a bounded initial time iff

$$\exists \mu \in \mathbf{B}^n, \forall u \in \widetilde{S}^{(m)}, \exists t_0 \in \mathbf{R}, \forall x \in f(u), \forall t < t_0, x(t) = \mu;$$

i) f has a constant initial state and a fixed initial time iff

$$\exists \mu \in \mathbf{B}^n, \exists t_0 \in \mathbf{R}, \forall u \in \widetilde{S}^{(m)}, \forall x \in f(u), \forall t < t_0, x(t) = \mu.$$

Proof. e) We must show that the conjunction of (4.2) and (5.2) on one hand and

$$\forall u \in \widetilde{S}^{(m)}, \exists \mu \in \mathbf{B}^n, \exists t_0 \in \mathbf{R}, \forall x \in f(u), \forall t < t_0, x(t) = \mu \tag{5.7}$$

on the other hand, are equivalent. This fact is obvious if f is null. Thus we can suppose that f is non null and it is sufficient to consider some admissible arbitrary input $u \in \widetilde{S}^{(m)}$.

(4.2) and (5.2) $\Longrightarrow$ (5.7).

From (4.2) we have the existence of a unique $\mu \in \mathbf{B}^n$ depending on u such that $\forall x \in f(u), x(-\infty + 0)$ exists and $x(-\infty + 0) = \mu$. Thus $f(u) \subset S^{(n)}$ and $f(u) \cap S^{(n)} = f(u)$. From (5.2) we infer that

$$\exists t_0 \in \mathbf{R}, \forall x \in f(u), \forall t < t_0, x(t) = \mu,$$

where t_0 depends on u and the statement

$$\exists \mu \in \mathbf{B}^n, \exists t_0 \in \mathbf{R}, \forall x \in f(u), \forall t < t_0, x(t) = \mu$$

is also true as μ and t_0 depend on u only. Relation (5.7) takes place.

(5.7) $\Longrightarrow$ (4.2) and (5.2).

(5.7) $\Longrightarrow$ (4.2) is obvious. On the other hand, for $u \in \widetilde{S}^{(m)}$ admissible and arbitrary like before, there is a unique $\mu \in \mathbf{B}^n$ depending on u such that

$$\exists t_0 \in \mathbf{R}, \forall x \in f(u), \forall t < t_0, x(t) = \mu.$$

In particular, the statement

$$\exists t_0 \in \mathbf{R}, \forall x \in f(u) \cap S^{(n)}, \forall t < t_0, x(t) = \mu$$

is true, as well as

$$\exists t_0 \in \mathbf{R}, \forall x \in f(u) \cap S^{(n)}, \exists \mu \in \mathbf{B}^n, \forall t < t_0, x(t) = \mu,$$

i.e. (5.2). □

THEOREM 26. *The following non-exclusive possibilities exist for the absolutely stable pseudo-system f:*

a) f is absolutely stable with an unbounded final time iff

$$\forall u \in \widetilde{S}^{(m)}, \forall x \in f(u), \exists \mu \in \mathbf{B}^n, \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = \mu;$$

b) f is absolutely stable with a bounded final time iff

$$\forall u \in \widetilde{S}^{(m)}, \exists t_f \in \mathbf{R}, \forall x \in f(u), \exists \mu \in \mathbf{B}^n, \forall t \geq t_f, x(t) = \mu;$$

c) f is absolutely stable with a fixed final time iff

$$\exists t_f \in \mathbf{R}, \forall u \in \widetilde{S}^{(m)}, \forall x \in f(u), \exists \mu \in \mathbf{B}^n, \forall t \geq t_f, x(t) = \mu;$$

d) f is absolutely race-free stable with an unbounded final time iff

$$\forall u \in \widetilde{S}^{(m)}, \exists \mu \in \mathbf{B}^n, \forall x \in f(u), \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = \mu;$$

e) f is absolutely race-free stable with a bounded final time iff

$$\forall u \in \widetilde{S}^{(m)}, \exists \mu \in \mathbf{B}^n, \exists t_f \in \mathbf{R}, \forall x \in f(u), \forall t \geq t_f, x(t) = \mu;$$

f) f is absolutely race-free stable with a fixed final time iff

$$\exists t_f \in \mathbf{R}, \forall u \in \widetilde{S}^{(m)}, \exists \mu \in \mathbf{B}^n, \forall x \in f(u), \forall t \geq t_f, x(t) = \mu;$$

g) f is absolutely constantly stable with an unbounded final time iff

$$\exists \mu \in \mathbf{B}^n, \forall u \in \widetilde{S}^{(m)}, \forall x \in f(u), \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = \mu;$$

h) f is absolutely constantly stable with a bounded final time iff

$$\exists \mu \in \mathbf{B}^n, \forall u \in \widetilde{S}^{(m)}, \exists t_f \in \mathbf{R}, \forall x \in f(u), \forall t \geq t_f, x(t) = \mu;$$

i) f is absolutely constantly stable with a fixed final time iff

$$\exists \mu \in \mathbf{B}^n, \exists t_f \in \mathbf{R}, \forall u \in \widetilde{S}^{(m)}, \forall x \in f(u), \forall t \geq t_f, x(t) = \mu.$$

REMARK 17. *In the conditions of Theorems 25 and 26 the following implications hold:*

$$\begin{array}{ccccc} i) & \Longrightarrow & h) & \Longrightarrow & g) \\ \Downarrow & & \Downarrow & & \Downarrow \\ f) & \Longrightarrow & e) & \Longrightarrow & d) \\ \Downarrow & & \Downarrow & & \Downarrow \\ c) & \Longrightarrow & b) & \Longrightarrow & a) \end{array}$$

6. Initial state function and final state function

DEFINITION 34. *Let be the pseudo-system* $f : \widetilde{S}^{(m)} \to P(\widetilde{S}^{(n)})$. *If it has initial states, the function* $\phi_0 : \widetilde{S}^{(m)} \to P(\mathbf{B}^n)$ *defined by*

$$\forall u \in \widetilde{S}^{(m)}, \phi_0(u) = \{x(-\infty + 0) | x \in f(u)\}$$

is called the ***initial state function*** *of* f *and the set*

$$\Theta_0 = \bigcup_{u \in \widetilde{S}^{(m)}} \phi_0(u)$$

is called the ***set of the initial states*** *of* f.

DEFINITION 35. *Consider the pseudo-system* f. *If it has final states, the function* $\phi_f : \widetilde{S}^{(m)} \to P(\mathbf{B}^n)$ *defined by*

$$\forall u \in \widetilde{S}^{(m)}, \phi_f(u) = \{x(\infty - 0) | x \in f(u)\}$$

is called the ***final state function*** *of* f, *while the set*

$$\Theta_f = \bigcup_{u \in \widetilde{S}^{(m)}} \phi_f(u)$$

is called the ***set of the final states*** *of* f.

EXAMPLE 18. *The constant function* $\widetilde{S}^{(m)} \to \widetilde{S}^{(n)}$ *equal to* $\mu \in \mathbf{B}^n$ *is a pseudo-system with a constant initial state* μ *and a fixed initial time. It is also absolutely constantly stable with fixed final time. The functions* ϕ_0, ϕ_f *and the sets* Θ_0, Θ_f *are defined and equal to* $\{\mu\}$.

NOTATION 10. *If* $\forall u \in \widetilde{S}^{(m)}, \phi_0(u)$ *has exactly one element, the usual notation of the initial state function is* $\phi_0 : \widetilde{S}^{(m)} \to \mathbf{B}^n$. *Similarly, for the final state function: if* $\forall u \in \widetilde{S}^{(m)}, \phi_f(u)$ *has exactly one element, we use the notation of the uni-valued functions* $\phi_f : \widetilde{S}^{(m)} \to \mathbf{B}^n$.

THEOREM 27. *Let* f *be a pseudo-system with initial states.*

a) If its initial states are race-free, then $\forall u \in \widetilde{S}^{(m)}, \phi_0(u)$ *has at most one element.*

b) If f *has a constant initial state* μ, *then* $\phi_0(u) = \{\mu\}$ *is true for any admissible* u; *for* $f = \emptyset$ *we have* $\Theta_0 = \emptyset$ *and for* $f \neq \emptyset$ *we have* $\Theta_0 = \{\mu\}$.

PROOF. a) Suppose that f has race-free initial states and let be $u \in \widetilde{S}^{(m)}$. If $f(u) = \emptyset$, then $\phi_0(u) = \emptyset$ and if $f(u) \neq \emptyset$, then there is a unique $\mu \in \mathbf{B}^n$, depending on u, such that $\forall x \in f(u), x(-\infty + 0) = \mu$ and $\phi_0(u) = \{\mu\}$.

b) Suppose that f has a constant initial state μ. If f is null, then $\forall u \in \widetilde{S}^{(m)}, \phi_0(u) = \emptyset$ and $\Theta_0 = \emptyset$, otherwise for any admissible u we have $\forall x \in f(u), x(-\infty + 0) = \mu$ and $\phi_0(u) = \{\mu\}$, thus $\Theta_0 = \{\mu\}$. □

THEOREM 28. *Consider the pseudo-system* f *with final states.*

a) If its final states are race-free, then $\forall u \in \widetilde{S}^{(m)}, \phi_f(u)$ *has at most one element.*

b) If f *has a constant final state* μ, *then* $\phi_f(u) = \{\mu\}$ *is true for any admissible* u; *if admissible inputs do not exist, then* $\Theta_f = \emptyset$ *and if admissible inputs exist, then* $\Theta_f = \{\mu\}$.

CHAPTER 4

Systems

The systems are particular pseudo-systems, namely those non-null pseudo-systems satisfying the property that the admissible inputs and the possible states have initial values. Nevertheless, there is an asymmetry here between the attributes initial (states, time) and final (states, time). This is justified by the fact that we use to reason temporally by choosing an initial time instant and the increasing sense of the time axis. Even if many notions characterizing the systems may be defined also for the pseudo-systems, we prefer to present them as related to the systems, as the systems are closer to our modeling needs. We define and study some fundamental notions: subsystems, dual systems, inverse systems, Cartesian product of systems, parallel and serial connection of systems, intersection, union and morphisms of systems.

1. Definition of the systems

Definition 36. *For the pseudo-system $f : \widetilde{S}^{(m)} \to P(\widetilde{S}^{(n)})$, the set U_f of the admissible inputs defined by*

$$U_f = \{u | u \in \widetilde{S}^{(m)}, f(u) \neq \emptyset\}$$

is also called the ***support (set) of*** *f.*

Definition 37. *The (asynchronous) pseudo-system f is called* ***(asynchronous) system*** *if*

a) $U_f \neq \emptyset$,

b) $U_f \subset S^{(m)}$,

c) $\forall u \in U_f, f(u) \subset S^{(n)}$.

Notation 11. *We identify the system f with the function $f_1 : U \to P^*(S^{(n)})$, where $U = U_f$, defined by $\forall u \in U, f_1(u) = f(u)$. This identification leads to the usual notation of the systems under the closed form i.e. $f : U \to P^*(S^{(n)})$, where $U \subset S^{(m)}$ is non-empty. If $\forall u \in U, f(u)$ has a single element, then we use the notation $f : U \to S^{(n)}$ of the uni-valued functions.*

Remark 18. *In implicit form, the systems consist in one or several equations or inequalities with existing solutions $x \in S^{(n)}$ that depend on the parameter $u \in U$. We keep in mind that any $u \in \widetilde{S}^{(m)} \setminus U$ gives no solutions $x \in \widetilde{S}^{(n)}$ and that for any $u \in U$, no solutions $x \in \widetilde{S}^{(n)} \setminus S^{(n)}$ exist.*

The systems are those non-null pseudo-systems f for which the admissible inputs and the possible states have initial values (implying that f has initial states). The concept creates an asymmetry between the initial states and the final states because:

- it is natural to consider the inputs as commands, a deliberate manner of acting on the circuit modeled by f in view of producing a certain effect. But this is made

after choosing an initial time instant t_0 from which we order our actions in the increasing sense of the time axis (not in both senses);

- it is natural to associate with the requirement $U \subset S^{(m)}$ (Definition 37 b) and the explanations from the previous paragraph) a requirement (Definition 37 c)) that is dual to absolute stability: the system orders its reactions from an initial time instant, in the increasing sense of the time axis (not in both senses).

The way from two senses on the time axis to one sense and the existence of the initial time instant were anticipated at Remark 7 by the fact that the sequences $(t_z)_{z\in\mathbf{Z}}$ consistent with the differentiable functions are replaced by $(t_k)_{k\in\mathbf{N}}$ sequences.

EXAMPLE 19. *The fact that in Ch. 3 Example 17, the double inequality (3.1) defines a $S \to P^*(S)$ system is obvious because*

$$\forall u \in S, \forall \delta \in (0,d], \bigcap_{\xi\in[t-d,t)} u(\xi) \le u(t-\delta) \le \bigcup_{\xi\in[t-d,t)} u(\xi).$$

Thus having initial values, $x(t) = u(t-\delta)$ satisfies it whenever $\delta \in (0,d]$. Moreover, when u has the initial value $u(-\infty+0)$, any solution x of this double inequality has the initial value $x(-\infty+0) = u(-\infty+0)$. In addition, in order to make the example correct, we must ask that $\forall u \in \widetilde{S} \setminus S$, the inequality has no solutions.

NOTATION 12. *Let $f : \widetilde{S}^{(m)} \to P(\widetilde{S}^{(n)})$ be a pseudo-system with the property that*

$$\exists u \in S^{(m)}, f(u) \cap S^{(n)} \neq \emptyset. \tag{1.1}$$

We denote by $[f] : U \to P^*(S^{(n)})$ the function defined in the following way

$$U = \{u | u \in S^{(m)}, f(u) \cap S^{(n)} \neq \emptyset\}, \tag{1.2}$$

$$\forall u \in U, [f](u) = f(u) \cap S^{(n)}. \tag{1.3}$$

THEOREM 29. *$[f]$ is a system.*

PROOF. The relation $U \neq \emptyset$ follows from (1.1) and (1.2), $U \subset S^{(m)}$ is a consequence of (1.2) and $\forall u \in U, [f](u) \subset S^{(n)}$ follows from (1.3). Thus $[f]$ is a system. □

DEFINITION 38. *For any pseudo-system f satisfying the property (1.1), $[f]$ is called the **system induced by** f.*

THEOREM 30. *The pseudo-system f is a system iff $f = [f]$.*

PROOF. $\Longleftarrow$ is obvious, since $[f]$ is a system.

$\Longrightarrow$ There are admissible inputs and let u be such an input. Because f is a system, u has an initial value. From $f(u) \subset S^{(n)}$, we have that $f(u) = f(u) \cap S^{(n)} = [f](u)$ and, due to the fact that u was arbitrarily chosen, we infer $\forall u \in U, f(u) = [f](u)$, i.e. $f = [f]$, where $U = U_f$. □

2. Initial states and final states

THEOREM 31. *Suppose that the pseudo-system $f : \widetilde{S}^{(m)} \to P(\widetilde{S}^{(n)})$ is a system and its support set is $U \in P^*(S^{(m)})$. Then the following statements are equivalent:*

a) Ch. 3, statement (4.1) and

$$\forall u \in U, \forall x \in f(u), \exists \mu \in \mathbf{B}^n, \exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = \mu; \tag{2.1}$$

b) Ch. 3, statement (4.2) and

$$\forall u\in U,\exists\mu\in\mathbf{B}^n,\forall x\in f(u),\exists t_0\in\mathbf{R},\forall t<t_0,x(t)=\mu; \tag{2.2}$$

c) Ch. 3, statement (4.3) and

$$\exists\mu\in\mathbf{B}^n,\forall u\in U,\forall x\in f(u),\exists t_0\in\mathbf{R},\forall t<t_0,x(t)=\mu; \tag{2.3}$$

d) Ch. 3, statement (4.4) and

$$\forall u\in U,\forall x\in f(u),\exists\mu\in\mathbf{B}^n,\exists t_f\in\mathbf{R},\forall t\geq t_f,x(t)=\mu; \tag{2.4}$$

e) Ch. 3, statement (4.5) and

$$\forall u\in U,\exists\mu\in\mathbf{B}^n,\forall x\in f(u),\exists t_f\in\mathbf{R},\forall t\geq t_f,x(t)=\mu; \tag{2.5}$$

f) Ch. 3, statement (4.6) and

$$\exists\mu\in\mathbf{B}^n,\forall u\in U,\forall x\in f(u),\exists t_f\in\mathbf{R},\forall t\geq t_f,x(t)=\mu. \tag{2.6}$$

PROOF. Because $\forall u\in\widetilde{S}^{(m)}\setminus U$, the statement $x\in f(u)$ is false, the corresponding property is trivially fulfilled, such that the only points where the truth of (4.1),...,(4.6) from Ch. 3 needs to be discussed are $u\in U$. □

REMARK 19. *For any system f, the property (2.1) is true.*

Theorem 31 says that we can limit the analysis of the systems, from the initial and final states point of view, to the $U\to P^*(S^{(n)})$ functions, as expected.

3. Initial time and final time

THEOREM 32. *Let the pseudo-system $f:\widetilde{S}^{(m)}\to P(\widetilde{S}^{(n)})$ have the support $U\subset\widetilde{S}^{(m)}$. If f is a system, then the following equivalencies hold:*

a) Ch. 3, statement (5.1) and

$$\forall u\in U,\forall x\in f(u),\exists\mu\in\mathbf{B}^n,\exists t_0\in\mathbf{R},\forall t<t_0,x(t)=\mu; \tag{3.1}$$

b) Ch. 3, statement (5.2) and

$$\forall u\in U,\exists t_0\in\mathbf{R},\forall x\in f(u),\exists\mu\in\mathbf{B}^n,\forall t<t_0,x(t)=\mu; \tag{3.2}$$

c) Ch. 3, statement (5.3) and

$$\exists t_0\in\mathbf{R},\forall u\in U,\forall x\in f(u),\exists\mu\in\mathbf{B}^n,\forall t<t_0,x(t)=\mu; \tag{3.3}$$

d) Ch. 3, statement (5.4) and

$$\forall u\in U,\forall x\in f(u)\cap S_c^{(n)},\exists\mu\in\mathbf{B}^n,\exists t_f\in\mathbf{R},\forall t\geq t_f,x(t)=\mu; \tag{3.4}$$

e) Ch. 3, statement (5.5) and

$$\forall u\in U,\exists t_f\in\mathbf{R},\forall x\in f(u)\cap S_c^{(n)},\exists\mu\in\mathbf{B}^n,\forall t\geq t_f,x(t)=\mu; \tag{3.5}$$

f) Ch. 3, statement (5.6) and

$$\exists t_f\in\mathbf{R},\forall u\in U,\forall x\in f(u)\cap S_c^{(n)},\exists\mu\in\mathbf{B}^n,\forall t\geq t_f,x(t)=\mu. \tag{3.6}$$

PROOF. Reasoning is similar with that of Theorem 31. In (3.1),...,(3.3) we have used the fact that $\forall u\in U,f(u)\cap S^{(n)}=f(u)$. □

REMARK 20. *The property (3.1) coincides with (2.1) and is always fulfilled. The property (3.4) is always fulfilled too.*

From now on, in the case of the systems we shall use the statements (2.1),..., (2.6), (3.1),..., (3.6).

4. Initial state function and set of initial states

THEOREM 33. *For any system f, the initial state function ϕ_0 and the set of the initial states Θ_0 exist.*

PROOF. This holds for each pseudo-system with initial states. □

NOTATION 13. *We identify the initial state function $\phi_0 : \widetilde{S}^{(m)} \to P(\mathbf{B}^n)$ with the function $\phi_{10} : U \to P^*(\mathbf{B}^n)$ defined by $\forall u \in U, \phi_{10}(u) = \phi_0(u)$ where $U = U_f$ is the support set of f. This identification allows us to use the notation $\phi_0 : U \to P^*(\mathbf{B}^n)$ and, when $\forall u \in U, x(-\infty + 0)$ is unique, the notation $\phi_0 : U \to \mathbf{B}^n$.*

REMARK 21. *Unlike the initial states of f, that always exist, the final states may not exist. If f has final states, the final state function $\phi_f : U \to P^*(\mathbf{B}^n)$ and the set of final states Θ_f are defined.*

5. Subsystems

THEOREM 34. *Consider the pseudo-systems $f, g : \widetilde{S}^{(m)} \to P(\widetilde{S}^{(n)})$ with the support sets denoted by U, V.*

a) The following statements are equivalent

$$\forall u \in \widetilde{S}^{(m)}, f(u) \subset g(u),$$

$$U \subset V \text{ and } \forall u \in U, f(u) \subset g(u).$$

b) If one of the statements from a) is true, g is a system and $U \neq \emptyset$, then f is a system.

PROOF. b) follows from the fact that $U \neq \emptyset, U \subset V \subset S^{(m)}$ and $\forall u \in U, f(u) \subset g(u) \subset S^{(n)}$. □

DEFINITION 39. *Let be the systems $f : U \to P^*(S^{(n)})$, $g : V \to P^*(S^{(n)})$, $U, V \in P^*(S^{(m)})$. If*

$$U \subset V \text{ and } \forall u \in U, f(u) \subset g(u)$$

we say that f is a ***subsystem*** *of g or that it is included in g and the usual notation for this is $f \subset g$.*

REMARK 22. *Intuitively, the fact that f is a subsystem of g means that the modeling of a circuit is made more precisely by f than by g, possibly after considering a smaller set of admissible inputs. The relation $\subset$ is a relation of partial order between the $U \to P^*(S^{(n)})$ systems, U runs over $P^*(S^{(m)})$, where the first element does not exist and the last element $S^{(n)} : S^{(m)} \to P^*(S^{(n)})$ is given by*

$$\forall u \in S^{(m)}, S^{(n)}(u) = S^{(n)}.$$

EXAMPLE 20. *Let be the system f and take some arbitrary $\mu \in \Theta_0$. The system $f_\mu : U_\mu \to P^*(S^{(n)})$ defined by*

$$U_\mu = \{u | u \in U, \mu \in \phi_0(u)\},$$

$$\forall u \in U_\mu, f_\mu(u) = \{x | x \in f(u), x(-\infty + 0) = \mu\}$$

is a subsystem of f, called the ***restriction*** *of f at the initial (value of the) state μ. Remark that f_μ is initialized and μ is its constant initial state.*

THEOREM 35. *Let be the system g and $f \subset g$ be an arbitrary subsystem. If g has race-free initial states (constant initial state), then f has race-free initial states (constant initial state) too.*

PROOF. If one of the previous properties is true for the states in $g(u)$, then it is true for the states in the subset $f(u) \subset g(u)$ also, $u \in U$. □

THEOREM 36. *Let be $f \subset g$. If g has final states (race-free final states, a constant final state), then f has final states (race-free final states, a constant final state) too.*

THEOREM 37. *Given the systems $f \subset g$, if g has a bounded initial time (a fixed initial time), then f has a bounded initial time (a fixed initial time).*

PROOF. Like previously, if one of the above properties is true for the states in $g(u)$, then it is true for the states in $f(u) \subset g(u), u \in U$. □

THEOREM 38. *Let f be a subsystem of g. If g has a bounded final time (a fixed final time), then f has a bounded final time (a fixed final time).*

THEOREM 39. *If $f \subset g$, then we denote by $\gamma_0 : V \to P^*(\mathbf{B}^n)$ the initial state function of g and by $\Gamma_0 \subset \mathbf{B}^n$ the set of the initial states of g. We have $\forall u \in U, \phi_0(u) \subset \gamma_0(u)$ and $\Theta_0 \subset \Gamma_0$.*

PROOF. As $U \subset V$ and $\forall u \in U, f(u) \subset g(u)$, the initial values of the states in $f(u)$ are among the initial values of the states in $g(u)$, $\phi_0(u) \subset \gamma_0(u)$ making $\Theta_0 \subset \Gamma_0$ also true. □

THEOREM 40. *If g has final states and $f \subset g$, we denote by $\gamma_f : V \to P^*(\mathbf{B}^n)$ the final state function of g and by $\Gamma_f \subset \mathbf{B}^n$ the set of the final states of g. We have $\forall u \in U, \phi_f(u) \subset \gamma_f(u)$ and $\Theta_f \subset \Gamma_f$.*

PROOF. The system f has final states from Theorem 36, thus ϕ_f and Γ_f exist. The rest of the proof is similar to the proof of Theorem 39. □

6. Dual systems

NOTATION 14. *For any $u \in S^{(m)}$ we denote by $\overline{u} \in S^{(m)}$ its complement made coordinately:*

$$\overline{u}(t) = (\overline{u_1(t)}, ..., \overline{u_m(t)}).$$

NOTATION 15. *If $U \subset S^{(m)}$ is a space of functions, we denote by U^* the set*

$$U^* = \{\overline{u} | u \in U\}. \tag{6.1}$$

THEOREM 41. *Let be the pseudo-system $f : \widetilde{S}^{(m)} \to P(\widetilde{S}^{(n)})$ for which $f^* : \widetilde{S}^{(m)} \to P(\widetilde{S}^{(n)})$ is the pseudo-system defined like this:*

$$\forall u \in \widetilde{S}^{(m)}, f^*(u) = \{\overline{x} | x \in f(\overline{u})\}.$$

By denoting by U the support set of f, we have that the support set of f^* is U^*. If f is a system, then f^* is a system.

PROOF. From (6.1) we get

$$U^* = \{\overline{u} | u \in \widetilde{S}^{(m)}, f(u) \neq \emptyset\} = \{u | u \in \widetilde{S}^{(m)}, f(\overline{u}) \neq \emptyset\} = \{u | u \in \widetilde{S}^{(m)}, f^*(u) \neq \emptyset\},$$

i.e. U^* is the support set of the pseudo-system f^*. At this moment suppose that f is a system. From $U \neq \emptyset$ we infer that $U^* \neq \emptyset$. The fact that $\forall u \in U, u(-\infty + 0)$ exists shows the truth of $\forall u \in U^*, u(-\infty + 0)$ exists, thus $U^* \subset S^{(m)}$. Because $\forall u \in U, f(u) \subset S^{(n)}$ we have $\forall u \in U^*, f(\overline{u}) \subset S^{(n)}$, thus $\forall u \in U^*, \{\overline{x} | x \in f(\overline{u})\} \subset S^{(n)}$ and, eventually, $\forall u \in U^*, f^*(u) \subset S^{(n)}$. f^* is a system. □

DEFINITION 40. *Let be the system* $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$. *The system* $f^* : U^* \to P^*(S^{(n)})$ *defined by*

$$\forall u \in U^*, f^*(u) = \{\overline{x} | x \in f(\overline{u})\},$$

where U^* *satisfies (6.1), is called the* ***dual system*** *of* f.

REMARK 23. *To the types of duality previously presented we add the duality between* $0, 1 \in \mathbf{B}$ *that Definition 40 makes use of. If* f *models some circuit, then* f^* *models the dual of that circuit (with AND logical gates instead of OR logical gates etc.) and it has many properties that can be inferred from those of* f.

We note that $\forall u \in U^*, f^*(u) = (f(\overline{u}))^*$. *This fact will be used later on.*

THEOREM 42. $(f^*)^* = f$.

PROOF. $(U^*)^* = \{\overline{u} | u \in U^*\} = \{u | \overline{u} \in U^*\} = U$ and we note that $\forall u \in U$, $(f^*)^*(u) = \{\overline{x} | x \in f^*(\overline{u})\} = \{x | \overline{x} \in f^*(\overline{u})\} = f(u)$. □

THEOREM 43. *For the system* f, *the following statements are equivalent:*
a) f *has race-free initial states (a constant initial state);*
b) f^* *has race-free initial states (a constant initial state).*

PROOF. We show that f has race-free initial states $\Longleftrightarrow$ f^* has race-free initial states:

$$\forall u \in U, \exists \mu \in \mathbf{B}^n, \forall x \in f(u), \exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = \mu \Longleftrightarrow$$

$$\Longleftrightarrow \forall u \in U, \exists \mu \in \mathbf{B}^n, \forall \overline{x} \in f^*(\overline{u}), \exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = \mu \Longleftrightarrow$$

$$\Longleftrightarrow \forall \overline{u} \in U^*, \exists \overline{\mu} \in \mathbf{B}^n, \forall \overline{x} \in f^*(\overline{u}), \exists t_0 \in \mathbf{R}, \forall t < t_0, \overline{x}(t) = \overline{\mu} \Longleftrightarrow$$

$$\Longleftrightarrow \forall u \in U^*, \exists \mu \in \mathbf{B}^n, \forall x \in f^*(u), \exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = \mu.$$

In the previous statements, $\forall \overline{x} \in f^*(\overline{u})$ is the notation for $\forall x, \overline{x} \in f^*(\overline{u})$, while $\exists \overline{\mu} \in \mathbf{B}^n$ is the notation for $\exists \mu, \overline{\mu} \in \mathbf{B}^n$ etc. □

THEOREM 44. *For the system* f, *the following statements are equivalent:*
a) f *has final states (race-free final states, a constant final state);*
b) f^* *has final states (race-free final states, a constant final state).*

THEOREM 45. *The following properties are equivalent for* f :
a) f *has a bounded initial time (a fixed initial time);*
b) f^* *has a bounded initial time (a fixed initial time).*

PROOF. Similarly with the proof of Theorem 43, we show that f has a fixed initial time $\Longleftrightarrow$ f^* has a fixed initial time:

$$\exists t_0 \in \mathbf{R}, \forall u \in U, \forall x \in f(u), \exists \mu \in \mathbf{B}^n, \forall t < t_0, x(t) = \mu \Longleftrightarrow$$

$$\Longleftrightarrow \exists t_0 \in \mathbf{R}, \forall u \in U, \forall \overline{x} \in f^*(\overline{u}), \exists \mu \in \mathbf{B}^n, \forall t < t_0, x(t) = \mu \Longleftrightarrow$$

$$\Longleftrightarrow \exists t_0 \in \mathbf{R}, \forall \overline{u} \in U^*, \forall \overline{x} \in f^*(\overline{u}), \exists \overline{\mu} \in \mathbf{B}^n, \forall t < t_0, \overline{x}(t) = \overline{\mu} \Longleftrightarrow$$

$$\Longleftrightarrow \exists t_0 \in \mathbf{R}, \forall u \in U^*, \forall x \in f^*(u), \exists \mu \in \mathbf{B}^n, \forall t < t_0, x(t) = \mu,$$

where $\forall \overline{x} \in f^*(\overline{u}), \forall \overline{u} \in U^*$... are notations similar to those from the proof of Theorem 43. □

THEOREM 46. *Let be the system* f. *The following properties are equivalent:*
a) f *has a bounded final time (a fixed final time);*
b) f^* *has a bounded final time (a fixed final time).*

THEOREM 47. *Denote by* $\phi_0^* : U^* \to P^*(\mathbf{B}^n)$ *the initial state function of* f^* *and by* Θ_0^* *the set of the initial states of* f^*. *We have*

$$\forall u \in U^*, \phi_0^*(u) = \{\overline{\mu} | \mu \in \phi_0(\overline{u})\},$$

$$\Theta_0^* = \{\overline{\mu} | \mu \in \Theta_0\}.$$

PROOF. The assertions of the theorem are obtained from the fact that

$$\forall u \in U^*, \phi_0^*(u) = \{x(-\infty + 0) | x \in f^*(u)\} =$$

$$= \{\overline{x(-\infty + 0)} | \overline{x} \in f^*(u)\} = \{\overline{x(-\infty + 0)} | x \in f(\overline{u})\} = \{\overline{\mu} | \mu \in \phi_0(\overline{u})\}$$

□

THEOREM 48. *If* f *has final states, we denote by* $\phi_f^* : U^* \to P^*(\mathbf{B}^n)$ *the final state function of* f^* *and by* Θ_f^* *the set of the final states of* f^*. *We have*

$$\forall u \in U^*, \phi_f^*(u) = \{\overline{\mu} | \mu \in \phi_f(\overline{u})\},$$

$$\Theta_f^* = \{\overline{\mu} | \mu \in \Theta_f\}.$$

PROOF. We use Theorem 44 to show that ϕ_f^*, Θ_f^* exist and the duality with Theorem 47. □

THEOREM 49. *For the systems* $f : U \to P^*(S^{(n)})$, $g : V \to P^*(S^{(n)})$, $U, V \in P^*(S^{(m)})$ *we have* $f \subset g \Longleftrightarrow f^* \subset g^*$.

PROOF. We get the following sequence of equivalencies:

$$f \subset g \Longleftrightarrow U \subset V \text{ and } \forall u \in U, f(u) \subset g(u) \Longleftrightarrow$$

$$\Longleftrightarrow \forall u \in U, u \in V \text{ and } f(u) \subset g(u) \Longleftrightarrow$$

$$\Longleftrightarrow \forall u \in U, u \in V \text{ and } \{x | x \in f(u)\} \subset \{x | x \in g(u)\} \Longleftrightarrow$$

$$\Longleftrightarrow \forall u \in U, u \in V \text{ and } \{\overline{x} | x \in f(u)\} \subset \{\overline{x} | x \in g(u)\} \Longleftrightarrow$$

$$\Longleftrightarrow \forall u \in U, u \in V \text{ and } f^*(\overline{u}) \subset g^*(\overline{u}) \Longleftrightarrow$$

$$\Longleftrightarrow \forall \overline{u} \in U^*, \overline{u} \in V^* \text{ and } f^*(\overline{u}) \subset g^*(\overline{u}) \Longleftrightarrow$$

$$\Longleftrightarrow \forall u \in U^*, u \in V^* \text{ and } f^*(u) \subset g^*(u) \Longleftrightarrow$$

$$\Longleftrightarrow U^* \subset V^* \text{ and } \forall u \in U^*, f^*(u) \subset g^*(u) \Longleftrightarrow f^* \subset g^*$$

□

7. Inverse systems

THEOREM 50. *Given the pseudo-system* $f : \widetilde{S}^{(m)} \to P(\widetilde{S}^{(n)})$, *the support of the pseudo-system* $f^{-1} : \widetilde{S}^{(n)} \to P(\widetilde{S}^{(m)})$,

$$\forall x \in \widetilde{S}^{(n)}, f^{-1}(x) = \{u | u \in \widetilde{S}^{(m)}, x \in f(u)\}$$

is

$$X = \bigcup_{u \in \widetilde{S}^{(m)}} f(u).$$

If f *is a system, then* f^{-1} *is a system too.*

PROOF. Obviously, X is the support set of f^{-1}. Suppose that f is a system. Then $f \neq \emptyset$ implies $X \neq \emptyset$. The fact that $\forall u \in \widetilde{S}^{(m)}, f(u) \subset S^{(n)}$ (in this inclusion $f(u)$ is empty for some u) shows that $X \subset S^{(n)}$. From $\forall u \in \widetilde{S}^{(m)}, f(u) \neq \emptyset \Longrightarrow u \in S^{(m)}$ we infer $\forall x \in \widetilde{S}^{(n)}, f^{-1}(x) \subset S^{(m)}$ (in this inclusion $f^{-1}(x)$ is empty for some x). All the requirements of Definition 37 are fulfilled, so that f^{-1} is a system. □

DEFINITION 41. *Let be the system* $f : U \to P(S^{(n)}), U \in P^*(S^{(m)})$. *The system* $f^{-1} : X \to P^*(S^{(m)})$ *given by*

$$X = \bigcup_{u \in U} f(u),$$

$$\forall x \in X, f^{-1}(x) = \{u | u \in U, x \in f(u)\}$$

is called the ***inverse*** *of* f.

REMARK 24. f^{-1} *is the inverse of* f *considered as a relation* $f \subset \widetilde{S}^{(m)} \times \widetilde{S}^{(n)}$. *The idea of its construction is that of inverting the cause-effect relation: with each possible effect* x *it associates these admissible inputs* u *that could have caused it.*

Working with f *may be thought as referring to the analysis of a circuit, while working with* f^{-1} *may be thought of as referring to the control of a circuit.*

In some properties that follow we shall use the truth of the equivalence $u \in f^{-1}(x) \Longleftrightarrow x \in f(u)$.

THEOREM 51. *For the system* f, *we have* $(f^{-1})^{-1} = f$.

PROOF. Denoting by U' the support set of $(f^{-1})^{-1}$, we can write

$$U' = \bigcup_{x \in X} f^{-1}(x) = \{u | \exists x \in X, u \in f^{-1}(x)\} =$$

$$= \{u | \exists u' \in U, \exists x \in f(u'), u \in f^{-1}(x)\} = \{u | u \in U, \exists u' \in U, \exists x \in f(u') \cap f(u)\} =$$
$$= \{u | u \in U, \exists u' \in U, f(u') \cap f(u) \neq \emptyset\} = U.$$

Thus the supports of $(f^{-1})^{-1}$ and f coincide. For any $u \in U$ we have

$$(f^{-1})^{-1}(u) = \{x | x \in X, u \in f^{-1}(x)\} = \{x | x \in f(u)\} = f(u).$$

□

THEOREM 52. *Denote by* $\phi_0^{-1} : X \to P^*(\mathbf{B}^m)$ *the initial state function and by* Θ_0^{-1} *the set of the initial states of* f^{-1}. *The following statements are true*

$$\forall x \in X, \phi_0^{-1}(x) = \{u(-\infty + 0) | u \in U, x \in f(u)\},$$

$$\Theta_0^{-1} = \{u(-\infty + 0) | u \in U\}.$$

THEOREM 53. *Suppose that f^{-1} has final states and we use the notation $\phi_f^{-1} : X \to P^*(\mathbf{B}^m)$, Θ_f^{-1} for its final state function and for its set of final states respectively. We have*

$$\forall x \in X, \phi_f^{-1}(x) = \{u(\infty - 0) | u \in U, x \in f(u)\},$$
$$\Theta_f^{-1} = \{u(\infty - 0) | u \in U\}.$$

THEOREM 54. *If $g : V \to P^*(S^{(n)}), V \in P^*(S^{(m)})$ is some system and $f \subset g$, then $f^{-1} \subset g^{-1}$ and $(f^*)^{-1} \subset (g^*)^{-1}$ take place.*

PROOF. With the notations $X = \bigcup_{u \in U} f(u)$, $Y = \bigcup_{u \in V} g(u)$ we have:

$$f \subset g \Longleftrightarrow (U \subset V \text{ and } \forall u \in U, f(u) \subset g(u)) \Longrightarrow$$
$$\Longrightarrow (X \subset Y \text{ and } \forall u \in U, \forall x \in X, x \in f(u) \Longrightarrow x \in g(u)) \Longleftrightarrow$$
$$\Longleftrightarrow (X \subset Y \text{ and } \forall x \in X, \forall u \in U, x \in f(u) \Longrightarrow x \in g(u)) \Longleftrightarrow$$
$$\Longleftrightarrow (X \subset Y \text{ and } \forall x \in X, \forall u \in U, u \in f^{-1}(x) \Longrightarrow u \in g^{-1}(x)) \Longleftrightarrow$$
$$\Longleftrightarrow (X \subset Y \text{ and } \forall x \in X, f^{-1}(x) \subset g^{-1}(x)) \Longleftrightarrow f^{-1} \subset g^{-1}.$$

On the other hand, $f \subset g$ implies $f^* \subset g^*$ (by Theorem 49) and from the previous item we get $(f^*)^{-1} \subset (g^*)^{-1}$. □

THEOREM 55. $(f^{-1})^* = (f^*)^{-1}$.

PROOF. The support of $(f^{-1})^*$ is X^* and the support of $(f^*)^{-1}$ is

$$\bigcup_{u \in U^*} f^*(u) = \{\overline{x} | \exists u \in U^*, x \in f(\overline{u})\} = \{\overline{x} | \exists u \in U, x \in f(u)\} = X^*,$$

thus the supports of the two systems coincide. For all $x \in X^*$ we get that

$$(f^{-1})^*(x) = \{\overline{u} | u \in f^{-1}(\overline{x})\} = \{\overline{u} | u \in U, \overline{x} \in f(u)\} = \{\overline{u} | u \in U, x \in f^*(\overline{u})\} =$$
$$= \{u | u \in U^*, x \in f^*(u)\} = \{u | u \in (f^*)^{-1}(x)\} = (f^*)^{-1}(x).$$

□

8. Cartesian product

THEOREM 56. *Consider the pseudo-systems $f : \widetilde{S}^{(m)} \to P(\widetilde{S}^{(n)})$, $f' : \widetilde{S}^{(m')} \to P(\widetilde{S}^{(n')})$ and define the pseudo-system $f \times f' : \widetilde{S}^{(m+m')} \to P^*(\widetilde{S}^{(n+n')})$ by*

$$\forall u \times u' \in \widetilde{S}^{(m+m')}, (f \times f')(u \times u') = f(u) \times f'(u').$$

If U, U' are the support sets of f, f', then $U \times U'$ is the support set of $f \times f'$. If f, f' are systems, then $f \times f'$ is a system.

PROOF. Obviously, $U \times U'$ is the support set of $f \times f'$. Suppose that f, f' are systems. We have that $U \neq \emptyset, U' \neq \emptyset$ imply $U \times U' \neq \emptyset$. Furthermore, $U \subset S^{(m)}, U' \subset S^{(m')}$ imply $U \times U' \subset S^{(m)} \times S^{(m')} = S^{(m+m')}$ and $\forall u \in U, f(u) \subset S^{(n)}$, $\forall u' \in U', f'(u') \subset S^{(n')}$ imply $\forall u \times u' \in U \times U', f(u) \times f'(u') \subset S^{(n)} \times S^{(n')} = S^{(n+n')}$. Therefore , $f \times f'$ is a system. □

DEFINITION 42. *Consider the systems $f : U \to P^*(S^{(n)})$, $f' : U' \to P^*(S^{(n')})$, where $U \in P^*(S^{(m)})$ and $U' \in P^*(S^{(m')})$. The **Cartesian product** of f and f' is the system $f \times f' : U \times U' \to P^*(S^{(n+n')})$ defined as*

$$\forall u \times u' \in U \times U', (f \times f')(u \times u') = f(u) \times f'(u').$$

REMARK 25. *The Cartesian product of two systems is the system that represents f and f' acting independently on each other. It models two circuits that are not interconnected.*

Let $f : U \to P^*(S^{(n)}), f' : U' \to P^*(S^{(n')}), f'' : U'' \to P^*(S^{(n'')})$ be three systems, where $U \in P^*(S^{(m)}), U' \in P^*(S^{(m')})$ and $U'' \in P^*(S^{(m'')})$. The associativity of the law '$\times$' may be thought of by identifying the systems $(f \times f') \times f''$ and $f \times (f' \times f'')$; any of them is denoted by $f \times f' \times f''$. The support set of $f \times f' \times f''$ is denoted by $U \times U' \times U'' \in P^*(S^{(m+m'+m'')})$, its range is $P^*(S^{(n+n'+n'')})$, its inputs are denoted by $u \times u' \times u'' \in U \times U' \times U''$ and its states are denoted by $x \times x' \times x'' \in (f \times f' \times f'')(u \times u' \times u'')$.

THEOREM 57. *The systems f and f' have race-free initial states (a constant initial state) iff $f \times f'$ has race-free initial states (a constant initial state).*

PROOF. For example, the conjunction of the statements

$$\exists \mu \in \mathbf{B}^n, \forall u \in U, \forall x \in f(u), \exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = \mu,$$

$$\exists \mu' \in \mathbf{B}^{n'}, \forall u' \in U', \forall x' \in f'(u'), \exists t_0^{'} \in \mathbf{R}, \forall t < t_0^{'}, x'(t) = \mu'$$

implies

$$\exists (\mu, \mu') \in \mathbf{B}^n \times \mathbf{B}^{n'}, \forall u \times u' \in U \times U', \forall x \times x' \in (f \times f')(u \times u'),$$

$$\exists t_0^{''} \in \mathbf{R}, \forall t < t_0^{''}, (x(t), x'(t)) = (\mu, \mu'),$$

where each time we can take $t_0^{''} = \min\{t_0, t_0'\}$.

The other implications are obvious at this moment. □

THEOREM 58. *The systems f and f' have final states (race-free final states, a constant final state) iff $f \times f'$ has final states (race-free final states, a constant final state).*

THEOREM 59. *Let be the systems f, f'. The following statements are equivalent:*
a) f and f' have a bounded initial time (a fixed initial time);
b) $f \times f'$ has a bounded initial time (a fixed initial time).

PROOF. For example, the conjunction of the statements

$$\exists t_0 \in \mathbf{R}, \forall u \in U, \forall x \in f(u), \exists \mu \in \mathbf{B}^n, \forall t < t_0, x(t) = \mu,$$

$$\exists t_0' \in \mathbf{R}, \forall u' \in U', \forall x' \in f'(u'), \exists \mu' \in \mathbf{B}^{n'}, \forall t < t_0', x'(t) = \mu'$$

is equivalent to the statement

$$\exists t_0^{''} \in \mathbf{R}, \forall u \times u' \in U \times U', \forall x \times x' \in (f \times f')(u \times u'),$$

$$\exists (\mu, \mu') \in \mathbf{B}^n \times \mathbf{B}^{n'}, \forall t < t_0^{''}, (x(t), x'(t)) = (\mu, \mu').$$

□

THEOREM 60. *The systems f and f' have a bounded final time (a fixed final time) iff $f \times f'$ has a bounded final time (a fixed final time).*

THEOREM 61. *Let the systems f and f' be defined as before. Denote by ϕ_0, ϕ_0' their initial state functions and by $(\phi \times \phi')_0 : U \times U' \to P^*(\mathbf{B}^{n+n'})$ the initial state function of $f \times f'$. Denote by $\Theta_0, \Theta_0^{'}$ the sets of the initial states of f and f' and let $(\Theta \times \Theta')_0$ be the set of the initial states of $f \times f'$. We have*

$$\forall u \times u' \in U \times U', (\phi \times \phi')_0(u \times u') = \phi_0(u) \times \phi_0'(u),$$

$$(\Theta\times\Theta')_0=\Theta_0\times\Theta_0^{'}.$$

PROOF. We obtain: $\forall u\times u'\in U\times U'$,

$$(\phi\times\phi')_0(u\times u')=\{(x(-\infty+0),x'(-\infty+0))|x\times x'\in(f\times f')(u\times u')\}=$$
$$=\{(x(-\infty+0),x'(-\infty+0))|x\times x'\in f(u)\times f'(u')\}=$$
$$=\{(x(-\infty+0),x'(-\infty+0))|x\in f(u),x'\in f'(u')\}=$$
$$=\{x(-\infty+0)|x\in f(u)\}\times\{x'(-\infty+0)|x'\in f'(u')\}=\phi_0(u)\times\phi_0'(u'),$$
$$(\Theta\times\Theta')_0=\bigcup_{u\times u'\in U\times U'}(\phi\times\phi')_0(u\times u')=\bigcup_{u\times u'\in U\times U'}\phi_0(u)\times\phi_0'(u')=$$
$$=\bigcup_{u\in U}\phi_0(u)\times\bigcup_{u'\in U'}\phi_0'(u')=\Theta_0\times\Theta_0^{'}.$$

□

THEOREM 62. *If f,f' have final states, we denote by ϕ_f,ϕ_f' their final state functions and by $(\phi\times\phi')_f:U\times U'\to P^*(\mathbf{B}^{n+n'})$ the final state function of $f\times f'$. Denote by $\Theta_f,\Theta_f^{'}$ the sets of the final states of f and f' and by $(\Theta\times\Theta')_f$ the set of the final states of $f\times f'$. We have*

$$\forall u\times u'\in U\times U',(\phi\times\phi')_f(u\times u')=\phi_f(u)\times\phi_f'(u'),$$
$$(\Theta\times\Theta')_f=\Theta_f\times\Theta_f^{'}.$$

THEOREM 63. *Let be the systems $f:U\to P^*(S^{(n)})$, $g:V\to P^*(S^{(n)})$, $f':U'\to P^*(S^{(n')})$, $g':V'\to P^*(S^{(n')})$ with $U,V\in P^*(S^{(m)})$ and $U',V'\in P^*(S^{(m')})$. We have that $f\subset g$ and $f'\subset g'$ iff $f\times f'\subset g\times g'$.*

PROOF. We have

$$f\subset g \text{ and } f'\subset g'$$
$$\Longleftrightarrow (U\subset V \; and \; \forall u\in U,f(u)\subset g(u)) \; and \; (U'\subset V' \; and \; \forall u'\in U',f'(u')\subset g'(u'))$$
$$\Longleftrightarrow U\subset V \; and \; U'\subset V' \; and \; \forall u\times u'\in U\times U',f(u)\subset g(u) \; and \; f'(u')\subset g'(u')$$
$$\Longleftrightarrow U\times U'\subset V\times V' \; and \; \forall u\times u'\in U\times U',f(u)\times f'(u')\subset g(u)\times g'(u')$$
$$\Longleftrightarrow U\times U'\subset V\times V' \; and \; \forall u\times u'\in U\times U',(f\times f')(u\times u')\subset(g\times g')(u\times u')$$
$$\Longleftrightarrow f\times f'\subset g\times g'.$$

□

THEOREM 64. *For any systems f,f' we have $(f\times f')^*=f^*\times f'^*$.*

PROOF. First of all we note that $(U\times U')^*=U^*\times U'^*$. Furthermore, for all $u\times u'\in(U\times U')^*$, we can write

$$(f\times f')^*(u\times u')=\{\overline{x\times x'}|x\times x'\in(f\times f')(\overline{u\times u'})\}=$$
$$=\{\overline{x}\times\overline{x'}|x\times x'\in(f\times f')(\overline{u}\times\overline{u'})\}=\{\overline{x}\times\overline{x'}|x\times x'\in f(\overline{u})\times f'(\overline{u'})\}=$$
$$=\{\overline{x}\times\overline{x'}|x\in f(\overline{u}),x'\in f'(\overline{u'})\}=\{\overline{x}\times\overline{x'}|\overline{x}\in f^*(u),\overline{x'}\in f'^*(u')\}=$$
$$=\{x\times x'|x\in f^*(u),x'\in f'^*(u')\}=\{x\times x'|x\times x'\in f^*(u)\times f'^*(u')\}=$$
$$=\{x\times x'|x\times x'\in(f^*\times f'^*)(u\times u')\}=(f^*\times f'^*)(u\times u').$$

□

THEOREM 65. *Let be f and f'. We have $(f\times f')^{-1}=f^{-1}\times f'^{-1}$.*

PROOF. Denote by W, W' the supports of $(f\times f')^{-1}$, $f^{-1}\times f'^{-1}$. We have

$$W=\bigcup_{u\times u'\in U\times U'}(f\times f')(u\times u')=\bigcup_{u\times u'\in U\times U'}f(u)\times f'(u')=$$

$$=\bigcup_{u\in U}f(u)\times\bigcup_{u'\in U'}f'(u')=W'.$$

Thus, for any $x\times x'\in W$, we can write

$$\begin{aligned}(f\times f')^{-1}(x\times x') &= \{u\times u'|u\times u'\in U\times U', x\times x'\in(f\times f')(u\times u')\}\\ &= \{u\times u'|u\times u'\in U\times U', x\times x'\in f(u)\times f'(u')\}\\ &= \{u\times u'|u\in U, u'\in U', x\in f(u), x'\in f'(u')\}\\ &= \{u\times u'|u\in f^{-1}(x), u'\in f'^{-1}(x')\}\\ &= \{u\times u'|u\times u'\in f^{-1}(x)\times f'^{-1}(x')\}\\ &= \{u\times u'|u\times u'\in(f^{-1}\times f'^{-1})(x\times x')\}\\ &= (f^{-1}\times f'^{-1})(x\times x').\end{aligned}$$

□

9. Parallel connection

THEOREM 66. *The pseudo-systems* $f:\widetilde{S}^{(m)}\to P(\widetilde{S}^{(n)}), f_1':\widetilde{S}^{(m)}\to P(\widetilde{S}^{(n')})$ *are given for which we define* $(f,f_1'):\widetilde{S}^{(m)}\to P(\widetilde{S}^{(n+n')})$ *by*

$$\forall u\in\widetilde{S}^{(m)}, (f,f_1')(u)=(f\times f_1')(u\times u).$$

If U, U_1' *are the support sets of* f, f_1' *then the support of* (f,f_1') *is* $U\cap U_1'$. *If* f, f_1' *are systems satisfying* $U\cap U_1'\neq\emptyset$, *then* (f,f_1') *is a system.*

PROOF. Obviously $U\cap U_1'$ is the support of (f,f_1'). Suppose that f and f_1' are systems and $U\cap U_1'\neq\emptyset$. From $U\subset S^{(m)}, U_1'\subset S^{(m)}$ we infer $U\cap U_1'\subset S^{(m)}$ and, moreover, $\forall u\in U, f(u)\subset S^{(n)}$, $\forall u\in U_1', f_1'(u)\subset S^{(n')}$ imply $\forall u\in U\cap U_1', f(u)\times f_1'(u)\subset S^{(n)}\times S^{(n')}=S^{(n+n')}$. □

DEFINITION 43. *Consider the systems* $f:U\to P^*(S^{(n)})$, $f_1':U_1'\to P^*(S^{(n')})$, $U, U_1'\in P^*(S^{(m)})$ *with* $U\cap U_1'\neq\emptyset$. *The system* $(f,f_1'):U\cap U_1'\to P^*(S^{(n+n')})$ *defined by*

$$\forall u\in U\cap U_1', (f,f_1')(u)=(f\times f_1')(u\times u)$$

is called the ***parallel connection*** *of the systems* f *and* f_1'.

REMARK 26. *The parallel connection of two systems* f *and* f_1' *is the system that represents* f, f_1' *acting independently on each other under the same input. The study of the parallel connection of the systems is made in quite similar terms with the study of the Cartesian product of systems from the previous section.*

Suppose that $f:U\to P^*(S^{(n)}), f_1':U_1'\to P^*(S^{(n')}), f_1'':U_1''\to P^*(S^{(n'')})$ *are three systems where* $U, U_1', U_1''\in P^*(S^{(m)})$. *If* $U\cap U_1'\cap U_1''\neq\emptyset$, *we identify* $((f,f_1'),f_1'')$ *with* $(f,(f_1',f_1''))$; *any of them is denoted by* (f,f_1',f_1''). *The support set of* (f,f_1',f_1'') *is* $U\cap U_1'\cap U_1''\in P^*(S^{(m)})$, *its range is* $P^*(S^{(n+n'+n'')})$, *its inputs are* $u\in U\cap U_1'\cap U_1''$ *and its states are* $x\times x'\times x''\in(f,f_1',f_1'')(u)$. *This is the associativity of the law of parallel connection of the systems.*

10. Serial connection

THEOREM 67. *Let be the pseudo-systems* $f : \widetilde{S}^{(m)} \to P(\widetilde{S}^{(n)})$ *with the support* $U \subset \widetilde{S}^{(m)}$ *and* $h : \widetilde{S}^{(n)} \to P(\widetilde{S}^{(p)})$ *with the support* $X \subset \widetilde{S}^{(n)}$. *We ask that* $\bigcup_{u \in U} f(u) \subset X$ *is true and we define the pseudo-system* $h \circ f : \widetilde{S}^{(m)} \to P(\widetilde{S}^{(p)})$ *in the following manner:*

$$\forall u \in \widetilde{S}^{(m)}, (h \circ f)(u) = \bigcup_{x \in f(u)} h(x).$$

We have that the support set of $h \circ f$ *is* U. *If* f, h *are systems then* $h \circ f$ *is a system.*

PROOF. Obviously, U is the support set of $h \circ f$. Suppose that f, h are systems, thus $U \neq \emptyset$. We have $U \subset S^{(m)}$ and

$$\forall u \in U, (h \circ f)(u) = \bigcup_{x \in f(u)} h(x) \subset \bigcup_{x \in X} h(x) \subset S^{(p)}$$

therefore $h \circ f$ is a system. □

DEFINITION 44. *Consider the systems* $f : U \to P^*(S^{(n)})$, $U \in P^*(S^{(m)})$ *and* $h : X \to P^*(S^{(p)})$, $X \in P^*(S^{(n)})$ *and suppose that the condition* $\bigcup_{u \in U} f(u) \subset X$ *is fulfilled. The system* $h \circ f : U \to P^*(S^{(p)})$ *defined as*

$$\forall u \in U, (h \circ f)(u) = \bigcup_{x \in f(u)} h(x)$$

is called the ***serial connection*** *of the systems* h *and* f.

REMARK 27. *The system* $h \circ f$ *coincides with the composition of* h *and* f, *thought of as relations. In the situation when* $\bigcup_{u \in U} f(u) \subset X$, *we interpret* $h \circ f$ *as the sequential action of* f *and* h: f *acts first and* h *second, the states of* f *being the inputs of* h. *Thus, the possible states of* f *become the possible inputs of* h, *representing a certain loss of precision that occurs when modeling large circuits.*

Consider the systems f, h *and* $k : Z \to P^*(S^{(q)}), Z \in P^*(S^{(p)})$ *with* $\bigcup_{u \in U} f(u) \subset X$ *and* $\bigcup_{x \in X} h(x) \subset Z$ *true. The associativity of the serial connection of* k, h *and* f *consists in noting that*

$$\forall u \in U, ((k \circ h) \circ f)(u) = \bigcup_{x \in f(u)} (k \circ h)(x) = \bigcup_{x \in f(u)} \bigcup_{y \in h(x)} k(y) =$$

$$= \bigcup_{y \in \bigcup_{x \in f(u)} h(x)} k(y) = \bigcup_{y \in (h \circ f)(u)} k(y) = (k \circ (h \circ f))(u).$$

On the other hand if $1_U : U \to S^{(m)}$ *is the canonical injection and* $1_{S^{(n)}} : S^{(n)} \to S^{(n)}$ *is the identity, we note that*

$$\forall u \in U, (f \circ 1_U)(u) = \bigcup_{v = 1_U(u)} f(v) = f(u),$$

$$\forall u \in U, (1_{S^{(n)}} \circ f)(u) = \bigcup_{x \in f(u)} 1_{S^{(n)}}(x) = f(u).$$

We mention a version of Definition 44 that we have used in previous works: instead of $\bigcup\limits_{u\in U} f(u)\subset X$ *we ask that* $\bigcup\limits_{u\in U} f(u)\cap X\neq\emptyset$ *is fulfilled and* $h\circ f:W\rightarrow P^*(S^{(n)})$ *is defined as*

$$W=\{u|u\in U,f(u)\cap X\neq\emptyset\},$$
$$\forall u\in W,(h\circ f)(u)=\bigcup_{x\in f(u)\cap X} h(x).$$

THEOREM 68. *Let be the systems* f,h *such that* $\bigcup\limits_{u\in U} f(u)\subset X$. *If* h *has a constant initial state, then* $h\circ f$ *has a constant initial state too.*

PROOF. From

$$\exists\nu\in\mathbf{B}^p,\forall x\in X,\forall y\in h(x),\exists t_0\in\mathbf{R},\forall t<t_0,y(t)=\nu$$

we infer

$$\exists\nu\in\mathbf{B}^p,\forall x\in\bigcup_{u\in U} f(u),\forall y\in h(x),\exists t_0\in\mathbf{R},\forall t<t_0,y(t)=\nu,$$
$$\exists\nu\in\mathbf{B}^p,\forall u\in U,\forall x\in f(u),\forall y\in h(x),\exists t_0\in\mathbf{R},\forall t<t_0,y(t)=\nu,$$
$$\exists\nu\in\mathbf{B}^p,\forall u\in U,\forall y\in (h\circ f)(u),\exists t_0\in\mathbf{R},\forall t<t_0,y(t)=\nu.$$

□

THEOREM 69. *If* h *has final states (a constant final state) and the system* $h\circ f$ *is defined, then* $h\circ f$ *has final states (a constant final state).*

THEOREM 70. *Let be the systems* f *and* h *with* $\bigcup\limits_{u\in U} f(u)\subset X$. *If* h *has a fixed initial time, then* $h\circ f$ *has a fixed initial time.*

PROOF. From

$$\exists t_0\in\mathbf{R},\forall x\in X,\forall y\in h(x),\exists\nu\in\mathbf{B}^p,\forall t<t_0,y(t)=\nu$$

we get

$$\exists t_0\in\mathbf{R},\forall x\in\bigcup_{u\in U} f(u),\forall y\in h(x),\exists\nu\in\mathbf{B}^p,\forall t<t_0,y(t)=\nu,$$
$$\exists t_0\in\mathbf{R},\forall u\in U,\forall x\in f(u),\forall y\in h(x),\exists\nu\in\mathbf{B}^p,\forall t<t_0,y(t)=\nu,$$
$$\exists t_0\in\mathbf{R},\forall u\in U,\forall y\in (h\circ f)(u),\exists\nu\in\mathbf{B}^p,\forall t<t_0,y(t)=\nu.$$

□

THEOREM 71. *If* h *has a fixed final time and* $\bigcup\limits_{u\in U} f(u)\subset X$, *then* $h\circ f$ *has a fixed final time.*

THEOREM 72. *We consider the systems* f *and* h *having the property that the serial connection system* $h\circ f$ *exists. Denote by* η_0,δ_0 *and* Δ_0 *the initial state functions of* $h,h\circ f$ *and the set of initial states of* $h\circ f$ *respectively. The following formulae are true:*

$$\forall u\in U,\delta_0(u)=\bigcup_{x\in f(u)}\eta_0(x),$$
$$\Delta_0=\bigcup_{u\in U}\bigcup_{x\in f(u)}\eta_0(x).$$

PROOF. We have

$$\forall u \in U, \delta_0(u) = \{y(-\infty+0)|y \in (h \circ f)(u)\} = \{y(-\infty+0)|y \in \bigcup_{x \in f(u)} h(x)\} =$$

$$= \bigcup_{x \in f(u)} \{y(-\infty+0)|y \in h(x)\} = \bigcup_{x \in f(u)} \eta_0(x).$$

We conclude that

$$\Delta_0 = \bigcup_{u \in U} \delta_0(u) = \bigcup_{u \in U} \bigcup_{x \in f(u)} \eta_0(x).$$

□

THEOREM 73. *Let be the systems f and h such that $h \circ f$ exists. Suppose that h has final states and use the notations η_f, δ_f and Δ_f for the final state functions of $h, h \circ f$ and for the set of final states of $h \circ f$ respectively. The following formulae are true:*

$$\forall u \in U, \delta_f(u) = \bigcup_{x \in f(u)} \eta_f(x),$$

$$\Delta_f = \bigcup_{u \in U} \bigcup_{x \in f(u)} \eta_f(x).$$

THEOREM 74. *Let us consider the systems $f : U \to P^*(S^{(n)})$, $g : V \to P^*(S^{(n)})$, $U, V \in P^*(S^{(m)})$ and $h : X \to P^*(S^{(p)})$, $h_1 : X_1 \to P^*(S^{(p)})$, $X, X_1 \in P^*(S^{(n)})$. We have:*

a) if $\bigcup_{u \in V} g(u) \subset X$ and $f \subset g$ then $\bigcup_{u \in U} f(u) \subset X$ and $h \circ f \subset h \circ g$;

b) if $\bigcup_{u \in U} f(u) \subset X$ and $h \subset h_1$ then $\bigcup_{u \in U} f(u) \subset X_1$ and $h \circ f \subset h_1 \circ f$.

PROOF. a) The hypothesis $\bigcup_{u \in V} g(u) \subset X$, $U \subset V$ and $\forall u \in U, f(u) \subset g(u)$ shows that

$$\bigcup_{u \in U} f(u) \subset \bigcup_{u \in U} g(u) \subset \bigcup_{u \in V} g(u) \subset X$$

and we have

$$\forall u \in U, (h \circ f)(u) = \bigcup_{x \in f(u)} h(x) \subset \bigcup_{x \in g(u)} h(x) = (h \circ g)(u).$$

b) The hypothesis states that $\bigcup_{u \in U} f(u) \subset X, X \subset X_1$ and $\forall x \in X, h(x) \subset h_1(x)$, wherefrom

$$\bigcup_{u \in U} f(u) \subset X \subset X_1,$$

$$\forall u \in U, (h \circ f)(u) = \bigcup_{x \in f(u)} h(x) \subset \bigcup_{x \in f(u)} h_1(x) = (h_1 \circ f)(u).$$

□

THEOREM 75. *Let be the systems f and h with $\bigcup_{u \in U} f(u) \subset X$. We have that $(h \circ f)^*$ and $h^* \circ f^*$ exist and $(h \circ f)^* = h^* \circ f^*$.*

Proof. The condition of existence of $(h\circ f)^*$ coincides with the condition of existence of $h\circ f$. It is fulfilled. The condition of existence of $h^*\circ f^*$ is obtained by passing in the inclusion $\bigcup\limits_{u\in U} f(u)\subset X$ to the complementary of all functions:

$$\bigcup_{u\in U^*} f^*(u)=\bigcup_{u\in U} f^*(\overline{u})=\bigcup_{u\in U}(f(u))^*=(\bigcup_{u\in U} f(u))^*\subset X^*.$$

Both $(h\circ f)^*$ and $h^*\circ f^*$ have the domain U^*, thus for any $u\in U^*$ we have:

$$(h\circ f)^*(u)=((h\circ f)(\overline{u}))^*=(\bigcup_{x\in f(\overline{u})} h(x))^*=\bigcup_{x\in f(\overline{u})}(h(x))^*=$$

$$=\bigcup_{x\in f(\overline{u})} h^*(\overline{x})=\bigcup_{\overline{x}\in f^*(u)} h^*(\overline{x})=(h^*\circ f^*)(u).$$

□

Theorem 76. *For any system f, the systems $f^{-1}\circ f$ and $f\circ f^{-1}$ have the support sets U and $X=\bigcup\limits_{u\in U} f(u)$ and the following statements hold:*

$$\forall u\in U,(f^{-1}\circ f)(u)=\{u'|u'\in U,f(u)\cap f(u')\neq\emptyset\},$$
$$\forall x\in X,(f\circ f^{-1})(x)=\{x'|x'\in X,f^{-1}(x)\cap f^{-1}(x')\neq\emptyset\}.$$

Proof. Because $\bigcup\limits_{u\in U} f(u)=X$, $f^{-1}\circ f$ exists and has the domain U. We have

$$\forall u\in U,(f^{-1}\circ f)(u)=\bigcup_{x\in f(u)} f^{-1}(x)=\bigcup_{x\in f(u)}\{u'|u'\in U,x\in f(u')\}=$$

$$=\{u'|u'\in U,\exists x\in f(u')\cap f(u)\}=\{u'|u'\in U,f(u')\cap f(u)\neq\emptyset\}$$

and similarly for the other statement. □

Theorem 77. *Let be $f:U\to P^*(S^{(n)}),U\in P^*(S^{(m)})$ and $h:X\to P^*(S^{(p)})$, $X\in P^*(S^{(n)})$, such that the condition $\bigcup\limits_{u\in U} f(u)=X$ is fulfilled. We have $(h\circ f)^{-1}=f^{-1}\circ h^{-1}$.*

Proof. Denote $Y=\bigcup\limits_{x\in X} h(x)$. From $\bigcup\limits_{u\in U} f(u)=X$ we infer that $h\circ f$ exists, thus $(h\circ f)^{-1}:Y\to P^*(S^{(m)})$ exists and from $\bigcup\limits_{y\in Y} h^{-1}(y)=X$ we infer that $f^{-1}\circ h^{-1}:Y\to P^*(S^{(m)})$ exists. We can write:

$$\forall y\in Y,(h\circ f)^{-1}(y)=\{u|u\in U,y\in(h\circ f)(u)\}=\{u|u\in U,\exists x,x\in f(u),y\in h(x)\}=$$

$$=\{u|u\in U,\exists x,x\in h^{-1}(y),u\in f^{-1}(x)\}=\bigcup_{x\in h^{-1}(y)} f^{-1}(x)=(f^{-1}\circ h^{-1})(y).$$

□

Theorem 78. *Consider the systems $f:U\to P^*(S^{(n)})$, $U\in P^*(S^{(m)})$, $f':U'\to P^*(S^{(n')})$, $U'\in P^*(S^{(m')})$, $h:X\to P^*(S^{(p)})$, $X\in P^*(S^{(n)})$ and $h':X'\to P^*(S^{(p')})$, $X'\in P^*(S^{(n')})$. If $\bigcup\limits_{u\in U} f(u)\subset X$ and $\bigcup\limits_{u'\in U'} f'(u')\subset X'$ are true, then the following formula*

$$(h\times h')\circ(f\times f')=(h\circ f)\times(h'\circ f')$$

holds.

PROOF. Because

$$\bigcup_{u\times u'\in U\times U'}(f\times f')(u\times u')=\bigcup_{u\times u'\in U\times U'}f(u)\times f'(u')=\bigcup_{u\in U}f(u)\times\bigcup_{u'\in U'}f'(u')\subset X\times X'$$

we obtain that $(h\times h')\circ(f\times f')$ exists. But $h\circ f$ and $h'\circ f'$ exist themselves, from the hypothesis, wherefrom $(h\circ f)\times(h'\circ f')$ exists. We can write

$$\forall u\times u'\in U\times U',((h\times h')\circ(f\times f'))(u\times u')=\bigcup_{x\times x'\in(f\times f')(u\times u')}(h\times h')(x\times x')=$$

$$=\bigcup_{x\times x'\in f(u)\times f'(u')}h(x)\times h'(x')=\bigcup_{x\in f(u)}h(x)\times\bigcup_{x'\in f'(u')}h'(x')=$$

$$=(h\circ f)(u)\times(h'\circ f')(u')=((h\circ f)\times(h'\circ f'))(u\times u').$$

□

THEOREM 79. *Let be the systems $f:U\to P^*(S^{(n)})$, $U\in P^*(S^{(m)})$, $f_1':U_1'\to P^*(S^{(n')})$, $U_1'\in P^*(S^{(m)})$, $h:X\to P^*(S^{(p)})$, $X\in P^*(S^{(n)})$, $h':X'\to P^*(S^{(p')})$, $X'\in P^*(S^{(n')})$. We suppose that $U\cap U_1'\neq\emptyset$ and the inclusions $\bigcup_{u\in U\cap U_1'}f(u)\subset X$, $\bigcup_{u\in U\cap U_1'}f_1'(u)\subset X'$ are true. In these conditions we have*

$$(h\times h')\circ(f,f_1')=(h\circ f,h'\circ f_1').$$

PROOF. (f,f_1') is defined and has the support $U\cap U_1'$. $(h\times h')\circ(f,f_1')$ is defined because

$$\bigcup_{u\in U\cap U_1'}(f,f_1')(u)=\bigcup_{u\in U\cap U_1'}f(u)\times f_1'(u)=\bigcup_{u\in U\cap U_1'}f(u)\times\bigcup_{u\in U\cap U_1'}f_1'(u)\subset$$

$$\subset\bigcup_{u\in U}f(u)\times\bigcup_{u\in U_1'}f_1'(u)\subset X\times X'.$$

The systems $h\circ f$ and $h'\circ f_1'$ are defined, from the hypothesis and have the supports U and U_1', thus the system $(h\circ f,h'\circ f_1')$ exists and has the support $U\cap U_1'$ etc. □

11. The Complement, an open problem

DEFINITION 45. *Let be $f:U\to P^*(S^{(n)}),U\in P^*(S^{(m)})$. The system $Cf:W\to P^*(S^{(n)})$ defined by*

$$W=\{u|u\in U,f(u)\neq S^{(n)}\}\cup(S^{(m)}\setminus U),$$

$$\forall u\in W,Cf(u)=\begin{cases}S^{(n)}\setminus f(u),\ u\in U\\ S^{(n)},\ u\notin U\end{cases}$$

where $W\neq\emptyset$, is called the ***complement*** *of f.*

REMARK 28. *Intuitively, if $x\in f(u)$ are those states that model a circuit, then $x\in Cf(u)$ are the states that do not model the circuit.*

Because both the naturalness and the utility of the complement Cf are not obvious at this moment, we leave its use as an open problem.

12. Intersection

THEOREM 80. *Consider the pseudo-systems* $f, g : \widetilde{S}^{(m)} \to P(\widetilde{S}^{(n)})$ *for which we define the pseudo-system* $f \cap g : \widetilde{S}^{(m)} \to P(\widetilde{S}^{(n)})$ *by*

$$\forall u \in \widetilde{S}^{(m)}, (f \cap g)(u) = f(u) \cap g(u).$$

Denote by U *and* V *the supports of* f *and* g. *We have that the support of* $f \cap g$ *is*

$$W = \{u | u \in U \cap V, f(u) \cap g(u) \neq \emptyset\}. \tag{12.1}$$

If f *and* g *are systems and* $\exists u \in U \cap V, f(u) \cap g(u) \neq \emptyset$, *then* $f \cap g$ *is a system.*

PROOF. Suppose that f, g are systems and $W \neq \emptyset$. From $W \subset U \subset S^{(m)}$ and $\forall u \in W, (f \cap g)(u) = f(u) \cap g(u) \subset f(u) \subset S^{(n)}$ we infer that $f \cap g$ is a system. $\square$

DEFINITION 46. *Let be the systems* $f : U \to P^*(S^{(n)})$, $g : V \to P^*(S^{(n)})$ *with* $U, V \in P^*(S^{(m)})$. *If* $\exists u \in U \cap V, f(u) \cap g(u) \neq \emptyset$, *the system* $f \cap g : W \to P^*(S^{(n)})$ *defined by*

$$\forall u \in W, (f \cap g)(u) = f(u) \cap g(u),$$

where W *satisfies (12.1), is called the* ***intersection*** *of* f *and* g.

REMARK 29. *The intersection of the pseudo-systems represents the gain of information (of precision) in the modeling of a circuit that follows by considering the simultaneous validity of two (consistent!) models.*

In the special case when the system g *is constant:* $g : S^{(m)} \to P^*(S^{(n)})$, $\forall u \in S^{(m)}$, $g(u) = X$, *where* $X \subset S^{(n)}$ *is some space of functions, then* $f \cap X : W \to P^*(S^{(n)})$ *is the system given by*

$$W = \{u | u \in U, f(u) \cap X \neq \emptyset\},$$

$$\forall u \in W, (f \cap X)(u) = f(u) \cap X.$$

We interpret the system $f \cap X$ *in the following manner: when* f *models a circuit,* $f \cap X$ *represents a gain of information following from the statement of a requirement that does not depend on* u.

EXAMPLE 21. *We give some possibilities of choosing, in the intersection* $f \cap g$, *the constant system* $g = X$:

a) the initial value of the states is null;

b) the coordinates $x_1, ..., x_n$ *of the states are monotonous;*

c) at each time instant, at least one coordinate of the state should be 1:

$$X = \{x | x \in S^{(n)}, x_1(t) \cup ... \cup x_n(t) = 1\};$$

d) the state is allowed to switch with at most one coordinate at a time:

$$X = \{x | x \in S^{(n)}, \forall t, x(t-0) \neq x(t) \Longrightarrow \exists ! i \in \{1, ..., n\}, Dx_i(t) = 1\};$$

e) X *represents a stuck at 1 fault:*

$$\exists i \in \{1, ..., n\}, X = \{x | x \in S^{(n)}, x_i(t) = 1\}.$$

This last choice of X *is interesting in designing systems for testability, respectively in designing redundant systems.*

THEOREM 81. *Let be the systems* $f : U \to P^*(S^{(n)})$ *and* $g : V \to P^*(S^{(n)})$, $U, V \in P^*(S^{(m)})$. *If* $\exists u \in U \cap V, f(u) \cap g(u) \neq \emptyset$ *and* f *has race-free initial states (a constant initial state), then* $f \cap g$ *has race-free initial states (a constant initial state).*

PROOF. $f \cap g \subset f$ and the statement of the theorem follows from Theorem 35. □

THEOREM 82. *If f has final states (race-free final states, a constant final state) and $f \cap g$ exists, then $f \cap g$ has final states (race-free final states, a constant final state).*

THEOREM 83. *If f has a bounded initial time (a fixed initial time) and $f \cap g$ exists, then $f \cap g$ has a bounded initial time (a fixed initial time).*

PROOF. $f \cap g \subset f$ and the results follow from Theorem 37. □

THEOREM 84. *If f has a bounded final time (a fixed final time) and $f \cap g$ exists, then $f \cap g$ has a bounded final time (a fixed final time).*

THEOREM 85. *Let be the systems f, g. We have $(\phi \cap \gamma)_0 : W \to P^*(\mathbf{B}^n)$,*

$$\forall u \in W, (\phi \cap \gamma)_0(u) = \phi_0(u) \cap \gamma_0(u),$$

$$(\Theta \cap \Gamma)_0 = \bigcup_{u \in W} (\phi \cap \gamma)_0(u).$$

We have supposed that the domain W of $f \cap g$ is non-empty and we have denoted by $\phi_0, \gamma_0, (\phi \cap \gamma)_0$ the initial state functions of $f, g, f \cap g$ and by $(\Theta \cap \Gamma)_0$ the set of initial states of $f \cap g$.

PROOF. We can write that $\forall u \in W$,

$$(\phi \cap \gamma)_0(u) = \{x(-\infty + 0) | x \in (f \cap g)(u)\} = \{x(-\infty + 0) | x \in f(u) \cap g(u)\} =$$

$$= \{x(-\infty + 0) | x \in f(u)\} \cap \{x(-\infty + 0) | x \in g(u)\} = \phi_0(u) \cap \gamma_0(u).$$

□

THEOREM 86. *If f, g have final states, then we have $(\phi \cap \gamma)_f : W \to P^*(\mathbf{B}^n)$,*

$$\forall u \in W, (\phi \cap \gamma)_f(u) = \phi_f(u) \cap \gamma_f(u),$$

$$(\Theta \cap \Gamma)_f = \bigcup_{u \in W} (\phi \cap \gamma)_f(u).$$

We have supposed that $W \neq \emptyset$ and the notations are obvious and similar with those from the previous theorem.

THEOREM 87. *Let be the systems $f : U \to P^*(S^{(n)})$, $f_1 : U_1 \to P^*(S^{(n)})$, $g : V \to P^*(S^{(n)})$ with $U, U_1, V \in P^*(S^{(m)})$. If $f \subset f_1$ and if $f \cap g$ exists, then $f_1 \cap g$ exists and the inclusion $f \cap g \subset f_1 \cap g$ is true.*

PROOF. Denote by W the set from (12.1) and by W_1 the set

$$W_1 = \{u | u \in U_1 \cap V, f_1(u) \cap g(u) \neq \emptyset\}.$$

From the fact that $U \subset U_1$, $\forall u \in U, f(u) \subset f_1(u)$ and $W \neq \emptyset$, we infer $W \subset W_1$, $W_1 \neq \emptyset$ and, furthermore, we have $\forall u \in W, (f \cap g)(u) = f(u) \cap g(u) \subset f_1(u) \cap g(u) = (f_1 \cap g)(u)$. □

THEOREM 88. *If $f \cap g$ exists, then $(f \cap g)^*$, $f^* \cap g^*$ exist and*

$$(f \cap g)^* = f^* \cap g^*.$$

Proof. Denote by W the domain (12.1) of $f \cap g$. The domain of $(f \cap g)^*$ is W^* and the domain W_1 of $f^* \cap g^*$ is

$$W_1 = \{u|u \in U^* \cap V^*, f^*(u) \cap g^*(u) \neq \emptyset\} =$$

$$= \{u|\overline{u} \in U \cap V, \{\overline{x}|x \in f(\overline{u})\} \cap \{\overline{x}|x \in g(\overline{u})\} \neq \emptyset\} =$$

$$= \{\overline{u}|u \in U \cap V, \{\overline{x}|x \in f(u)\} \cap \{\overline{x}|x \in g(u)\} \neq \emptyset\} =$$

$$= \{\overline{u}|u \in U \cap V, \{x|x \in f(u)\} \cap \{x|x \in g(u)\} \neq \emptyset\} = W^*.$$

Moreover, for any $u \in W^*$, we infer

$$(f \cap g)^*(u) = \{\overline{x}|x \in (f \cap g)(\overline{u})\} = \{\overline{x}|x \in f(\overline{u}) \cap g(\overline{u})\} =$$

$$= \{\overline{x}|x \in f(\overline{u})\} \cap \{\overline{x}|x \in g(\overline{u})\} = f^*(u) \cap g^*(u) = (f^* \cap g^*)(u).$$

□

Theorem 89. *Let be the systems $f : U \to P^*(S^{(n)}), g : V \to P^*(S^{(n)}), U, V \in P^*(S^{(m)})$. If $\exists u \in U \cap V, f(u) \cap g(u) \neq \emptyset$, then the systems $(f \cap g)^{-1}$, $f^{-1} \cap g^{-1}$ exist and they have the same support set*

$$X = \bigcup_{u \in U \cap V} (f(u) \cap g(u)).$$

Furthermore, we have

$$(f \cap g)^{-1} = f^{-1} \cap g^{-1}.$$

Proof. Obviously X is the support set of $(f \cap g)^{-1}$. We can write

$$X = \bigcup_{u \in U \cap V} (f(u) \cap g(u)) = \{x|\exists u \in U \cap V, x \in f(u) \cap g(u)\} =$$

$$= \{x|x \in \bigcup_{v \in U} f(v) \cap \bigcup_{v \in V} g(v), \exists u \in U \cap V, u \in f^{-1}(x) \text{ and } u \in g^{-1}(x)\} =$$

$$= \{x|x \in \bigcup_{v \in U} f(v) \cap \bigcup_{v \in V} g(v), f^{-1}(x) \cap g^{-1}(x) \neq \emptyset\}.$$

Thus X is the support of $f^{-1} \cap g^{-1}$ too. We have $\forall x \in X$,

$$(f \cap g)^{-1}(x) = \{u|u \in U \cap V, x \in (f \cap g)(u)\} = \{u|u \in U \cap V, x \in f(u) \cap g(u)\} =$$

$$= \{u|u \in U \cap V, x \in f(u)\} \cap \{u|u \in U \cap V, x \in g(u)\} =$$

$$= \{u|u \in U, x \in f(u)\} \cap \{u|u \in V, x \in g(u)\} = f^{-1}(x) \cap g^{-1}(x) = (f^{-1} \cap g^{-1})(x).$$

□

Theorem 90. *Let be the systems $f : U \to P^*(S^{(n)})$, $g : V \to P^*(S^{(n)})$, $U, V \in P^*(S^{(m)})$ and $f' : U' \to P^*(S^{(n')})$, $U' \in P^*(S^{(m')})$. If $\exists u \in U \cap V, f(u) \cap g(u) \neq \emptyset$, then the systems $(f \cap g) \times f', (f \times f') \cap (g \times f')$ are defined and $W \times U'$ is their common support, where we have put*

$$W = \{u|u \in U \cap V, f(u) \cap g(u) \neq \emptyset\}.$$

We have the equality

$$(f \cap g) \times f' = (f \times f') \cap (g \times f').$$

PROOF. We show that $W \times U'$, the support set of $(f \cap g) \times f'$, is also the support set of $(f \times f') \cap (g \times f')$:

$$\{u \times u' | u \times u' \in (U \times U') \cap (V \times U'), (f \times f')(u \times u') \cap (g \times f')(u \times u') \neq \emptyset\} =$$
$$= \{u \times u' | u \times u' \in (U \cap V) \times U', (f(u) \times f'(u')) \cap (g(u) \times f'(u')) \neq \emptyset\} =$$
$$= \{u \times u' | u \in U \cap V, u' \in U', f(u) \cap g(u) \neq \emptyset \text{ and } f'(u') \neq \emptyset\} =$$
$$= \{u \times u' | u \in W, u' \in U'\} = W \times U'.$$

Furthermore, for any $u \times u' \in W \times U'$, we have

$$((f \cap g) \times f')(u \times u') = (f \cap g)(u) \times f'(u') = (f(u) \cap g(u)) \times f'(u') =$$
$$= (f(u) \times f'(u')) \cap (g(u) \times f'(u')) = (f \times f')(u \times u') \cap (g \times f')(u \times u') =$$
$$= ((f \times f') \cap (g \times f'))(u \times u').$$

$\square$

THEOREM 91. *Consider the systems $f : U \to P^*(S^{(n)})$, $g : V \to P^*(S^{(n)})$, $f_1' : U_1' \to P^*(S^{(n')})$, with $U, V, U_1' \in P^*(S^{(m)})$. Suppose that $\exists u \in U \cap V \cap U_1'$, such that $f(u) \cap g(u) \neq \emptyset$. Then the set*

$$W' = \{u | u \in U \cap V \cap U_1', f(u) \cap g(u) \neq \emptyset\}$$

is the non-empty support of the systems $(f \cap g, f_1')$, $(f, f_1') \cap (g, f_1')$ and the equality

$$(f \cap g, f_1') = (f, f_1') \cap (g, f_1')$$

holds true.

PROOF. The set W' is the support of $(f \cap g, f_1')$ and we show that it is also the support of $(f, f_1') \cap (g, f_1')$. Denote by

$$W'' = \{u | u \in (U \cap U_1') \cap (V \cap U_1'), (f, f_1')(u) \cap (g, f_1')(u) \neq \emptyset\}$$

the support set of $(f, f_1') \cap (g, f_1')$ for which we have

$$W'' = \{u | u \in U \cap V \cap U_1', (f(u) \times f_1'(u)) \cap (g(u) \times f_1'(u)) \neq \emptyset\} =$$
$$= \{u | u \in U \cap V \cap U_1', (f(u) \cap g(u)) \times f_1'(u) \neq \emptyset\} =$$
$$= \{u | u \in U \cap V \cap U_1', f(u) \cap g(u) \neq \emptyset\}.$$

Thus $W'' = W'$. For any $u \in W'$ we have

$$(f \cap g, f_1')(u) = ((f \cap g) \times f_1')(u \times u) = (f \cap g)(u) \times f_1'(u) = (f(u) \cap g(u)) \times f_1'(u) =$$
$$= (f(u) \times f_1'(u)) \cap (g(u) \times f_1'(u)) = ((f \times f_1')(u \times u)) \cap ((g \times f_1')(u \times u)) =$$
$$= (f, f_1')(u) \cap (g, f_1')(u) = ((f, f_1') \cap (g, f_1'))(u).$$

$\square$

REMARK 30. *A result similar with the one from Theorem 90 states the truth of the formula*

$$f \times (f' \cap g') = (f \times f') \cap (f \times g')$$

and then, by Theorem 90, we get the result

$$(f \cap g) \times (f' \cap g') = (f \times f') \cap (f \times g') \cap (g \times f') \cap (g \times g').$$

Like in Theorem 91, we can prove that

$$(f, f_1' \cap g_1') = (f, f_1') \cap (f, g_1')$$

is true and then, by Theorem 91, we obtain

$$(f \cap g, f_1' \cap g_1') = (f, f_1') \cap (f, g_1') \cap (g, f_1') \cap (g, g_1').$$

THEOREM 92. *Let be the systems* $f : U \to P^*(S^{(n)})$, $g : V \to P^*(S^{(n)})$, $U, V \in P^*(S^{(m)})$ *and* $h : X \to P^*(S^{(p)})$, $h_1 : X_1 \to P^*(S^{(p)})$, $X, X_1 \in P^*(S^{(n)})$.

a) If $\bigcup\limits_{u \in U} f(u) \subset X$, $\bigcup\limits_{u \in V} g(u) \subset X$ *and the set*

$$W = \{u | u \in U \cap V, f(u) \cap g(u) \neq \emptyset\}$$

is non-empty, then the systems $h \circ (f \cap g)$, $(h \circ f) \cap (h \circ g)$ *exist and we have*

$$h \circ (f \cap g) \subset (h \circ f) \cap (h \circ g).$$

b) We ask that $\bigcup\limits_{u \in U} f(u) \subset \{x | x \in X \cap X_1, h(x) \cap h_1(x) \neq \emptyset\}$ *be true. Then the systems* $(h \cap h_1) \circ f$, $(h \circ f) \cap (h_1 \circ f)$ *are defined and the inclusion*

$$(h \cap h_1) \circ f \subset (h \circ f) \cap (h_1 \circ f)$$

is true.

PROOF. a) We shall use in the following the fact that $\forall u \in U \cap V$, we have

$$\bigcup_{x \in f(u) \cap g(u)} h(x) \subset \bigcup_{x \in f(u)} h(x), \quad \bigcup_{x \in f(u) \cap g(u)} h(x) \subset \bigcup_{x \in g(u)} h(x),$$

(in the left hand side of the inclusions we may have $\emptyset$), wherefrom

$$\bigcup_{x \in f(u) \cap g(u)} h(x) \subset \bigcup_{x \in f(u)} h(x) \cap \bigcup_{x \in g(u)} h(x).$$

We show the existence of $h \circ (f \cap g)$:

$$\bigcup_{u \in W} (f \cap g)(u) = \bigcup_{u \in W} (f(u) \cap g(u)) \subset \bigcup_{u \in W} (f(u) \cup g(u)) =$$

$$= \bigcup_{u \in W} f(u) \cup \bigcup_{u \in W} g(u) \subset \bigcup_{u \in U} f(u) \cup \bigcup_{u \in V} g(u) \subset X.$$

We show the existence of $(h \circ f) \cap (h \circ g)$. Let us denote by W' its support set. Because

$$W = \{u | u \in U \cap V, f(u) \cap g(u) \neq \emptyset\} = \{u | u \in U \cap V, \bigcup_{x \in f(u) \cap g(u)} h(x) \neq \emptyset\} \subset$$

$$\subset \{u | u \in U \cap V, \bigcup_{x \in f(u)} h(x) \cap \bigcup_{x \in g(u)} h(x) \neq \emptyset\} =$$

$$= \{u | u \in U \cap V, (h \circ f)(u) \cap (h \circ g)(u) \neq \emptyset\} = W'$$

we can see that the non-emptyness of W implies the non-emptyness of W', thus $(h \circ f) \cap (h \circ g)$ exists.

We conclude that $\forall u \in W$,

$$(h \circ (f \cap g))(u) = \bigcup_{x \in f(u) \cap g(u)} h(x) \subset \bigcup_{x \in f(u)} h(x) \cap \bigcup_{x \in g(u)} h(x) =$$

$$= (h \circ f)(u) \cap (h \circ g)(u) = ((h \circ f) \cap (h \circ g))(u).$$

b) Because

$$\bigcup_{u\in U} f(u)\subset\{x|x\in X\cap X_1, h(x)\cap h_1(x)\neq\emptyset\}\subset X,$$

$$\bigcup_{u\in U} f(u)\subset\{x|x\in X\cap X_1, h(x)\cap h_1(x)\neq\emptyset\}\subset X_1$$

we get the existence of $h\circ f$ and $h_1\circ f$.

We shall use in the following the fact that $\forall u\in U$, from

$$\bigcup_{x\in f(u)}(h\cap h_1)(x)\subset\bigcup_{x\in f(u)}h(x),\ \bigcup_{x\in f(u)}(h\cap h_1)(x)\subset\bigcup_{x\in f(u)}h_1(x)$$

(in the left hand side of the inclusions we may have $\emptyset$) we infer

$$\bigcup_{x\in f(u)}(h\cap h_1)(x)\subset\bigcup_{x\in f(u)}h(x)\cap\bigcup_{x\in f(u)}h_1(x).$$

The set $\{x|x\in X\cap X_1, h(x)\cap h_1(x)\neq\emptyset\}$ is non-empty and represents the support of $h\cap h_1$, thus the hypothesis has implied the existence of $(h\cap h_1)\circ f$. Furthermore:

$$\forall u\in U,((h\circ f)\cap(h_1\circ f))(u)=(h\circ f)(u)\cap(h_1\circ f)(u)=\bigcup_{x\in f(u)}h(x)\cap\bigcup_{x\in f(u)}h_1(x)\supset$$

$$\supset\bigcup_{x\in f(u)}(h(x)\cap h_1(x))\neq\emptyset$$

from the hypothesis, meaning that the support of $(h\circ f)\cap(h_1\circ f)$ is U.

Eventually, we obtain: $\forall u\in U$,

$$((h\cap h_1)\circ f)(u)=\bigcup_{x\in f(u)}(h\cap h_1)(x)\subset$$

$$\subset\bigcup_{x\in f(u)}h(x)\cap\bigcup_{x\in f(u)}h_1(x)=(h\circ f)(u)\cap(h_1\circ f)(u)=((h\circ f)\cap(h_1\circ f))(u).$$

□

13. Union

THEOREM 93. *Let be the pseudo-systems $f,g:\widetilde{S}^{(m)}\to P(\widetilde{S}^{(n)})$ and the pseudo-system $f\cup g:\widetilde{S}^{(m)}\to P(\widetilde{S}^{(n)})$ be defined by*

$$\forall u\in\widetilde{S}^{(m)},(f\cup g)(u)=f(u)\cup g(u).$$

For U,V, the supports of f,g, we have that the support of $f\cup g$ is $U\cup V$. If f,g are systems, then $f\cup g$ is a system.

PROOF. Suppose that f,g are systems. We have $U\neq\emptyset\Longrightarrow U\cup V\neq\emptyset$; $U\subset S^{(m)}$ and $V\subset S^{(m)}$ imply $U\cup V\subset S^{(m)}$; $\forall u\in U,f(u)\subset S^{(n)},\forall v\in V,g(v)\subset S^{(n)}$ imply

$$\forall u\in U\cup V,(f\cup g)(u)=\begin{cases} f(u), if\ u\in U\setminus V\\ g(u), if\ u\in V\setminus U\\ f(u)\cup g(u), if\ u\in U\cap V\end{cases}\subset S^{(n)}.$$

Thus $f\cup g$ is a system. □

Definition 47. *The* ***union*** *of the systems f and g is the system $f \cup g : U \cup V \to P^*(S^{(n)})$ defined by*

$$\forall u \in U \cup V, (f \cup g)(u) = \begin{cases} f(u), if\ u \in U \setminus V \\ g(u), if\ u \in V \setminus U \\ f(u) \cup g(u), if\ u \in U \cap V \end{cases}.$$

If $U \cap V = \emptyset$, then $f \cup g$ is called the ***disjoint union*** *of f and g.*

Remark 31. *The union of the systems is the dual concept to that of intersection. It represents the loss of information (of precision) which in general follows from the validity of one of two models. However, the disjoint union means no loss of information.*

We have also the special case when in the union $f \cup g$ the system g is constant, namely $g : S^{(m)} \to P^*(S^{(n)})$, $\forall u \in S^{(m)}$, $g(u) = X$, with $X \subset S^{(n)}$. In this situation $f \cup X : S^{(m)} \to P^*(S^{(n)})$ is defined by

$$\forall u \in S^{(m)}, (f \cup X)(u) = \begin{cases} X, if\ u \in S^{(m)} \setminus U \\ f(u) \cup X, if\ u \in U \end{cases}.$$

The interpretation of $f \cup X$ reads: when f is the model of an asynchronous circuit, X represents perturbations independent of u.

Example 22. *Let be the system $f : U \to P^*(S^{(n)})$, $U \in P^*(S^{(m)})$. If $\Theta_0 = \{\mu^1, ..., \mu^k\} \subset \mathbf{B}^n$, then $f_{\mu^1} : U_{\mu^1} \to P^*(S^{(n)}), ..., f_{\mu^k} : U_{\mu^k} \to P^*(S^{(n)})$ are the restrictions of f at the initial values of the states $\mu^1, ..., \mu^k$ (see Example 20). We have $f = f_{\mu^1} \cup ... \cup f_{\mu^k}$.*

Example 23. *In the union $f \cup g$ we presume that $U \cap V \neq \emptyset$ and f, g model two different circuits, the first considered 'good, without errors' and the second 'bad, with errors' (or 'bad, with a certain error'). The* ***testing problem*** *consists in finding an input $u \in U \cap V$ such that $f(u) \cap g(u) = \emptyset$ and after its application to $f \cup g$ and the measurement of the state $x \in (f \cup g)(u)$, we can say if $x \in f(u)$ and the tested circuit is 'good' or perhaps $x \in g(u)$ and the tested circuit is 'bad'.*

Theorem 94. *a) If f, g have race-free initial states and $\forall u \in U \cap V, f(u) \cap g(u) \neq \emptyset$, then $f \cup g$ has race-free initial states.*

b) If f, g have constant initial states and $\bigcup_{u \in U} f(u) \cap \bigcup_{u \in V} g(u) \neq \emptyset$, then $f \cup g$ has a constant initial state.

Proof. a) The hypothesis states the truth of the following properties

$$\forall u \in U, \exists \mu \in \mathbf{B}^n, \forall x \in f(u), \exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = \mu,$$

$$\forall u \in V, \exists \mu \in \mathbf{B}^n, \forall x \in g(u), \exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = \mu,$$

$$\forall u \in U \cap V, f(u) \cap g(u) \neq \emptyset.$$

If $(U \setminus V) \cup (V \setminus U) \neq \emptyset$, then $\forall u \in (U \setminus V) \cup (V \setminus U)$ the statement is true because it states separately for f and g that they have race-free initial states. If $U \cap V \neq \emptyset$, then we infer that μ towards which $x \in f(u) \cup g(u)$ converges for $u \in U \cap V$ as $t \to -\infty$ depends on u only, but not on the fact that $x \in f(u)$ or $x \in g(u)$. We have that

$$\forall u \in U \cup V, \exists \mu \in \mathbf{B}^n, \forall x \in (f \cup g)(u), \exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = \mu$$

is true.

b) Because $\bigcup\limits_{u\in U} f(u)\cap \bigcup\limits_{u\in V} g(u)\neq\emptyset$, in the statements

$$\exists\mu\in\mathbf{B}^n,\forall u\in U,\forall x\in f(u),\exists t_0\in\mathbf{R},\forall t<t_0,x(t)=\mu,$$

$$\exists\mu'\in\mathbf{B}^n,\forall u\in V,\forall x\in g(u),\exists t_0\in\mathbf{R},\forall t<t_0,x(t)=\mu'$$

the two constants μ and μ', whose existence is unique, coincide. □

THEOREM 95. *a) If f,g have final states, then $f\cup g$ has final states.*
b) If f,g have race-free final states and $\forall u\in U\cap V, f(u)\cap g(u)\neq\emptyset$, then $f\cup g$ has race-free final states.
c) If f,g have constant final states and $\bigcup\limits_{u\in U} f(u)\cap \bigcup\limits_{u\in V} g(u)\neq\emptyset$, then $f\cup g$ has a constant final state.

THEOREM 96. *If f,g have a bounded initial time (a fixed initial time), then $f\cup g$ has a bounded initial time (a fixed initial time).*

PROOF. We suppose that f,g have a bounded initial time and let $u\in U\cup V$ be arbitrary. Then $t_0',t_0''\in\mathbf{R}$ exist, depending on u, such that

$$\forall x\in f(u),\exists\mu\in\mathbf{B}^n,\forall t<t_0',x(t)=\mu,$$

$$\forall x\in g(u),\exists\mu\in\mathbf{B}^n,\forall t<t_0'',x(t)=\mu$$

where, if in one of the previous statements $f(u)=\emptyset$ or $g(u)=\emptyset, t_0'$ and t_0'' may be arbitrarily chosen. The time $t_0=\min\{t_0',t_0''\}$ satisfies the relation

$$\forall x\in f(u)\cup g(u),\exists\mu\in\mathbf{B}^n,\forall t<t_0,x(t)=\mu.$$

□

THEOREM 97. *If f,g have a bounded final time (a fixed final time), then $f\cup g$ has a bounded final time (a fixed final time).*

THEOREM 98. *For the systems f,g, we have $(\phi\cup\gamma)_0:U\cup V\to P^*(\mathbf{B}^n)$,*

$$\forall u\in U\cup V,(\phi\cup\gamma)_0(u)=\begin{cases}\phi_0(u),\ u\in U\setminus V\\ \gamma_0(u),\ u\in V\setminus U\\ \phi_0(u)\cup\gamma_0(u),\ u\in U\cap V\end{cases},$$

$$(\Theta\cup\Gamma)_0=\bigcup_{u\in U\cup V}(\phi\cup\gamma)_0(u).$$

We have denoted by $\phi_0,\gamma_0,(\phi\cup\gamma)_0$ the initial state functions of $f,g,f\cup g$ and with $(\Theta\cup\Gamma)_0$ the set of initial states of $f\cup g$.

PROOF. We have three possibilities for an arbitrary $u\in U\cup V: u\in U\setminus V, u\in V\setminus U$ and $u\in U\cap V$. If, for example, $u\in U\setminus V$, then

$$(\phi\cup\gamma)_0(u)=\{x(-\infty+0)|x\in(f\cup g)(u)\}=\{x(-\infty+0)|x\in f(u)\}=\phi_0(u).$$

□

THEOREM 99. *We suppose that f,g have final states. We have $(\phi\cup\gamma)_f:U\cup V\to P^*(\mathbf{B}^n)$,*

$$\forall u\in U\cup V,(\phi\cup\gamma)_f(u)=\begin{cases}\phi_f(u),\ u\in U\setminus V\\ \gamma_f(u),\ u\in V\setminus U\\ \phi_f(u)\cup\gamma_f(u),\ u\in U\cap V\end{cases},$$

$$(\Theta \cup \Gamma)_f = \bigcup_{u \in U \cup V} (\phi \cup \gamma)_f(u),$$

where the notations are obvious and similar to those from the previous theorem.

THEOREM 100. *Consider the systems* $f : U \to P^*(S^{(n)})$, $f_1 : U_1 \to P^*(S^{(n)})$, $g : V \to P^*(S^{(n)})$, *with* $U, U_1, V \in P^*(S^{(m)})$. *If* $f \subset f_1$, *then* $f \cup g \subset f_1 \cup g$.

PROOF. From $U \subset U_1$ we infer that $U \cup V \subset U_1 \cup V$. It is shown that $\forall u \in U \cup V, (f \cup g)(u) \subset (f_1 \cup g)(u)$ is true in all the three situations $u \in U \setminus V, u \in V \setminus U$ and $u \in U \cap V$. □

THEOREM 101. *We have*

$$(f \cup g)^* = f^* \cup g^*.$$

PROOF. We remark that the equal supports of the two systems are $(U \cup V)^* = U^* \cup V^*$. Let be an arbitrary $u \in U^* \cup V^*$. If $u \in U^* \setminus V^*$, then $f^*(u) = (f^* \cup g^*)(u)$ and the fact that $\overline{u} \in U \setminus V$ implies $(f \cup g)(\overline{u}) = f(\overline{u})$. Thus

$$(f \cup g)^*(u) = \{\overline{x} | x \in (f \cup g)(\overline{u})\} = \{\overline{x} | x \in f(\overline{u})\} = f^*(u) = (f^* \cup g^*)(u).$$

If $u \in V^* \setminus U^*$, the situation is similar. At this moment suppose that $u \in U^* \cap V^*$, implying $f^*(u) \cup g^*(u) = (f^* \cup g^*)(u)$, $\overline{u} \in U \cap V$, $(f \cup g)(\overline{u}) = f(\overline{u}) \cup g(\overline{u})$ and we have

$$(f \cup g)^*(u) = \{\overline{x} | x \in (f \cup g)(\overline{u})\} = \{\overline{x} | x \in f(\overline{u}) \cup g(\overline{u})\} =$$
$$= \{\overline{x} | x \in f(\overline{u})\} \cup \{\overline{x} | x \in g(\overline{u})\} = f^*(u) \cup g^*(u) = (f^* \cup g^*)(u).$$

In all the three cases the equality was proved to be true. □

THEOREM 102. *Consider the systems* $f : U \to P^*(S^{(n)}), g : V \to P^*(S^{(n)})$, $U, V \in P^*(S^{(m)})$. *The systems* $(f \cup g)^{-1}$, $f^{-1} \cup g^{-1}$ *have the supports equal to*

$$X = \bigcup_{u \in U} f(u) \cup \bigcup_{u \in V} g(u)$$

and the following equality

$$(f \cup g)^{-1} = f^{-1} \cup g^{-1}$$

is true.

PROOF. The support of $(f \cup g)^{-1}$ is $\bigcup\limits_{u \in U \cup V} (f \cup g)(u)$ and it coincides with X, that obviously is the support of $f^{-1} \cup g^{-1}$. For any $x \in X$ we have

$$(f \cup g)^{-1}(x) = \{u | u \in U \cup V, x \in (f \cup g)(u)\} = \{u | u \in U \setminus V, x \in f(u)\} \cup$$
$$\cup \{u | u \in V \setminus U, x \in g(u)\} \cup \{u | u \in U \cap V, x \in f(u)\} \cup \{u | u \in U \cap V, x \in g(u)\} =$$
$$= \{u | u \in U, x \in f(u)\} \cup \{u | u \in V, x \in g(u)\}$$

and, on the other hand,

$$(f^{-1} \cup g^{-1})(x) = \begin{cases} f^{-1}(x), x \in \bigcup\limits_{u \in U} f(u) \setminus \bigcup\limits_{u \in V} g(u) \\ g^{-1}(x), x \in \bigcup\limits_{u \in V} g(u) \setminus \bigcup\limits_{u \in U} f(u) \\ f^{-1}(x) \cup g^{-1}(x), x \in \bigcup\limits_{u \in U} f(u) \cap \bigcup\limits_{u \in V} g(u) \end{cases} =$$

$$= \begin{cases} \{u|u \in U, x \in f(u)\}, x \in \bigcup_{u\in U} f(u) \setminus \bigcup_{u\in V} g(u) \\ \{u|u \in V, x \in g(u)\}, x \in \bigcup_{u\in V} g(u) \setminus \bigcup_{u\in U} f(u) \\ \{u|u \in U, x \in f(u)\} \cup \{u|u \in V, x \in g(u)\}, x \in \bigcup_{u\in U} f(u) \cap \bigcup_{u\in V} g(u) \end{cases}.$$

We can check the equality from the statement of the theorem in the three cases: $x \in \bigcup_{u\in U} f(u) \setminus \bigcup_{u\in V} g(u)$, $x \in \bigcup_{u\in V} g(u) \setminus \bigcup_{u\in U} f(u)$ and $x \in \bigcup_{u\in U} f(u) \cap \bigcup_{u\in V} g(u)$. □

THEOREM 103. *Let be the systems $f : U \to P^*(S^{(n)})$, $g : V \to P^*(S^{(n)})$, $U, V \in P^*(S^{(m)})$ and $f' : U' \to P^*(S^{(n')})$, $U' \in P^*(S^{(m')})$. The common support of $(f \cup g) \times f'$, $(f \times f') \cup (g \times f')$ is $(U \cup V) \times U' = (U \times U') \cup (V \times U')$ and the following equality*

$$(f \cup g) \times f' = (f \times f') \cup (g \times f')$$

holds true.

PROOF. For $\forall u \times u' \in (U \cup V) \times U'$ we have one of the following possibilities:

Case $u \times u' \in (U \setminus V) \times U' = (U \times U') \setminus (V \times U')$,

$$((f \cup g) \times f')(u \times u') = (f \cup g)(u) \times f'(u') = f(u) \times f'(u') = (f \times f')(u \times u') =$$
$$= ((f \times f') \cup (g \times f'))(u \times u');$$

Case $u \times u' \in (V \setminus U) \times U'$ is similar;

Case $u \times u' \in (U \cap V) \times U' = (U \times U') \cap (V \times U')$,

$$((f \cup g) \times f')(u \times u') = (f \cup g)(u) \times f'(u') = (f(u) \cup g(u)) \times f'(u') =$$
$$= (f(u) \times f'(u')) \cup (g(u) \times f'(u')) = (f \times f')(u \times u') \cup (g \times f')(u \times u') =$$
$$= ((f \times f') \cup (g \times f'))(u \times u').$$

□

THEOREM 104. *Let be the systems $f : U \to P^*(S^{(n)})$, $g : V \to P^*(S^{(n)})$, $f'_1 : U'_1 \to P^*(S^{(n')})$, with $U, V, U'_1 \in P^*(S^{(m)})$. If $U \cap U'_1 \neq \emptyset$, $V \cap U'_1 \neq \emptyset$, then the common support of the systems $(f \cup g, f'_1)$, $(f, f'_1) \cup (g, f'_1)$ is $(U \cup V) \cap U'_1 = (U \cap U'_1) \cup (V \cap U'_1)$ and we have*

$$(f \cup g, f'_1) = (f, f'_1) \cup (g, f'_1).$$

REMARK 32. *The last two theorems have consequences concerning the truth of the formulae*

$$f \times (f' \cup g') = (f \times f') \cup (f \times g'),$$
$$(f \cup g) \times (f' \cup g') = (f \times f') \cup (f \times g') \cup (g \times f') \cup (g \times g')$$

and of the formulae

$$(f, f'_1 \cup g'_1) = (f, f'_1) \cup (f, g'_1),$$
$$(f \cup g, f'_1 \cup g'_1) = (f, f'_1) \cup (f, g'_1) \cup (g, f'_1) \cup (g, g'_1).$$

THEOREM 105. *Let be the systems $f : U \to P^*(S^{(n)})$, $g : V \to P^*(S^{(n)})$, $U, V \in P^*(S^{(m)})$ and $h : X \to P^*(S^{(p)})$, $h_1 : X_1 \to P^*(S^{(p)})$, $X, X_1 \in P^*(S^{(n)})$.*

a) Suppose that $\bigcup_{u\in U} f(u) \cup \bigcup_{u\in V} g(u) \subset X$. Then $h \circ (f \cup g)$, $(h \circ f) \cup (h \circ g)$ exist and the following equality

$$h \circ (f \cup g) = (h \circ f) \cup (h \circ g)$$

is true.

b) If $\bigcup_{u\in U} f(u)\subset X$, $\bigcup_{u\in U} f(u)\subset X_1$ *then the systems* $(h\cup h_1)\circ f$, $(h\circ f)\cup(h_1\circ f)$ *are defined and we have*

$$(h\cup h_1)\circ f=(h\circ f)\cup(h_1\circ f).$$

Proof. a) As $\bigcup_{u\in U\cup V}(f\cup g)(u)=\bigcup_{u\in U} f(u)\cup\bigcup_{u\in V} g(u)\subset X$, we infer the existence of $h\circ(f\cup g)$. The inclusions $\bigcup_{u\in U} f(u)\subset X$, $\bigcup_{u\in V} g(u)\subset X$ show that $h\circ f, h\circ g, (h\circ f)\cup(h\circ g)$ are also defined and the common support of $h\circ(f\cup g)$ and $(h\circ f)\cup(h\circ g)$ is $U\cup V$. The next equalities are true:

$$\forall u\in U\cup V,(h\circ(f\cup g))(u)=\bigcup_{x\in(f\cup g)(u)} h(x)=\begin{cases}\bigcup_{x\in f(u)} h(x),u\in U\setminus V\\ \bigcup_{x\in g(u)} h(x),u\in V\setminus U\\ \bigcup_{x\in f(u)\cup g(u)} h(x),u\in U\cap V\end{cases}=$$

$$=\begin{cases}\bigcup_{x\in f(u)} h(x),u\in U\setminus V\\ \bigcup_{x\in g(u)} h(x),u\in V\setminus U\\ \bigcup_{x\in f(u)} h(x)\cup\bigcup_{x\in g(u)} h(x),u\in U\cap V\end{cases}=$$

$$=\begin{cases}(h\circ f)(u),u\in U\setminus V\\ (h\circ g)(u),u\in V\setminus U\\ (h\circ f)(u)\cup(h\circ g)(u),u\in U\cap V\end{cases}=((h\circ f)\cup(h\circ g))(u).$$

b) From the hypothesis $\bigcup_{u\in U} f(u)\subset X\cup X_1$, thus $(h\cup h_1)\circ f$ exists. On the other hand $h\circ f$, $h_1\circ f$, $(h\circ f)\cup(h_1\circ f)$ exist themselves. The common support of the systems $(h\cup h_1)\circ f,(h\circ f)\cup(h_1\circ f)$ is U. We can see that

$$\forall u\in U,((h\cup h_1)\circ f)(u)=\bigcup_{x\in f(u)}(h\cup h_1)(x)=\bigcup_{x\in f(u)} h(x)\cup\bigcup_{x\in f(u)} h_1(x)=$$

$$=(h\circ f)(u)\cup(h_1\circ f)(u)=((h\circ f)\cup(h_1\circ f))(u).$$

□

14. Morphisms

Definition 48. *Let be the systems* $f:U\to P^*(S^{(n)}),U\subset S^{(m)}$ *and* $f':U'\to P^*(S^{(n')}),U'\subset S^{(m')}$. *We call the* ***morphism*** *(of systems) from* f *to* f'*, and we denote it by* $(\omega,\Omega):f\to f'$*, a couple of functions* $\omega:U\to U'$, $\Omega:P^*(S^{(n)})\to P^*(S^{(n')})$ *with the property that the following diagram is commutative*

$$\begin{array}{ccc} U & \xrightarrow{f} & P^*(S^{(n)})\\ \omega\downarrow & & \downarrow\Omega\\ U' & \xrightarrow{f'} & P^*(S^{(n')})\end{array}$$

Remark 33. *The existence of a morphism* $f\to f'$ *shows that* f' *reproduces the properties of* f*. The origin of the word 'morphism' is the greek word 'morphe' (Le Petit Robert) which means 'form'. Thus* f' *is another form of* f*.*

Example 24. *For any system f, the following diagram*

$$\begin{array}{ccc} U & \xrightarrow{f} & P^*(S^{(n)}) \\ 1_U \downarrow & & \downarrow 1_{P^*(S^{(n)})} \\ U & \xrightarrow{f} & P^*(S^{(n)}) \end{array}$$

is commutative. It defines a morphism $(1_U, 1_{P^*(S^{(n)})}) : f \to f$, called the ***identical morphism****. The usual notation for the identical morphism is 1_f.*

Example 25. *Define the functions*

$$\omega : U \to U^*, \forall u \in U, \omega(u) = \overline{u},$$

$$\Omega : P^*(S^{(n)}) \to P^*(S^{(n)}), \forall X \in P^*(S^{(n)}), \Omega(X) = X^*.$$

Because the following diagram

$$\begin{array}{ccc} U & \xrightarrow{f} & P^*(S^{(n)}) \\ \omega \downarrow & & \downarrow \Omega \\ U^* & \xrightarrow{f^*} & P^*(S^{(n)}) \end{array}$$

is commutative

$$\forall u \in U, (f^* \circ \omega)(u) = f^*(\omega(u)) = f^*(\overline{u}) = \{\overline{x}|x \in f(u)\} = \Omega(f(u)) = (\Omega \circ f)(u),$$

we infer that $(\omega, \Omega) : f \to f^*$ is a morphism of systems. We remark that a morphism $f^* \to f$ can be defined in the same manner.

Exercise 1. *Let be the system $f : S^{(m)} \to P^*(S^{(n)})$. The reader may want to construct the morphisms $f \to f'$, where f' is one of the following systems:*

a) $f' : S^{(m)} \to P^*(S^{(n)}), \forall u \in S^{(m)}, f'(u) = \{\overline{x}|x \in f(u)\}$;

b) $f' : S^{(m)} \to P^*(S^{(n)}), \forall u \in S^{(m)}, f'(u) = \{a \cdot x|x \in f(u)\}$, where $a \in S$ and we have used the notation

$$(a \cdot x)(t) = (a(t) \cdot x_1(t), ..., a(t) \cdot x_n(t));$$

c) $f' : S^{(m)} \to P^*(S^{(n)}), \forall u \in S^{(m)}, f'(u) = \{a \cup x|x \in f(u)\}$, where $a \in S$ and we have used the notation

$$(a \cup x)(t) = (a(t) \cup x_1(t), ..., a(t) \cup x_n(t));$$

d) $f' : S^{(m)} \to P^*(S^{(n)}), \forall u \in S^{(m)}, f'(u) = f(u) \cap X$, where $X \subset S^{(n)}$ satisfies $\forall u \in S^{(m)}, f(u) \cap X \neq \emptyset$;

e) $f' : S^{(m)} \to P^*(S^{(n)}), \forall u \in S^{(m)}, f'(u) = f(u) \cup X$, where $X \subset S^{(n)}$;

f) $f' : S^{(m+1)} \to P^*(S^{(n)}), \forall (u_1, ..., u_m, u_{m+1}) \in S^{(m+1)}, f'(u_1, ..., u_m, u_{m+1}) = f(u_1, ..., u_m)$;

g) $f' : S^{(m-1)} \to P^*(S^{(n)}), \forall u \in S^{(m)}, f'(u_1, ..., \widehat{u}_i, ..., u_m) = f(u_1, ..., u_i, ..., u_m)$ where $m > 1$, $i \in \{1, ..., m\}$, f does not depend on u_i and we have used the notation $\widehat{u}_i$ to show the fact that the coordinate u_i is missing.

Other morphisms $f \times f' \to f$, $f \times f' \to f'$, $(f, f') \to f$, $(f, f') \to f$' exist, that the reader is invited to write.

Theorem 106. *Let be the systems $f : U \to P^*(S^{(n)})$, $U \in P^*(S^{(m)})$, $f' : U' \to P^*(S^{(n')})$, $U' \in P^*(S^{(m')})$, $f'' : U'' \to P^*(S^{(n'')})$, $U'' \in P^*(S^{(m'')})$ and the morphisms $(\omega, \Omega) : f \to f'$, $(\omega', \Omega') : f' \to f''$. We have the morphism $(\omega', \Omega') \circ (\omega, \Omega) : f \to f''$ defined by*

$$(\omega', \Omega') \circ (\omega, \Omega) = (\omega' \circ \omega, \Omega' \circ \Omega), \tag{14.1}$$

where, in the right side of (14.1), $'\circ'$ is the usual composition of the functions.

PROOF. The hypothesis states the commutativity of the diagrams

$$\begin{array}{ccc} U & \xrightarrow{f} & P^*(S^{(n)}) \\ \omega\downarrow & & \downarrow\Omega \\ U' & \xrightarrow{f'} & P^*(S^{(n')}) \end{array}$$

$$\begin{array}{ccc} U' & \xrightarrow{f'} & P^*(S^{(n')}) \\ \omega'\downarrow & & \downarrow\Omega' \\ U'' & \xrightarrow{f''} & P^*(S^{(n'')}) \end{array}$$

and the conclusion refers to the commutativity of the diagram

$$\begin{array}{ccc} U & \xrightarrow{f} & P^*(S^{(n)}) \\ \omega'\circ\omega\downarrow & & \downarrow\Omega'\circ\Omega \\ U'' & \xrightarrow{f''} & P^*(S^{(n'')}) \end{array}$$

From the hypothesis we infer the following:

$$\forall u\in U,(f''\circ(\omega'\circ\omega))(u)=((f''\circ\omega')\circ\omega)(u)=((\Omega'\circ f')\circ\omega)(u)=$$
$$=(\Omega'\circ(f'\circ\omega))(u)=(\Omega'\circ(\Omega\circ f))(u)=((\Omega'\circ\Omega)\circ f)(u).$$

Thus $(\omega'\circ\omega,\Omega'\circ\Omega):f\to f''$ is a morphism of systems. □

DEFINITION 49. *If the systems f,f' have the property that the morphisms $(\omega,\Omega):f\to f'$, $(\omega',\Omega'):f'\to f$ exist such that*

$$(\omega',\Omega')\circ(\omega,\Omega)=1_f,$$
$$(\omega,\Omega)\circ(\omega',\Omega')=1_{f'},$$

then they are called ***isomorphic*** *and the morphisms (ω,Ω), (ω',Ω') are called the* ***isomorphisms****.*

REMARK 34. *The systems f and f^* are isomorphic. In Exercise 1, f is isomorphic with the systems f' from a), f) and, if f does not depend on u_i, then g) gives another example of system f' that is isomorphic with f.*

The asynchronous systems form a category Sys: its objects are the systems and its morphisms are defined like in Definition 48. For any $f\in ObSys$ the unit morphism is 1_f and for any morphisms $(\omega,\Omega):f\to f'$, $(\omega',\Omega'):f'\to f''$, their composition is defined like in (14.1).

We end this section by showing a way of constructing morphisms of systems suggested by the work [**21**] *of Moisil. Let $\pi:\mathbf{B}^n\to\mathbf{B}^n$ be a function. It induces the functions $\widetilde{\pi}:S^{(n)}\to S^{(n)},\widehat{\pi}:P^*(S^{(n)})\to P^*(S^{(n)})$ defined as:*

$$\forall t\in\mathbf{R},\forall x\in S^{(n)},\widetilde{\pi}(x)(t)=\pi(x(t)),$$
$$\forall X\in P^*(S^{(n)}),\widehat{\pi}(X)=\{\widetilde{\pi}(x)|x\in X\}.$$

The couples $(1_U,\widehat{\pi})$ are called ***morphisms under a given input****. When π is a bijection, they become isomorphisms.*

CHAPTER 5

General properties of the systems

Many sets and functions related to systems and various properties of systems are defined and analyzed. Particular types of systems, important for the author's treatment are described too. Several definitions are presented for the non-anticipation and injectivity, together with long comments and results for them.

1. Constant initial state function. Initialization

DEFINITION 50. *The system $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$* ***has a constant initial state function*** *if*

$$\exists k \in \{1, ..., 2^n\}, \exists \mu^1 \in \mathbf{B}^n, ..., \exists \mu^k \in \mathbf{B}^n, \forall u \in U, \phi_0(u) = \{\mu^1, ..., \mu^k\}.$$

REMARK 35. *The constancy of ϕ_0 means that we know among what values we can search, $\forall u \in U$, the initial states of f. The initialization, i.e. the property of a system of having a (constant) initial state is given by any of the equivalent statements*

$$\exists \mu \in \mathbf{B}^n, \forall u \in U, \forall x \in f(u), \exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = \mu,$$

$$\exists \mu \in \mathbf{B}^n, \forall u \in U, \phi_0(u) = \mu$$

and it represents one of the most important properties of the systems.

THEOREM 107. *Let be two systems f, g with the property $f \subset g$.*

a) If the two initial state functions ϕ_0, γ_0 are constant, then the inclusion of sets $\phi_0 \subset \gamma_0$ takes place.

b) If g is initialized, then f is initialized also and $\phi_0 = \gamma_0$, i.e. the two initial states coincide.

PROOF. a) Special case of Theorem 39.

b) In the inclusion $\phi_0 \subset \gamma_0$, ϕ_0 is non-empty and γ_0 has one element, thus the equality $\phi_0 = \gamma_0$ holds true. □

THEOREM 108. *If f has a constant initial state function ϕ_0 (if f is initialized), then f^* has a constant initial state function ϕ_0^* (then f^* is initialized) and $\phi_0^* = \{\overline{\mu} | \mu \in \phi_0\}$ is true ($\phi_0^* = \overline{\phi_0}$ is true).*

PROOF. We have

$$\phi_0 = \{\mu^1, ..., \mu^k\} \Longleftrightarrow \forall u \in U, \{\mu^1, ..., \mu^k\} = \{x(-\infty + 0) | x \in f(u)\}$$

$$\Longleftrightarrow \forall u \in U^*, \{\mu^1, ..., \mu^k\} = \{x(-\infty + 0) | x \in f(\overline{u})\}$$

$$\Longleftrightarrow \forall u \in U^*, \{\overline{\mu^1}, ..., \overline{\mu^k}\} = \{\overline{x(-\infty + 0)} | x \in f(\overline{u})\}$$

$$\Longleftrightarrow \forall u \in U^*, \{\overline{\mu^1}, ..., \overline{\mu^k}\} = \{x(-\infty + 0) | x \in f^*(u)\} \Longleftrightarrow \{\overline{\mu^1}, ..., \overline{\mu^k}\} = \phi_0^*.$$

□

THEOREM 109. *If the systems* $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$ *and* $f' : U' \to P^*(S^{(n')}), U' \in P^*(S^{(m')})$ *have the initial state functions* ϕ_0, ϕ'_0 *constant, then their product* $f \times f'$ *has the initial state function* $(\phi \times \phi')_0$ *constant. If* f *and* f' *are initialized, then* $f \times f'$ *is initialized too.*

PROOF. From Theorem 61 we have $(\phi \times \phi')_0 = \phi_0 \times \phi'_0$ the Cartesian product of sets. If ϕ_0 and ϕ'_0 have one element, then their Cartesian product $\phi_0 \times \phi'_0$ has one element. □

THEOREM 110. *Suppose that the systems* $f : U \to P^*(S^{(n)})$ *and* $f'_1 : U'_1 \to P^*(S^{(n')})$, $U, U'_1 \in P^*(S^{(m)})$ *satisfy* $U \cap U'_1 \neq \emptyset$. *If they have the initial state functions constant, then their parallel connection* (f, f'_1) *has the initial state function constant. In particular, if* f *and* f'_1 *are initialized, then* (f, f'_1) *is initialized.*

THEOREM 111. *The systems* f *and* $h : X \to P^*(S^{(p)})$, $X \in P^*(S^{(n)})$ *are given such that the inclusion* $\bigcup_{u \in U} f(u) \subset X$ *is true. If* h *has the initial state function constant, then* $h \circ f$ *has the initial state function constant. If* h *is initialized, then* $h \circ f$ *is initialized too.*

PROOF. Denote by $\eta_0 : X \to P^*(\mathbf{B}^p)$ the initial state function of h and by $\delta_0 : U \to P^*(\mathbf{B}^p)$ the initial state function $h \circ f$. The hypothesis states the existence of $k \in \{1, ..., 2^p\}$ and $\nu^1, ..., \nu^k \in \mathbf{B}^p$, such that

$$\forall x \in X, \eta_0(x) = \{y(-\infty + 0) | y \in h(x)\} = \{\nu^1, ..., \nu^k\},$$

wherefrom we infer, by Theorem 72, that

$$\forall u \in U, \delta_0(u) = \bigcup_{x \in f(u)} \eta_0(x) = \{\nu^1, ..., \nu^k\}.$$

The second statement follows, but this result was already proved in Theorem 68. □

THEOREM 112. *The systems* $f : U \to P^*(S^{(n)})$ *and* $g : V \to P^*(S^{(n)})$ *with* $U, V \in P^*(S^{(m)})$ *are given, satisfying the property that the set*

$$W = \{u | u \in U \cap V, f(u) \cap g(u) \neq \emptyset\}$$

is non-empty. If their initial state functions ϕ_0, γ_0 *are constant, then the initial state function* $(\phi \cap \gamma)_0$ *of the intersection* $f \cap g$ *is constant. If one of* f, g *is initialized, then* $f \cap g$ *is initialized too.*

PROOF. We apply Theorem 85, wherefrom we get $(\phi \cap \gamma)_0 = \phi_0 \cap \gamma_0$. If in this equality one of the sets ϕ_0 and γ_0 has one element, then the intersection $\phi_0 \cap \gamma_0$ has one element, thus the system $f \cap g$ is initialized. □

REMARK 36. *Let be the systems* f *and* g. *If* ϕ_0 *and* γ_0 *are constant, then in general the union* $f \cup g$ *does not have the initial state function* $(\phi \cup \gamma)_0$ *constant (see Theorem 98). But if* f, g *are initialized with* $\phi_0 = \gamma_0$, *then* $f \cup g$ *is initialized with* $(\phi \cup \gamma)_0 = \phi_0 = \gamma_0$. *A result of the same nature was proved in Theorem 94, b).*

2. Autonomy

DEFINITION 51. *We call the* ***autonomous system****, any of the following concepts:*

a) the system $f : U \to P^*(S^{(n)})$, $U \in P^*(S^{(m)})$ *given by the constant function*

$$\exists X \in P^*(S^{(n)}), \forall u \in U, f(u) = X;$$

b) the system f *in the special case when it has exactly one input,* $|U| = 1$[1]*;*

c) the set $X \in P^*(S^{(n)})$.

Implicitly, by autonomous system further we mean a). However, when significant differences occur among a), b), c), in order to avoid misunderstandings, we specify which concept of autonomy we use.

NOTATION 16. *The autonomous system* f *is generally denoted by* X*, or* $f = X$ *according to our convention of identification of the constant function with the constant.*

REMARK 37. *The autonomous systems may be considered to be without input since the states* $x \in X$ *are the same for all* $u \in U$*. Whence one of the meanings of Definition 51, c).*

EXAMPLE 26. *The total system defined by* $S^{(n)} : S^{(m)} \to P^*(S^{(n)}), \forall u \in S^{(m)}, S^{(n)}(u) = S^{(n)}$ *is autonomous.*

EXAMPLE 27. *The system* $f : S^{(m)} \to P^*(S), \forall u \in S^{(m)}, f(u) = \{x | x \in S, \overline{x(t-0)} \cdot x(t) = 0\}$ *of the monotonous decreasing functions is autonomous. Each of its states switches at most once from* 1 *to* 0*.*

EXAMPLE 28. *Let be the system* $f : S^{(m)} \to P^*(S)$ *defined by the inequalities*

$$\overline{x(t-0)} \cdot x(t) \leq \bigcap_{\xi \in [t, t+\delta_r]} x(\xi),$$

$$x(t-0) \cdot \overline{x(t)} \leq \bigcap_{\xi \in [t, t+\delta_f]} \overline{x(\xi)},$$

where $\delta_r \geq 0, \delta_f \geq 0$*. Its states* $x \in S$ *are called* ***absolutely inertial*** *and have the property that after switching from 0 to 1 they remain 1 more than* δ_r *time units and after switching from 1 to 0 they remain 0 more than* δ_f *time units. The system* f *is autonomous.*

EXAMPLE 29. *The equation*

$$\bigcap_{\xi \in [t, t+\delta_r]} x(\xi) = 0,$$

with $\delta_r > 0$*, defines an autonomous system* $f : S^{(m)} \to P^*(S)$ *with the states* $x \in S$ *having* $x(-\infty + 0) = 0$ *and 1-pulses of length* $\leq \delta_r$*.*

THEOREM 113. *Let be the autonomous system* $X \in P^*(S^{(n)})$*.*

a) The property of existence of the initial state

$$\forall x \in X, \exists \mu \in \mathbf{B}^n, \exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = \mu$$

holds true.

---

[1] $|\ |$ is the usual notation for the number of elements of a set; Definition 51, b) was given in [**11**].

b) The existence of the race-free initial states coincides with the existence of the constant initial state and is given by

$$\exists\mu \in \mathbf{B}^n, \forall x \in X, \exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = \mu.$$

PROOF. These properties are the same as (2.1),...,(2.3) from Ch. 4, where the quantification $\forall u \in U$ is missing because the states $x \in X$ do not depend on the input u. □

THEOREM 114. *If f is autonomous, in the sense that $X \in P^*(S^{(n)})$ exists with $\forall u \in U, f(u) = X$, then*

a) f has an unbounded initial time

$$\forall x \in X, \exists\mu \in \mathbf{B}^n, \exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = \mu;$$

b) the existence of both the bounded initial time and of the fixed initial time coincide with

$$\exists t_0 \in \mathbf{R}, \forall x \in X, \exists\mu \in \mathbf{B}^n, \forall t < t_0, x(t) = \mu.$$

PROOF. These are the properties (3.1),...,(3.3) from Ch. 4, where the quantification $\forall u \in U$ is missing. □

THEOREM 115. *Suppose that f is an autonomous system, $f = X$. The following equivalencies hold:*

a) f has initial states and an unbounded initial time iff

$$\forall x \in X, \exists\mu \in \mathbf{B}^n, \exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = \mu;$$

b) f has initial states and a bounded initial time iff

$$\exists t_0 \in \mathbf{R}, \forall x \in X, \exists\mu \in \mathbf{B}^n, \forall t < t_0, x(t) = \mu;$$

c) f has race-free initial states and an unbounded initial time iff

$$\exists\mu \in \mathbf{B}^n, \forall x \in X, \exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = \mu;$$

d) f has race-free initial states and a bounded initial time iff

$$\exists\mu \in \mathbf{B}^n, \exists t_0 \in \mathbf{R}, \forall x \in X, \forall t < t_0, x(t) = \mu.$$

THEOREM 116. *The initial state function of the autonomous system f is constant and equal to the set of the initial states.*

PROOF. By hypothesis there is the set X such that $\forall u \in U, f(u) = X$. From the definition of ϕ_0 we get

$$\forall u \in U, \phi_0(u) = \{x(-\infty + 0)|x \in f(u)\} = \{x(-\infty + 0)|x \in X\} = \Theta_0.$$

□

THEOREM 117. *The autonomous system $f : U \to P^*(S^{(n)}), U \subset S^{(m)}$ is given. Then f^* is autonomous.*

PROOF. The system $f^* : U^* \to P^*(S^{(n)})$ is defined by $\forall u \in U^*, f^*(u) = \{\overline{x}|x \in f(\overline{u})\} = \{\overline{x}|x \in X\} = X^*$, where we have considered that $\forall u \in U, f(u) = X$. □

THEOREM 118. *If f is autonomous, then f^{-1} is autonomous too.*

PROOF. If $\forall u \in U, f(u) = X$, then $\forall x \in X, f^{-1}(x) = U$. □

THEOREM 119. *The Cartesian product and the parallel connection of the autonomous systems are autonomous systems.*

Proof. The hypothesis states that $f : U \to P^*(S^{(n)}), U \subset S^{(m)}$, $f' : U' \to P^*(S^{(n')}), U' \subset S^{(m')}$ are two systems with $\forall u \in U, f(u) = X$ and $\forall u' \in U', f'(u') = X'$, where $X \in P^*(S^{(n)})$, $X' \in P^*(S^{(n')})$ are some spaces of functions. We get

$$\forall u \times u' \in U \times U', (f \times f')(u \times u') = f(u) \times f'(u') = X \times X'.$$

The situation is similar in the second case. □

Theorem 120. *Let be the systems* $f : U \to P^*(S^{(n)}), U \subset S^{(m)}$, $h : X \to P^*(S^{(p)}), X \subset S^{(n)}$, *where* $\bigcup_{u \in U} f(u) \subset X$.

a) If f is autonomous, then $h \circ f$ is autonomous.
b) If h is autonomous, then $h \circ f$ is autonomous.
c) If f and h are autonomous, then $h \circ f$ is autonomous.

Proof. a) By hypothesis there is $X_1 \in P^*(S^{(n)})$ such that $\forall u \in U, f(u) = X_1$ and $X \subset X_1$. In these circumstances we have

$$\forall u \in U, (h \circ f)(u) = \bigcup_{x \in f(u)} h(x) = \bigcup_{x \in X_1} h(x).$$

b) there is $Y \in P^*(S^{(p)})$ such that $\forall x \in X, h(x) = Y$. Then

$$\forall u \in U, (h \circ f)(u) = \bigcup_{x \in f(u)} h(x) = Y.$$

□

Theorem 121. *Let be the autonomous systems f, g. If $f \cap g$ exists, then it is autonomous.*

Proof. When X, X_1 exist such that $\forall u \in U, f(u) = X$, $\forall u \in V, g(u) = X_1$, $U \cap V \neq \emptyset$ and $X \cap X_1 \neq \emptyset$, we have

$$\forall u \in U \cap V, (f \cap g)(u) = f(u) \cap g(u) = X \cap X_1.$$

□

Theorem 122. *Let be the systems f, g. If one of the following requirements of autonomy is fulfilled:*
a) $\exists X \in P^*(S^{(n)}), \forall u \in U, f(u) = X, \forall v \in V, g(v) = X$;
b) $U = V$ *and* $\exists X \in P^*(S^{(n)}), \forall u \in U, f(u) = X, \exists X' \in P^*(S^{(n)}), \forall u \in U, g(u) = X'$,
then $f \cup g$ is autonomous.

3. Finite input space

Definition 52. *The system $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$ is said to have a **finite input space** (or a **finite input set**) if the set U is finite.*

Theorem 123. *If $|U| = k \geq 1$, then f is the disjoint union of k autonomous systems (in the sense of Definition 51, b)).*

Proof. By hypothesis $U = \{u^1, ..., u^k\}$ and define the autonomous systems $f_q : U_q \to P^*(S^{(n)})$ by

$$U_q = \{u^q\},$$
$$\forall u \in U_q, f_q(u) = f(u),$$

$q = \overline{1, k}$. We have obtained that $f = f_1 \cup ... \cup f_k$, which is a disjoint union. □

Remark 38. *We may interpret the admissible inputs $u \in U$ as commands. Then the finitude of U obviously shows the fact that the circuit may be run in a finite number of ways.*

The fact that the systems with finite input space are unions of autonomous systems is a good reason to consider the autonomous systems be important in the analysis of the systems.

If g has a finite input space and $f \subset g$, then f has a finite input space too. If f has a finite input space, then f^* has the same property. The Cartesian product of systems with a finite input space is a system with a finite input space. If the parallel connection (f, f_1') exists and f, f_1' have a finite input space, then (f, f_1') has a finite input space too. If f has a finite input space and $h \circ f$ exists, then $h \circ f$ has a finite input space too. If f has a finite input space and $f \cap g$ exists, then $f \cap g$ has a finite input space too. If f and g have finite input spaces, then $f \cup g$ has the same property.

4. Finite and deterministic systems

Definition 53. *The system $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$ is **finite** if it satisfies the property that $\forall u \in U, f(u)$ has a finite number of elements and it is **infinite** otherwise. The system f is **deterministic** if $\forall u \in U, f(u)$ has a single element and it is **non-deterministic** otherwise.*

Remark 39. *Unlike the previous section where the word 'finite' characterized the input set, here 'finite' refers to the sets of possible states. This is the implicit sense of the attribute 'finite' given to a system.*

When the system is given under the implicit form, the finitude means that the equations and inequalities have a finite number of solutions and determinism means that the solution is unique.

Finitude is useful when during modeling we work with 'the most favorable case', 'the most unfavorable case', 'the most frequent case' etc.

The deterministic systems can be identified with the $U \to S^{(n)}$ functions, as we have already done. Such models show a perfect knowledge of the behavior of a circuit.

Example 30. *The system $f : S \to S$ defined by*

$$\forall u \in S, f(u)(t) = \int_{-\infty}^{t} D_{01}u$$

is deterministic, where we have denoted

$$\int_{-\infty}^{t} D_{01}u = \begin{cases} 0, if\ |(-\infty, t] \cap supp\ D_{01}u| \text{ is even} \\ 1, if\ |(-\infty, t] \cap supp\ D_{01}u| \text{ is odd} \end{cases}.$$

and 0 was considered to be an even number. In order to detect the correctness of this example, first let us remark that $\forall t \in \mathbf{R}, \forall u \in S$, the set $(-\infty, t] \cap suppD_{01}u$ is finite. Thus it makes a sense to refer to its parity. Second, for all $u \in S$, the function in t: $\int_{-\infty}^{t} D_{01}u$ belongs to S indeed.

EXAMPLE 31. *For the Boolean function $F : \mathbf{B}^m \to \mathbf{B}$, we denote by $\partial_j F : \mathbf{B}^m \to \mathbf{B}$, $j \in \{1, ..., m\}$ its Boolean derivatives*

$$\forall \lambda \in \mathbf{B}^m, \partial_j F(\lambda) = F(\lambda_1, ..., \underset{j}{0}, ..., \lambda_m) \oplus F(\lambda_1, ..., \underset{j}{1}, ..., \lambda_m).$$

The system $f : S^{(m)} \to S^{(m)}$ defined by

$$\forall u \in S^{(m)}, f(u)(t) = (\partial_1 F(u(t)), ..., \partial_m F(u(t)))$$

is deterministic.

THEOREM 124. *If f is a finite system, then it has a bounded initial time.*

PROOF. In the property of existence of the initial state with unbounded initial time

$$\forall u \in U, \forall x \in f(u), \exists \mu \in \mathbf{B}^n, \exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = \mu,$$

we fix an arbitrary $u \in U$ and suppose that $f(u) = \{x^1, ..., x^k\}$. Then $\mu^1, ..., \mu^k \in \mathbf{B}^n$ and $t_0^1, ..., t_0^k \in \mathbf{R}$ exist such that

$$\forall t < t_0^1, x^1(t) = \mu^1, ..., \forall t < t_0^k, x^k(t) = \mu^k.$$

Since the number $t_0 = \min\{t_0^1, ..., t_0^k\}$ depends only on u, it satisfies

$$\forall x \in f(u), \exists \mu \in \mathbf{B}^n, \forall t < t_0, x(t) = \mu.$$

Because u was arbitrarily chosen, we conclude that

$$\forall u \in U, \exists t_0 \in \mathbf{R}, \forall x \in f(u), \exists \mu \in \mathbf{B}^n, \forall t < t_0, x(t) = \mu.$$

□

THEOREM 125. *Let f be deterministic. Then it has race-free initial states and ϕ_0 satisfies $\forall u \in U, |\phi_0(u)| = 1$.*

PROOF. In the statement concerning the existence of the initial values of the states of f we have

$$\forall u \in U, \forall x \in f(u), \exists \mu \in \mathbf{B}^n, \exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = \mu.$$

Since $\forall x \in f(u)$ and $\exists \mu \in \mathbf{B}^n$ commute, f has race-free initial states. We have, of course,

$$\forall u \in U, |\phi_0(u)| = |\{x(-\infty + 0) | x \in f(u)\}| = 1.$$

□

THEOREM 126. *Let be the systems $f : U \to P^*(S^{(n)})$, $g : V \to P^*(S^{(n)})$, $U, V \in P^*(S^{(m)})$ with $f \subset g$. If g is finite, then f is finite too. If g is deterministic, then f is deterministic also and $f = g$.*

PROOF. For $\forall u \in U, f(u)$ has at most as many elements as $g(u)$. If $\forall u \in V, g(u)$ has exactly one element, then $\forall u \in U, f(u)$ has exactly one element and $f(u) = g(u)$. □

THEOREM 127. *If the axiom of choice is true, then any system $g : V \to P^*(S^{(n)})$, $V \in P^*(S^{(m)})$ includes a finite (a deterministic) system $f : U \to P^*(S^{(n)})$, $U \in P^*(S^{(m)})$.*

Proof. We take $U \subset V$ arbitrarily and non-empty. For any $u \in U$, the axiom of choice allows choosing from the set $g(u)$ some x and defining, in this manner, a selective function $f(u) = \{x\}$. The system f is deterministic and $\forall u \in U, f(u) \subset g(u)$. The union of a finite number of such deterministic systems $f_1, ..., f_k \subset g$ is a finite system $f_1 \cup ... \cup f_k \subset g$ having the domain $U \cup ... \cup U = U$. □

Theorem 128. *If the system f is finite (deterministic), then its dual f^* is finite (deterministic) too.*

Proof. For any $u \in U$, the finite sets $f^*(\overline{u}) = \{\overline{x}|x \in f(u)\}$ and $f(u) = \{x|x \in f(u)\}$ have the same number of elements. □

Theorem 129. *Suppose that the systems $f : U \to P^*(S^{(n)})$, $U \in P^*(S^{(m)})$, $f' : U' \to P^*(S^{(n')})$, $U' \in P^*(S^{(m')})$ are finite (deterministic). Then the system $f \times f'$ is finite (deterministic) too.*

Proof. This follows from the equality

$$\forall u \times u' \in U \times U', |(f \times f')(u \times u')| = |f(u) \times f'(u')| = |f(u)| \cdot |f'(u')|.$$

□

Theorem 130. *Let be the finite (deterministic) systems $f : U \to P^*(S^{(n)})$, $f_1' : U_1' \to P^*(S^{(n')})$, $U, U_1' \in P^*(S^{(m)})$ with $U \cap U_1' \neq \emptyset$. The system (f, f_1') is finite (deterministic).*

Theorem 131. *Consider the finite (deterministic) systems $f : U \to P^*(S^{(n)})$, $U \in P^*(S^{(m)})$, $h : X \to P^*(S^{(p)})$, $X \in P^*(S^{(n)})$ with $\bigcup\limits_{u \in U} f(u) \subset X$. Then $h \circ f$ is finite (deterministic).*

Proof. In the formula

$$\forall u \in U, (h \circ f)(u) = \bigcup_{x \in f(u)} h(x)$$

the finite union of finite sets is a finite set. □

Theorem 132. *If one of the systems $f : U \to P^*(S^{(n)})$, $g : V \to P^*(S^{(n)})$, $U, V \in P^*(S^{(m)})$ is finite (deterministic), then $\exists u \in U \cap V, f(u) \cap g(u) \neq \emptyset$ implies that $f \cap g$ is finite (deterministic); and if both systems are finite, then $f \cup g$ is finite.*

Proof. If f is finite, then $\forall u \in U \cap V$, the set $(f \cap g)(u)$ has at most as many elements as $f(u)$, wherefrom we infer the first statement of the theorem. If f, g are both finite, then $\forall u \in U \cup V$, $(f \cup g)(u)$ represents a finite set or the union of two finite sets as $u \in U \setminus V, u \in V \setminus U$ or $u \in U \cap V$. □

Theorem 133. *The system f is autonomous and finite (deterministic) iff $\exists X \in P^*(S^{(n)})$ finite (with one element) such that $\forall u \in U, f(u) = X$.*

Proof. Obvious. □

5. Ideal combinational systems

NOTATION 17. *Let be the Boolean function* $F : \mathbf{B}^m \to \mathbf{B}^n$ *and the number* $d \in \mathbf{R}$. *Denote by* $F_d : S^{(m)} \to S^{(n)}$ *the deterministic system defined by*

$$\forall u \in S^{(m)}, F_d(u)(t) = F(u(t-d)).$$

In the special case when $d = 0$ *we write* F *instead* F_0.

REMARK 40. *This notation was previously used under the form: if* $F : \mathbf{B} \to \mathbf{B}$, $\forall \lambda \in \mathbf{B}$, $F(\lambda) = \overline{\lambda}$, *then* $F : S \to S$ *is the system*

$$\forall u \in S, \overline{u}(t) = F(u)(t) = F(u(t)) = \overline{u(t)},$$

while for $F : \mathbf{B}^2 \to \mathbf{B}$, $\forall \lambda \in \mathbf{B}^2$, $F(\lambda) = \lambda_1 \cup \lambda_2$, *we have the system* $F : S^{(2)} \to S$,

$$\forall u \in S^{(2)}, (u_1 \cup u_2)(t) = F(u)(t) = F(u(t)) = u_1(t) \cup u_2(t)$$

etc.

DEFINITION 54. *The deterministic system* $f : U \to S^{(n)}, U \in P^*(S^{(m)})$ *is called an* ***ideal combinational system*** *if there are the Boolean function* $F : \mathbf{B}^m \to \mathbf{B}^n$ *and the number* $d \in \mathbf{R}$ *such that* $f \subset F_d$, *in other words if*

$$\forall u \in U, f(u)(t) = F_d(u)(t).$$

In this case we say that f *is* ***generated by the function*** F *and* F *is called the* ***generator function*** *of* f. *If* $d \geq 0$, *then it is called the* ***transmission delay for transitions***, *or, shortly the* ***delay*** *of* f *and we also say that* f ***has the delay*** d.

REMARK 41. *The ideal combinational systems are uni-valued (usual) functions, where the correspondence between the input* u *and the state* x *is given by the equation* $x(t) = F(u(t-d))$. *These are the models of the combinational circuits.*

In general, an ideal combinational system has several generator functions and several parameters d *(for example if* F *is constant).*

Due to its determinism, an ideal combinational system has a bounded initial time and race-free initial states. Moreover it fulfills all the properties of the finite systems from the Section 4.

It is not necessary in Definition 54 to ask that $d \geq 0$; *this property is one of non-anticipation.*

EXAMPLE 32. *Recall Examples 11, 12, 13 and 15 from Ch. 3. These pseudo-systems induce ideal combinational systems, in the sense of Definition 54, with the generator functions: the identity* $1_{\mathbf{B}^m}$, *the projection* $\pi_j : \mathbf{B}^m \to \mathbf{B}$, $j \in \{1, ..., m\}$, *the constant function* $\mu : \mathbf{B}^m \to \mathbf{B}^n$, *plus the general case. In Example 31 the derivative* $\partial F : \mathbf{B}^m \to \mathbf{B}^m$, $\forall \lambda \in \mathbf{B}^m$, $\partial F(\lambda) = (\partial_1 F(\lambda), ..., \partial_m F(\lambda))$ *of the function* $F : \mathbf{B}^m \to \mathbf{B}$ *occurred, generating an ideal combinational system.*

THEOREM 134. *Consider* $F : \mathbf{B}^m \to \mathbf{B}^n$ *and* $d \in \mathbf{R}$. *Then*

$$\forall u \in S^{(m)}, F_d(u) = F(u \circ \tau^d) = F(u) \circ \tau^d.$$

PROOF. For any $u \in S^{(m)}$,

$$F(u \circ \tau^d)(t) = F((u \circ \tau^d)(t)) = F(u(t-d)) = F(u)(t-d) = (F(u) \circ \tau^d)(t).$$

□

THEOREM 135. *Let $f \subset F_d$ be an ideal combinational system. The equation*

$$\forall u \in U, \phi_0(u) = F(u(-\infty + 0))$$

is true.

THEOREM 136. *A subsystem of an ideal combinational system is ideal combinational.*

PROOF. From $g \subset F_d$ and $f \subset g$ we infer $f \subset F_d$. $\square$

THEOREM 137. $(F^*)_d = (F_d)^*$

PROOF. For any $u \in S^{(m)}$ we have

$$(F^*)_d(u) = F^*(u \circ \tau^d) = \overline{F(\overline{u \circ \tau^d})} = \overline{F(\overline{u} \circ \tau^d)} = \overline{F_d(\overline{u})} = (F_d)^*(u).$$

$\square$

NOTATION 18. *The previous theorem allows us to use the notation F_d^* for any of $(F^*)_d$ and $(F_d)^*$.*

THEOREM 138. *If $f \subset F_d$, then $f^* \subset F_d^*$.*

PROOF. The hypothesis states that $f : U \to S^{(n)}$ satisfies $\forall u \in U, f(u) = F_d(u)$ wherefrom $\forall u \in U^*, f^*(u) = \overline{f(\overline{u})} = \overline{F_d(\overline{u})} = F_d^*(u)$. $\square$

THEOREM 139. *The inverse of the system $F_d : S^{(m)} \to S^{(n)}$ satisfies*

$$\forall x \in S^{(n)}, (F_d)^{-1}(x) = F^{-1}(x \circ \tau^{-d}).$$

PROOF. In the equality

$$F(u(t - d)) = x(t),$$

where $u \in S^{(m)}, x \in S^{(n)}$, we make the substitution $t' = t - d$ and obtain

$$F(u(t')) = x(t' + d) = (x \circ \tau^{-d})(t').$$

Thus

$$\forall x \in S^{(n)}, (F_d)^{-1}(x) = \{u | u \in S^{(m)}, F_d(u) = x\} =$$
$$= \{u | u \in S^{(m)}, F(u) = x \circ \tau^{-d}\} = F^{-1}(x \circ \tau^{-d}).$$

$\square$

THEOREM 140. *Let be $f : U \to P^*(S^{(n)})$, $U \in P^*(S^{(m)})$, $f' : U' \to P^*(S^{(n')})$, $U' \in P^*(S^{(m')})$, $F : \mathbf{B}^m \to \mathbf{B}^n$, $F' : \mathbf{B}^{m'} \to \mathbf{B}^{n'}$ and $d \in \mathbf{R}$. If $f \subset F_d, f' \subset F'_d$, then $f \times f' \subset (F \times F')_d$, where we have denoted by $F \times F' : \mathbf{B}^m \times \mathbf{B}^{m'} \to \mathbf{B}^n \times \mathbf{B}^{n'}$ the function*

$$\forall (\lambda, \lambda') \in \mathbf{B}^m \times \mathbf{B}^{m'}, (F \times F')(\lambda, \lambda') = (F(\lambda), F'(\lambda')).$$

PROOF. The fact that $f \times f' \subset F_d \times F'_d$ follows from Theorem 63 and furthermore we have

$$\forall u \times u' \in S^{(m)} \times S^{(m')}, (F_d \times F'_d)(u \times u') = F_d(u) \times F'_d(u') = F(u \circ \tau^d) \times F'(u' \circ \tau^d) =$$
$$= (F \times F')(u \circ \tau^d \times u' \circ \tau^d) = (F \times F')((u \times u') \circ \tau^d) = (F \times F')_d(u \times u').$$

$\square$

THEOREM 141. *The systems* $f : U \to P^*(S^{(n)})$, $f_1' : U_1' \to P^*(S^{(n')})$, $U, U_1' \in P^*(S^{(m)})$ *are given with* $U \cap U_1' \neq \emptyset$ *as well as the Boolean functions* $F : \mathbf{B}^m \to \mathbf{B}^n$, $F_1' : \mathbf{B}^m \to \mathbf{B}^{n'}$ *and the number* $d \in \mathbf{R}$. *If* $f \subset F_d$, $f_1' \subset F_{1d}'$, *then* $(f, f_1') \subset (F, F_1')_d$, *where we have used the notation* $(F, F_1') : \mathbf{B}^m \to \mathbf{B}^n \times \mathbf{B}^{n'}$ *for the function*

$$\forall \lambda \in \mathbf{B}^m, (F, F_1')(\lambda) = (F(\lambda), F_1'(\lambda)).$$

THEOREM 142. *For the functions* $F : \mathbf{B}^m \to \mathbf{B}^n$, $H : \mathbf{B}^n \to \mathbf{B}^p$ *and the numbers* $d \in \mathbf{R}$, $d' \in \mathbf{R}$ *the following property*

$$H_{d'} \circ F_d = (H \circ F)_{d+d'}$$

holds true, where $H \circ F : \mathbf{B}^m \to \mathbf{B}^p$ *is the usual composition of the functions.*

PROOF. We remark first of all that the support of $H_{d'}$ is $S^{(n)}$, thus $H_{d'} \circ F_d$ is defined. By taking into account Theorem 134, we note that $\forall u \in S^{(m)}$ we have

$$(H_{d'} \circ F_d)(u) = H_{d'}(F_d(u)) = H_{d'}(F(u \circ \tau^d)) = H_{d'}(F(u) \circ \tau^d) = H(F(u) \circ \tau^d \circ \tau^{d'}) =$$

$$= H(F(u) \circ \tau^{d+d'}) = H(F(u)) \circ \tau^{d+d'} = (H \circ F)(u) \circ \tau^{d+d'} = (H \circ F)_{d+d'}(u).$$

□

THEOREM 143. *The systems* $f : U \to P^*(S^{(n)})$, $U \in P^*(S^{(m)})$, $h : X \to P^*(S^{(p)})$, $X \in P^*(S^{(n)})$ *are given and the requirement is* $\bigcup_{u \in U} f(u) \subset X$. *Let be also the functions* $F : \mathbf{B}^m \to \mathbf{B}^n$, $H : \mathbf{B}^n \to \mathbf{B}^p$ *and the numbers* $d \in \mathbf{R}$, $d' \in \mathbf{R}$. *If* $f \subset F_d$ *and* $h \subset H_{d'}$, *then* $h \circ f \subset (H \circ F)_{d+d'}$.

PROOF. We infer

$$\begin{array}{rcl} h \circ f & \subset & h \circ F_d \text{ (Theorem 74, a))} \\ & \subset & H_{d'} \circ F_d \text{ (Theorem 74, b))} \\ & = & (H \circ F)_{d+d'} \text{ (Theorem 142).} \end{array}$$

□

THEOREM 144. *Consider the systems* $f : U \to P^*(S^{(n)})$ *and* $g : V \to P^*(S^{(n)})$, $U, V \in P^*(S^{(m)})$ *satisfying* $f \subset F_d, g \subset F_d$.

a) If $U \cap V \neq \emptyset$, *then* $f \cap g$ *exists and* $f \cap g \subset F_d$.

b) $f \cup g \subset F_d$.

REMARK 42. *The morphisms characterizing the ideal combinational systems are those satisfying the commutativity of the diagrams*

$$\begin{array}{ccc} U & \xrightarrow{f} & S^{(n)} \\ \omega \downarrow & & \downarrow 1_{S^{(n)}} \\ S^{(m)} & \xrightarrow{F_d} & S^{(n)} \end{array}$$

where ω *is the canonical injection.*

The autonomy of these systems is given, for example, by the constant Boolean function.

6. Self-duality

DEFINITION 55. *The Boolean function $F : \mathbf{B}^m \to \mathbf{B}^n$ is called **self-dual** if*

$$\forall \lambda \in \mathbf{B}^m, F(\lambda) = F^*(\lambda)$$

is true.

EXAMPLE 33. *Two self-dual $F : \mathbf{B} \to \mathbf{B}$ functions are $F(\lambda) = \lambda$ and $F(\lambda) = \overline{\lambda}$, $\lambda \in \mathbf{B}$ and four $F : \mathbf{B}^2 \to \mathbf{B}$ self-dual functions are: $F(\lambda) = \lambda_1$, $F(\lambda) = \overline{\lambda_1}$, $F(\lambda) = \lambda_2$, $F(\lambda) = \overline{\lambda_2}$, $\lambda \in \mathbf{B}^2$. The function $F : \mathbf{B}^3 \to \mathbf{B}$, $F(\lambda_1, \lambda_2, \lambda_3) = \lambda_1 \oplus \lambda_2 \oplus \lambda_3$ is self-dual too.*

DEFINITION 56. *The set U is called **self-dual** if it satisfies one of the equivalent conditions:*

a) $U = U^*$;

b) $\forall u, u \in U \Longrightarrow \overline{u} \in U$.

EXAMPLE 34. *The set $S^{(m)}$ is self-dual.*

EXAMPLE 35. *Let $j \in \{1, ..., m\}$ and $\delta \geq 0$ be fixed. The set*

$$U = \{u | u \in S^{(m)},\ \overline{u_j(t-0)} \cdot u_j(t) \leq \bigcap_{\xi \in [t, t+\delta]} u_j(\xi),\ u_j(t-0) \cdot \overline{u_j(t)} \leq \bigcap_{\xi \in [t, t+\delta]} \overline{u_j(\xi)}\}$$

is self-dual.

LEMMA 1. *Consider the system $f : U \to P^*(S^{(n)})$, $U \in P^*(S^{(m)})$. The following statements are equivalent:*

a) $f = f^*$;

b) $U = U^*$ *and* $\forall u \in U, \forall x \in f(u), \overline{x} \in f(\overline{u})$;

c) $U = U^*$ *and* $\forall u \in U, f(\overline{u}) = \{\overline{x} | x \in f(u)\}$;

d) $U = U^*$ *and the diagram*

$$\begin{array}{ccc} U & \xrightarrow{f} & P^*(S^{(n)}) \\ \omega \downarrow & & \downarrow \Omega \\ U & \xrightarrow{f} & P^*(S^{(n)}) \end{array}$$

is commutative, where we have denoted

$$\forall u \in U, \omega(u) = \overline{u},$$

$$\forall X \in P^*(S^{(n)}), \Omega(X) = X^*,$$

i.e. $(\omega, \Omega) : f \to f$ is a morphism of systems.

PROOF. *a)* $\Longrightarrow$ *b)* $U = U^*$ is true and, on the other hand, $\forall u \in U, \forall x \in f(u), x \in \{\overline{y} | y \in f(\overline{u})\} = \{y | \overline{y} \in f(\overline{u})\}$ takes place.

b) $\Longrightarrow$ *c)* $\forall u \in U, \forall x \in f(\overline{u}), \overline{x} \in f(u)$, thus $f(\overline{u}) = \{x | x \in f(\overline{u})\} \subset \{x | \overline{x} \in f(u)\} = \{\overline{x} | x \in f(u)\}$. Conversely, $\forall u \in U, \forall x \in f(u)$ we get $\overline{x} \in f(\overline{u})$, thus $\{\overline{x} | x \in f(u)\} \subset f(\overline{u})$.

c) $\Longrightarrow$ *d)* is obvious.

d) $\Longrightarrow$ *a)* $\forall u \in U, f(\overline{u}) = f(\omega(u)) = (f \circ \omega)(u) = (\Omega \circ f)(u) = \Omega(f(u)) = \{\overline{x} | x \in f(u)\}$, in other words $\forall u \in U, f(u) = \{\overline{x} | x \in f(\overline{u})\} = f^*(u)$. □

DEFINITION 57. *The system f is called **self-dual**, or **symmetrical** (in the rising-falling sense) if one of the previous equivalent properties a),...,d) is true. Otherwise, f is called **asymmetrical** (in the rising-falling sense).*

REMARK 43. *The self-duality of f states that the form of x under the input u coincides with the form of $\overline{x}$ under the input $\overline{u}$ and the terminology of rising-falling symmetry is due to the fact that while $x(t)$ switches at the time instant t in the rising (falling) sense, $\overline{x}(t)$ switches at the time instant t in the falling (rising) sense:* $\forall u \in U, \forall x \in f(u), \forall i \in \{1, ..., n\}$,

$$\overline{x_i(t-0)} \cdot x_i(t) = \overline{x_i(t-0)} \cdot \overline{\overline{x_i(t)}},\ x_i(t-0) \cdot \overline{x_i(t)} = \overline{\overline{x_i(t-0)}} \cdot \overline{x_i(t)}.$$

EXAMPLE 36. *Let $F : \mathbf{B}^m \to \mathbf{B}^n$ be a self-dual function and $d \in \mathbf{R}$. The ideal combinational system $F_d : S^{(m)} \to S^{(n)}$ is self-dual.*

EXAMPLE 37. *The $S^{(2)} \to S$ system defined by the following inequality*

$$u_1(t) \cdot u_2(t) \leq x(t) \leq u_1(t) \cup u_2(t)$$

is self-dual. This is easily seen from Lemma 1, b): if x satisfies the inequality, then $\overline{x}$ satisfies

$$\overline{u_1(t)} \cdot \overline{u_2(t)} \leq \overline{x(t)} \leq \overline{u_1(t)} \cup \overline{u_2(t)}.$$

EXAMPLE 38. *The $S \to P^*(S)$ system*

$$\bigcap_{\xi\in[t-d,t-d+m]} u(\xi) \leq x(t) \leq \bigcup_{\xi\in[t-d,t-d+m]} u(\xi)$$

is self-dual, where $0 \leq m \leq d$. Like previously, this is seen from Lemma 1, b): if x satisfies that inequality, then $\overline{x}$ satisfies the following inequality

$$\bigcap_{\xi\in[t-d,t-d+m]} \overline{u(\xi)} \leq \overline{x(t)} \leq \bigcup_{\xi\in[t-d,t-d+m]} \overline{u(\xi)}.$$

THEOREM 145. *If f is self-dual, then $\phi_0 = \phi_0^*$ and $\Theta_0 = \Theta_0^*$, where we have denoted by ϕ_0^*, Θ_0^* the initial state function and the set of the initial states of f^*.*

PROOF. We have $U = U^*$ and $\forall u \in U$,

$$\phi_0(u) = \{x(-\infty+0) | x \in f(u)\} = \{x(-\infty+0) | x \in f^*(u)\} = \phi_0^*(u).$$

□

THEOREM 146. *If f is self-dual, then f^* is self-dual too.*

PROOF. From $U = U^*$ we get $U^* = (U^*)^*$ and from $f = f^*$ we infer $f^* = (f^*)^*$. □

THEOREM 147. *If f is self-dual, then f^{-1} is self-dual too.*

PROOF. We denote by $X = \bigcup\limits_{u\in U} f(u)$ the support set of f^{-1}. We prove that X is self-dual: $\forall x$,

$$x \in \bigcup_{u\in U} f(u) \Longrightarrow \exists u \in U, x \in f(u) \Longrightarrow \exists \overline{u} \in U, x \in f(u) \Longrightarrow \exists u \in U, x \in f(\overline{u}) \Longrightarrow$$

$$\Longrightarrow \exists u \in U, \overline{x} \in f^*(u) \Longrightarrow \exists u \in U, \overline{x} \in f(u) \Longrightarrow \overline{x} \in \bigcup_{u\in U} f(u).$$

For any $x \in X$ we have

$$(f^{-1})(x) = (f^*)^{-1}(x) = (f^{-1})^*(x)$$

and we have used Theorem 55. □

THEOREM 148. *If the systems* $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$ *and* $f' : U' \to P^*(S^{(n')}), U' \in P^*(S^{(m')})$ *are self-dual, then the system* $f \times f'$ *is self-dual too.*

PROOF. We show that $U \times U'$ is a self-dual set

$$U \times U' = \{u \times u' | u \in U, u' \in U'\} = \{u \times u' | \overline{u} \in U, \overline{u'} \in U'\} =$$
$$= \{\overline{u} \times \overline{u'} | u \in U, u' \in U'\} = \{\overline{u \times u'} | u \in U, u' \in U'\} = (U \times U')^*.$$

Furthermore

$$\forall u \times u' \in U \times U', (f \times f')(u \times u') = (f^* \times f'^*)(u \times u') = (f \times f')^*(u \times u')$$

and, in the last equality, we have used Theorem 64. □

THEOREM 149. *Suppose that the systems* $f : U \to P^*(S^{(n)})$, $f_1' : U_1' \to P^*(S^{(n')}), U, U_1' \in P^*(S^{(m)})$ *are self-dual and* $U \cap U_1' \neq \emptyset$. *Then the parallel connection* (f, f_1') *is self-dual.*

PROOF. We show that $U \cap U'$ is self-dual

$$U \cap U' = \{u | u \in U \text{ and } u \in U'\} = \{u | \overline{u} \in U \text{ and } \overline{u} \in U'\} =$$
$$= \{\overline{u} | u \in U \text{ and } u \in U'\} = \{\overline{u} | u \in U \cap U'\} = (U \cap U')^*$$

etc. □

THEOREM 150. *Let be the self-dual systems* $f : U \to P^*(S^{(n)})$, $U \in P^*(S^{(m)})$ *and* $h : X \to P^*(S^{(p)})$, $X \in P^*(S^{(n)})$. *If* $\bigcup_{u \in U} f(u) \subset X$, *then the system* $h \circ f$ *is self-dual.*

PROOF. By hypothesis $U = U^*$. Furthermore, by Theorem 75 we have that

$$\forall u \in U, (h \circ f)(u) = (h^* \circ f^*)(u) = (h \circ f)^*(u).$$

□

THEOREM 151. *Consider the self-dual systems* $f : U \to P^*(S^{(n)})$ *and* $g : V \to P^*(S^{(n)})$, *where* $U, V \in P^*(S^{(m)})$. *If the set*

$$W = \{u | u \in U \cap V, f(u) \cap g(u) \neq \emptyset\}$$

is non-empty, then the system $f \cap g$ *is self-dual.*

PROOF. We show that W is self-dual. Indeed,

$$W = \{u | u \in U \cap V, f(u) \cap g(u) \neq \emptyset\} = \{\overline{u} | \overline{u} \in U \cap V, f(\overline{u}) \cap g(\overline{u}) \neq \emptyset\} =$$
$$= \{\overline{u} | \overline{u} \in U \text{ and } \overline{u} \in V \text{ and } f^*(\overline{u}) \cap g^*(\overline{u}) \neq \emptyset\} =$$
$$= \{\overline{u} | u \in U \text{ and } u \in V \text{ and } \{\overline{x} | x \in f(u)\} \cap \{\overline{x} | x \in g(u)\} \neq \emptyset\} =$$
$$= \{\overline{u} | u \in U \cap V, f(u) \cap g(u) \neq \emptyset\} = W^*.$$

The system $f \cap g$ is self-dual because

$$\forall u \in W, (f \cap g)(u) = (f^* \cap g^*)(u) = (f \cap g)^*(u).$$

We have used Theorem 88. □

THEOREM 152. *If the systems* f, g *are self-dual, then* $f \cup g$ *is self-dual too.*

PROOF. We note that $U \cup V$ is self-dual. Then we apply Theorem 101. □

REMARK 44. *The autonomous system $f = X$ is self-dual iff both U, X are self-dual. If we consider Definition 51, b) of autonomy, then self-dual autonomous systems do not exist since U with $|U| = 1$ and $U = U^*$ is impossible. Definition 51, c) of autonomy gives the self-dual systems that fulfill $X = X^*$.*

7. Symmetry

NOTATION 19. *Let $S(\{1, ..., m\})$ be the notation of the symmetrical group of the set $\{1, ..., m\}$. Its elements are the bijections (the permutations) $\sigma : \{1, ..., m\} \to \{1, ..., m\}$.*

NOTATION 20. *Let be the bijection $\sigma \in S(\{1, ..., m\})$. For each $\lambda \in \mathbf{B}^m$, $\lambda = (\lambda_1, ..., \lambda_m)$ we denote by $\lambda_\sigma \in \mathbf{B}^m$ the vector $\lambda_\sigma = (\lambda_{\sigma(1)}, ..., \lambda_{\sigma(m)})$.*

DEFINITION 58. *The Boolean function $F : \mathbf{B}^m \to \mathbf{B}^n$ is called **symmetrical** if for any bijection σ we have*

$$\forall \lambda \in \mathbf{B}^m, F(\lambda) = F(\lambda_\sigma)$$

*and **asymmetrical** otherwise.*

NOTATION 21. *For any bijective function $\sigma \in S(\{1, ..., m\})$ and any $u \in S^{(m)}$ we denote by $u_\sigma \in S^{(m)}$ the function $u_\sigma = (u_{\sigma(1)}, ..., u_{\sigma(m)})$.*

DEFINITION 59. *The set $U \in P^*(S^{(m)})$ is called **invariant to permutations** if for any bijection σ we have*

$$\forall u, u \in U \Longrightarrow u_\sigma \in U.$$

EXAMPLE 39. *The sets $U \in P^*(S)$ are invariant to permutations and $S^{(m)}$ has the same property.*

DEFINITION 60. *The system f is called **symmetrical** if U is invariant to permutations and for any $\sigma \in S(\{1, ..., m\})$ we have $\forall u \in U, f(u) = f(u_\sigma)$. Otherwise, it is called **asymmetrical**.*

REMARK 45. *This definition is natural because all the simple logical gates: NOT, AND, OR, NAND, XOR are symmetrical (relative to the inputs) and their models may have the same property.*

EXAMPLE 40. *The autonomous systems are symmetrical whenever their input set is invariant to permutations.*

EXAMPLE 41. *All the systems with $m = 1$ are symmetrical.*

EXAMPLE 42. *If $F : \mathbf{B}^m \to \mathbf{B}^n$ is a symmetrical function and $d \in \mathbf{R}$ is an arbitrary number, the ideal combinational system $F_d : S^{(m)} \to S^{(n)}$ is symmetrical (we remind that the set $S^{(m)}$ is invariant to permutations).*

EXAMPLE 43. *The $f : S^{(m)} \to P^*(S)$ system given by*

$$\forall u \in S^{(m)}, f(u) = \{x | x(t) \geq u_1(t) \cdot ... \cdot u_m(t)\}$$

is symmetrical.

EXAMPLE 44. *The $S^{(m)} \to P^*(S)$ system*

$$\bigcap_{\xi \in [t-d,t)} (u_1(\xi) \cdot ... \cdot u_m(\xi)) \leq x(t) \leq \bigcup_{\xi \in [t-d,t)} (u_1(\xi) \cup ... \cup u_m(\xi))$$

with $d > 0$, is symmetrical.

THEOREM 153. *Let f be symmetrical. Then, for any bijection $\sigma \in S(\{1,...,m\})$, the initial state function ϕ_0 fulfills*

$$\forall u \in U, \phi_0(u) = \phi_0(u_\sigma).$$

PROOF. For any $\sigma \in S(\{1,...,m\})$ and any $u \in U$ we have

$$\phi_0(u) = \{x(-\infty+0)|x \in f(u)\} = \{x(-\infty+0)|x \in f(u_\sigma)\} = \phi_0(u_\sigma).$$

□

THEOREM 154. *If f is symmetrical, then so is f^*.*

PROOF. By hypothesis, U is invariant to permutations and we show that U^* is invariant to permutations. Indeed, for any bijection σ and any input u we have

$$u \in U^* \Longrightarrow \overline{u} \in U \Longrightarrow \overline{u}_\sigma \in U \Longrightarrow \overline{u_\sigma} \in U \Longrightarrow u_\sigma \in U^*.$$

Furthermore, $\forall \sigma \in S(\{1,...,m\})$, $\forall u \in U^*$

$$f^*(u) = \{\overline{x}|x \in f(\overline{u})\} = \{\overline{x}|x \in f(\overline{u}_\sigma)\} = \{\overline{x}|x \in f(\overline{u_\sigma})\} = f^*(u_\sigma).$$

□

THEOREM 155. *Let $f : U \to P^*(S^{(n)})$, $U \in P^*(S^{(m)})$ be a symmetrical system. The inverse system $f^{-1} : X \to P^*(S^{(m)})$, $X = \{x|\exists u \in U, x \in f(u)\}$ satisfies the property that $\forall x \in X, f^{-1}(x)$ is invariant to permutations.*

PROOF. For any bijection $\sigma \in S(\{1,...,m\})$ and any $x \in X$ we have

$$u \in f^{-1}(x) \Longrightarrow u \in U \text{ and } x \in f(u) \Longrightarrow u_\sigma \in U \text{ and } x \in f(u_\sigma) \Longrightarrow u_\sigma \in f^{-1}(x).$$

□

REMARK 46. *In general the Cartesian product of symmetrical systems is not a symmetrical system. In fact, if the support sets U, U' of f, f' are invariant to permutations, we cannot say whether $U \times U'$ is or is not invariant to permutations.*

THEOREM 156. *Suppose that the systems $f : U \to P^*(S^{(n)})$, $f_1' : U_1' \to P^*(S^{(n')})$, $U, U_1' \in P^*(S^{(m)})$ are symmetrical and $U \cap U_1' \neq \emptyset$. Then the parallel connection (f, f_1') is symmetrical.*

PROOF. Let be the permutation $\sigma \in S(\{1,...,m\})$ and the input $u \in U \cap U_1'$. The fact that $u_\sigma \in U, u_\sigma \in U_1'$ are both true implies $u_\sigma \in U \cap U_1'$, thus $U \cap U_1'$ is invariant to permutations. Moreover,

$$\forall u \in U \cap U_1', (f, f_1')(u) = \{x \times x'|x \in f(u), x' \in f_1'(u)\} =$$

$$= \{x \times x'|x \in f(u_\sigma), x' \in f_1'(u_\sigma)\} = (f, f_1')(u_\sigma).$$

□

THEOREM 157. *We require that the systems $f : U \to P^*(S^{(n)})$, $U \in P^*(S^{(m)})$ and $h : X \to P^*(S^{(p)})$, $X \in P^*(S^{(n)})$ fulfill the condition $\bigcup\limits_{u \in U} f(u) \subset X$. If f is symmetrical, then the serial connection $h \circ f$ is symmetrical too.*

PROOF. The set U is invariant to permutations. For any $\sigma \in S(\{1,...,n\})$ and any $u \in U$ we have

$$(h \circ f)(u) = \bigcup_{x \in f(u)} h(x) = \bigcup_{x \in f(u_\sigma)} h(x) = (h \circ f)(u_\sigma).$$

$\square$

THEOREM 158. *Let be the symmetrical systems $f : U \to P^*(S^{(n)})$ and $g : V \to P^*(S^{(n)})$, $U, V \in P^*(S^{(m)})$. If the set $W = \{u|u \in U \cap V, f(u) \cap g(u) \neq \emptyset\}$ is non-empty, then $f \cap g$ is symmetrical.*

PROOF. We show that W is invariant to permutations. Let be the bijection $\sigma \in S(\{1,...,m\})$. For any $u \in W$ we have $u \in U \cap V$ and $f(u) \cap g(u) \neq \emptyset$, thus $u_\sigma \in U \cap V$ and $f(u_\sigma) \cap g(u_\sigma) \neq \emptyset$ and, eventually, $u_\sigma \in W$. We obtain

$$(f \cap g)(u) = f(u) \cap g(u) = f(u_\sigma) \cap g(u_\sigma) = (f \cap g)(u_\sigma).$$

$\square$

THEOREM 159. *The union of the symmetrical systems is symmetrical.*

PROOF. We show that $U \cup V$ is invariant to permutations. For the bijection $\sigma \in S(\{1,...,m\})$ and $u \in U \cup V$, if $u \in U$ then $u_\sigma \in U$, thus $u_\sigma \in U \cup V$ etc. $\square$

THEOREM 160. *For the system f the following statements are equivalent:*

a) f is self-dual and symmetrical;

b) for any bijection $\sigma \in S(\{1,...,m\})$ and any $u \in U$ we have $\overline{u}_\sigma \in U$ and, moreover, for any bijection σ, any input $u \in U$ and any state x, $x \in f(u) \Longrightarrow \overline{x} \in f(\overline{u}_\sigma)$.

PROOF. Take an arbitrary bijection $\sigma \in S(\{1,...,m\})$ and an arbitrary $u \in U$.

a) $\Longrightarrow$ *b)* U is self-dual which shows that $\overline{u} \in U$ and the fact that U is invariant to permutations implies that $\overline{u}_\sigma \in U$. Let be an arbitrary $x \in f(u)$. The system f is self-dual therefore $\overline{x} \in f(\overline{u})$ and the fact that f is symmetrical indicates that $f(\overline{u}) = f(\overline{u}_\sigma)$, thus we have obtained $\overline{x} \in f(\overline{u}_\sigma)$.

b) $\Longrightarrow$ *a)* The relation $\overline{u} \in U$ is true for the identical bijection $1_{\{1,...,m\}}$, thus U is self-dual. For the identical bijection we have also $\forall x, x \in f(u) \Longrightarrow \overline{x} \in f(\overline{u})$ showing that f is self-dual. Because $\overline{u} \in U$ is true and we infer $u_\sigma \in U$, we have that U is invariant to permutations. From $\forall x, x \in f(u) \Longrightarrow \overline{x} \in f(\overline{u}_\sigma)$, $\overline{x} \in f(\overline{u}_\sigma) \Longrightarrow x \in f(u_\sigma)$ we get that $\forall x, x \in f(u) \Longrightarrow x \in f(u_\sigma)$ following $f(u) \subset f(u_\sigma)$. The inverse inclusion is shown like this: $f(u_\sigma) \subset f((u_\sigma)_{\sigma^{-1}}) = f(u)$, in other words $f(u) = f(u_\sigma)$, hence f is symmetrical. $\square$

8. Time invariance

DEFINITION 61. *The set $U \subset S^{(m)}$ is said to be **invariant to translations** if it fulfills*

$$\forall d \in \mathbf{R}, \forall u, u \in U \Longrightarrow u \circ \tau^d \in U.$$

EXAMPLE 45. *The set $U = S^{(m)}$ is invariant to translations.*

EXAMPLE 46. *The set $U = \{u|u \in S^{(m)}, u_1, ..., u_m$ are monotonous$\}$ is itself invariant to translations.*

EXAMPLE 47. *Let U be the set of the signals $u \in S^{(m)}$ satisfying, for some $\delta_r > 0, \delta_f > 0$, the properties:* $\forall j \in \{1,...,m\}$,

$$\overline{u_j(t-0)} \cdot u_j(t) \leq \bigcap_{\xi \in [t,t+\delta_r]} u_j(\xi),$$

$$u_j(t-0) \cdot \overline{u_j(t)} \leq \bigcap_{\xi \in [t,t+\delta_f]} \overline{u_j(\xi)}.$$

The set U is invariant to translations.

LEMMA 2. *Let be the system $f : U \to P^*(S^{(n)})$ where the set $U \in P^*(S^{(m)})$ is invariant to translations. The following statements are equivalent:*

a) $\forall d \in \mathbf{R}, \forall u \in U, \forall x \in f(u), x \circ \tau^d \in f(u \circ \tau^d)$;

b) $\forall d \in \mathbf{R}, \forall u \in U, f(u \circ \tau^d) = \{x \circ \tau^d | x \in f(u)\}$;

c) for all $d \in \mathbf{R}$, the diagrams

$$\begin{array}{ccc} U & \xrightarrow{f} & P^*(S^{(n)}) \\ \omega_d \downarrow & & \downarrow \Omega_d \\ U & \xrightarrow{f} & P^*(S^{(n)}) \end{array}$$

are commutative, i.e. $(\omega_d, \Omega_d) : f \to f$ is a morphism of systems. We have denoted

$$\omega_d : U \to U, \forall u \in U, \omega_d(u) = u \circ \tau^d,$$

$$\Omega_d : P^*(S^{(n)}) \to P^*(S^{(n)}), \forall X \in P^*(S^{(n)}), \Omega_d(X) = \{x \circ \tau^d | x \in X\}.$$

PROOF. Let $d \in \mathbf{R}$ and $u \in U$ arbitrary.

a) $\Longrightarrow$ *b)* From the fact that $x \in f(u)$ implies $x \circ \tau^d \in f(u \circ \tau^d)$, we have $\{x \circ \tau^d | x \in f(u)\} \subset f(u \circ \tau^d)$. We take some $y \in f(u \circ \tau^d)$. Because $y \circ \tau^{-d} \in f(u)$, we get $f(u \circ \tau^d) \subset \{y | y \circ \tau^{-d} \in f(u)\}$. From

$$\{x \circ \tau^d | x \in f(u)\} \subset f(u \circ \tau^d) \subset \{y | y \circ \tau^{-d} \in f(u)\} = \{x \circ \tau^d | x \in f(u)\}$$

we infer b).

b) $\Longrightarrow$ *a)* If $x \in f(u)$, then $x \circ \tau^d \in f(u \circ \tau^d)$, in other words *a)* is true.

b) $\Longrightarrow$ *c)* $(f \circ \omega_d)(u) = f(\omega_d(u)) = f(u \circ \tau^d) = \{x \circ \tau^d | x \in f(u)\} = \Omega_d(f(u)) = (\Omega_d \circ f)(u)$.

c) $\Longrightarrow$ *b)* $f(u \circ \tau^d) = f(\omega_d(u)) = (f \circ \omega_d)(u) = (\Omega_d \circ f)(u) = \Omega_d(f(u)) = \{x \circ \tau^d | x \in f(u)\}$. □

DEFINITION 62. *The system f is* ***time invariant*** *if U is invariant to translations and one of the previous equivalent conditions a), b), c) from Lemma 2 is fulfilled. If f is not time invariant, then we say that it is* ***time variant****.*

EXAMPLE 48. *We show that the system $f : S^{(m)} \to S$ defined by $\forall u \in S^{(m)}, f(u) = u_j \circ \tau^{d'}$ is time invariant, where $j \in \{1,...,m\}$ and $d' \in \mathbf{R}$. $S^{(m)}$ is invariant to translations and for $x = u_j \circ \tau^{d'}$ we have that*

$$\forall d \in \mathbf{R}, \forall u \in S^{(m)}, f(u \circ \tau^d) = (u \circ \tau^d)_j \circ \tau^{d'} = u_j \circ \tau^{d+d'} = (u_j \circ \tau^{d'}) \circ \tau^d = x \circ \tau^d.$$

EXAMPLE 49. *More general than previously, any ideal combinational system $F_{d'}$ is time invariant:*

$$\forall d \in \mathbf{R}, \forall u \in S^{(m)}, F_{d'}(u \circ \tau^d)(t) = F((u \circ \tau^d)(t-d')) = F(u(t-d-d')) =$$

$$= F_{d'}(u(t-d)) = F_{d'}(u)(t-d) = (F_{d'}(u) \circ \tau^d)(t).$$

EXAMPLE 50. *Let the system $f : S^{(m)} \to S$ be defined by the equation*

$$x(t) = \lim_{\xi \to \infty} \bigcup_{\rho \in (\xi,\infty)} (u_1(\rho) \cdot ... \cdot u_m(\rho)). \tag{8.1}$$

On one hand, for any $u \in S^{(m)}$ the function in $\xi : \bigcup_{\rho \in (\xi,\infty)} (u_1(\rho) \cdot ... \cdot u_m(\rho))$ switches at most once from 1 to 0 as ξ runs in the increasing sense of $\mathbf{R}$. Thus the limit $\lim_{\xi \to \infty} \bigcup_{\rho \in (\xi,\infty)} (u_1(\rho) \cdot ... \cdot u_m(\rho))$ always exists and (8.1) defines a system, indeed. On the other hand, $S^{(m)}$ is invariant to translations and since for any $d \in \mathbf{R}$ and any $u \in S^{(m)}$ we have

$$f(u \circ \tau^d) = \lim_{\xi \to \infty} \bigcup_{\rho \in (\xi,\infty)} (u_1(\rho - d) \cdot ... \cdot u_m(\rho - d)) =$$

$$= \lim_{\xi \to \infty} \bigcup_{\rho \in (\xi,\infty)} (u_1(\rho) \cdot ... \cdot u_m(\rho)) = x = x \circ \tau^d,$$

the fact that the system is time invariant is inferred.

EXAMPLE 51. *The system $f : S^{(m)} \to P^*(S)$ defined by the inequality*

$$u_1(t - d') \cdot ... \cdot u_m(t - d') \leq x(t)$$

with $d' \in \mathbf{R}$ fixed is time invariant and, in order to see this, let us take some $u \in S^{(m)}$ and $d \in \mathbf{R}$. If $x \in f(u)$, then

$$u_1(t - d - d') \cdot ... \cdot u_m(t - d - d') \leq x(t - d)$$

is obvious, i.e. $x \circ \tau^d \in f(u \circ \tau^d)$.

EXAMPLE 52. *The system $f : S^{(m)} \to P^*(S)$ defined by*

$$x(t) \leq u_1(t - d') \cup ... \cup u_m(t - d'),$$

with $d' \in \mathbf{R}$, is time invariant.

THEOREM 161. *If the system $f : U \to P^*(S^{(n)})$, $U \in P^*(S^{(m)})$ is time invariant, then its initial state function $\phi_0 : U \to P^*(\mathbf{B}^n)$ fulfills*

$$\forall d \in \mathbf{R}, \forall u \in U, \phi_0(u \circ \tau^d) = \phi_0(u).$$

PROOF. $\forall d \in \mathbf{R}, \forall u \in U,$

$$\phi_0(u \circ \tau^d) = \{x(-\infty + 0) | x \in f(u \circ \tau^d)\} = \{x(-\infty + 0) | x \in \{y \circ \tau^d | y \in f(u)\}\} =$$

$$= \{y \circ \tau^d(-\infty + 0) | y \in f(u)\} = \{y(-\infty + 0) | y \in f(u)\} = \phi_0(u).$$

□

THEOREM 162. *If f is time invariant with race-free initial states and fixed initial time, then $\forall u \in U, \forall x \in f(u), x$ is the constant function.*

PROOF. There is $t_0 \in \mathbf{R}$ such that for any $u \in U$ we have the initial state $\mu^0 \in \mathbf{B}^n$ with $\forall x \in f(u), \forall t < t_0, x(t) = \mu^0$. On the other hand, for some $v \in U$, there is $\mu'^0 \in \mathbf{B}^n$ with $\forall y \in f(v), \forall t < t_0, y(t) = \mu'^0$. For an arbitrary $d \in \mathbf{R}$ we choose $v = u \circ \tau^d$. From $f(v) = f(u \circ \tau^d) = \{x \circ \tau^d | x \in f(u)\}$ we infer

$$\forall x \in f(u), \forall t < t_0, \mu^0 = x(t) = x(t - d) = \mu'^0$$

and because d is arbitrary we get that x is constant. □

THEOREM 163. *If f is time invariant, then f^* is time invariant.*

Proof. First of all, remark that if U is invariant to translations, then U^* has the same property: for any $d \in \mathbf{R}$ and any u we have

$$u \in U^* \Longrightarrow \overline{u} \in U \Longrightarrow \overline{u} \circ \tau^d \in U \Longrightarrow \overline{u \circ \tau^d} \in U \Longrightarrow u \circ \tau^d \in U^*.$$

Furthermore, we take some arbitrary $d \in \mathbf{R}$ and $u \in U^*$. We can write

$$f^*(u \circ \tau^d) = \{\overline{x}|x \in f(\overline{u \circ \tau^d})\} = \{\overline{x}|x \in f(\overline{u} \circ \tau^d)\} = \{\overline{x}|x \in \{y \circ \tau^d|y \in f(\overline{u})\}\} =$$
$$= \{\overline{y \circ \tau^d}|y \in f(\overline{u})\} = \{x \circ \tau^d|\overline{x} \in f(\overline{u})\} = \{x \circ \tau^d|x \in f^*(u)\},$$

therefore f^* is time invariant. □

Theorem 164. *Suppose that the system f is time invariant. Then $f^{-1} : X \to P^*(S^{(m)})$, where $X = \bigcup_{u \in U} f(u) \subset S^{(n)}$, is time invariant.*

Proof. First we show that X is invariant to translations. Let be $x \in X$. In other words there is $u \in U$ such that $x \in f(u)$. For an arbitrary $d \in \mathbf{R}$, from the invariance of U to translations, we have $u \circ \tau^d \in U$ while from the time invariance of f, we have $x \circ \tau^d \in f(u \circ \tau^d)$. Thus $x \circ \tau^d \in X$.

Show now that f^{-1} is time invariant. Take some arbitrary d and u, x such that $u \in f^{-1}(x)$. This means that $u \in U$ and $x \in f(u)$. Because $u \circ \tau^d \in U$ and $x \circ \tau^d \in f(u \circ \tau^d)$, we have $u \circ \tau^d \in f^{-1}(x \circ \tau^d)$. □

Theorem 165. *The Cartesian product of time invariant systems is a time invariant system.*

Proof. Consider the time invariant systems $f : U \to P^*(S^{(n)})$, $U \in P^*(S^{(m)})$ and $f' : U' \to P^*(S^{(n')})$, $U' \in P^*(S^{(m')})$. Let $d \in \mathbf{R}$, $u \in U$, $u' \in U'$ be arbitrary. We infer

$$u \times u' \in U \times U' \Longrightarrow u \in U \text{ and } u' \in U' \Longrightarrow u \circ \tau^d \in U \text{ and } u' \circ \tau^d \in U' \Longrightarrow$$
$$\Longrightarrow u \circ \tau^d \times u' \circ \tau^d \in U \times U' \Longrightarrow (u \times u') \circ \tau^d \in U \times U'.$$

Thus $U \times U'$ is invariant to translations. For any $d \in \mathbf{R}$ and any $u \times u' \in U \times U'$ we get

$$(f \times f')((u \times u') \circ \tau^d) = (f \times f')(u \circ \tau^d \times u' \circ \tau^d) = f(u \circ \tau^d) \times f'(u' \circ \tau^d) =$$
$$= \{y \times y'|y \in f(u \circ \tau^d), y' \in f'(u' \circ \tau^d)\} =$$
$$= \{y \times y'|y \in \{x \circ \tau^d|x \in f(u)\}, y' \in \{x' \circ \tau^d|x' \in f'(u')\}\} =$$
$$= \{x \circ \tau^d \times x' \circ \tau^d|x \in f(u), x' \in f'(u')\} = \{(x \times x') \circ \tau^d|(x \times x') \in (f \times f')(u \times u')\}.$$

□

Theorem 166. *The parallel connection of time invariant systems is a time invariant system.*

Theorem 167. *Consider the time invariant systems $f : U \to P^*(S^{(n)})$, $U \in P^*(S^{(m)})$ and $h : X \to P^*(S^{(p)})$, $X \in P^*(S^{(n)})$ such that the inclusion $\bigcup_{u \in U} f(u) \subset X$ is true. Then the serial connection system $h \circ f : U \to P^*(S^{(p)})$ is time invariant.*

Proof. Let $d \in \mathbf{R}$, $u \in U$ and $y \in (h \circ f)(u)$ be arbitrary, showing the existence of some $x \in f(u)$ with $y \in h(x)$. We have $x \circ \tau^d \in f(u \circ \tau^d)$ and $y \circ \tau^d \in h(x \circ \tau^d)$ from the time invariance of f and h, giving the conclusion that $y \circ \tau^d \in (h \circ f)(u \circ \tau^d)$. □

THEOREM 168. *Let be the time invariant systems* $f : U \to P^*(S^{(n)})$, $g : V \to P^*(S^{(n)})$, $U, V \in P^*(S^{(m)})$. *If* $\exists u \in U \cap V, f(u) \cap g(u) \neq \emptyset$, *then* $f \cap g$ *is time invariant.*

PROOF. Denote $W = \{u | u \in U \cap V, f(u) \cap g(u) \neq \emptyset\}$ and show that this set is invariant to translations. Let be $d \in \mathbf{R}$ and $u \in W$. Thus $u \in U \cap V$ and $f(u) \cap g(u) \neq \emptyset$. We take an $x \in f(u) \cap g(u)$. From the invariance of U and the invariance of V to translations it follows $u \circ \tau^d \in U \cap V$ while from the time invariance of f and g it follows $x \circ \tau^d \in f(u \circ \tau^d) \cap g(u \circ \tau^d)$. Thus $f(u \circ \tau^d) \cap g(u \circ \tau^d) \neq \emptyset$ and $u \circ \tau^d \in W$.

The fact that $x \in (f \cap g)(u) \Longrightarrow x \circ \tau^d \in (f \cap g)(u \circ \tau^d)$ was already shown. Hence $f \cap g$ is time invariant. □

THEOREM 169. *The union of time invariant systems is time invariant.*

REMARK 47. *The time invariant autonomous systems have the support* U *and the set* $X \in P^*(S^{(n)})$ *invariant to translations. The version from Definition 51 c) of the concept of autonomy keeps only the requirement that* X *is invariant to translations.*

The following two theorems are similar to Theorem 160 and their proof is omitted.

THEOREM 170. *For the system* f, *the following statements are equivalent:*

a) f *is self-dual and time invariant;*

b) $\forall d \in \mathbf{R}, \forall u, u \in U \Longrightarrow \overline{u} \circ \tau^d \in U$ *and, furthermore,* $\forall d \in \mathbf{R}, \forall u \in U, \forall x, x \in f(u) \Longrightarrow \overline{x} \circ \tau^d \in f(\overline{u} \circ \tau^d)$.

THEOREM 171. *Let be the system* f. *The following properties are equivalent:*

a) f *is symmetrical and time invariant;*

b) $\forall d \in \mathbf{R}$, *for any bijection* $\sigma \in S(\{1, ..., m\}), \forall u, u \in U \Longrightarrow u_\sigma \circ \tau^d \in U$ *and, furthermore,* $\forall d \in \mathbf{R}$, *for any* $\sigma \in S(\{1, ..., m\}), \forall u \in U, \forall x, x \in f(u) \Longrightarrow x \circ \tau^d \in f(u_\sigma \circ \tau^d)$.

9. Non-anticipation, the first definition

DEFINITION 63. $f : U \to P^*(S^{(n)})$, $U \in P^*(S^{(m)})$ *is a* ***non-anticipatory system*** *(in* [**11**] *the attribute 'dynamic' is considered to be a synonym of 'non-anticipatory') if for all* $u \in U$ *and all* $x \in f(u)$ *it satisfies one of the following statements:*

a) x *is constant;*

b) u, x *are both variable and we have*

$$\min\{t | u(t-0) \neq u(t)\} \leq \min\{t | x(t-0) \neq x(t)\}, \tag{9.1}$$

i.e. the first input switch is prior to the first output switch. If f *does not fulfill the previous property, it is called* ***anticipatory***.

REMARK 48. *The non-anticipation (Definition 63) means that the system* f *is in equilibrium*[2], *as represented by the existence of the time interval* $(-\infty, t_0)$, *where* u *and* x *are constant:* $u_{|(-\infty,t_0)} = u(t_0 - 0)$ *and* $x_{|(-\infty,t_0)} = x(t_0 - 0)$; *then the only*

[2]Moisil calls this equilibrium the 'rest position'; all his systems start their evolution from the rest position, we have given a relevant example in this sense at page ix. The supposition of the existence of the rest condition 'considerably simplifies the argument'. If we have a circuit which has no rest position, we can replace it by another which has such a position, by introducing a new input, which is not included in the circuit: the connection to the mains ([**22**], page 521).

possibility to get out of this situation is the switch of the input. We conclude that for such systems the first switch should be that of the input and afterwards the first switch of the state may follow[3].

EXAMPLE 53. *The system f with $\forall u \in U, \forall x \in f(u), x$ is the constant function is non-anticipatory.*

EXAMPLE 54. *The system $\pi_j : S^{(m)} \to S, \forall u \in S^{(m)}, \pi_j(u_1, ..., u_j, ..., u_m) = u_j$, $j \in \{1, ..., m\}$ is non-anticipatory, because either u_j is constant, or u_j is variable implying that u is variable. In this last case we have*

$$\min\{t|u(t-0) \neq u(t)\} \leq \min\{t|u_j(t-0) \neq u_j(t)\}.$$

EXAMPLE 55. *The previous example is generalized by the statement that any ideal combinational system F_d with $d \geq 0$ is non-anticipatory. Indeed, for any t_0, $u_{|(-\infty,t_0)} = u(t_0 - 0)$ implies $F_d(u)_{|(-\infty,t_0)} = F_d(u)(t_0 - 0)$ and if $u(t_0 - 0) \neq u(t_0)$, then $F(u(t_0-d-0)), F(u(t_0-d))$ represent two values that may be equal or different.*

EXAMPLE 56. *The following $S \to S$ system*

$$x(t) = \bigcap_{\xi \in (-\infty, t]} u(\xi)$$

is non-anticipatory since, for all $u \in S$, either x is constant, or it is variable with exactly one switch from 1 to 0. In the second case we can write

$$x(-\infty + 0) = u(-\infty + 0) = 1,$$

$$\min\{t|x(t-0) \neq x(t)\} = \min\{t|x(t-0) \cdot \overline{x(t)} = 1\} =$$

$$= \min\{t|u(t-0) \cdot \overline{u(t)} = 1\} = \min\{t|u(t-0) \neq u(t)\}.$$

EXAMPLE 57. *The $S \to S$ system*

$$x(t) = \bigcup_{\xi \in [t, \infty)} u(\xi)$$

is non-anticipatory.

EXERCISE 2. *Is to be analyzed from the anticipation point of view the system $f : S^{(2)} \to P^*(S)$ described by*

$$u_1(t) \leq x(t) \leq u_1(t) \cup u_2(t)$$

in the following four cases: a) $u_1(-\infty+0) = 0, u_2(-\infty+0) = 0$; b) $u_1(-\infty+0) = 1, u_2(-\infty+0) = 0$; c) $u_1(-\infty+0) = 0, u_2(-\infty+0) = 1$; d) $u_1(-\infty+0) = 1, u_2(-\infty+0) = 1$.

THEOREM 172. *If $g : V \to P^*(S^{(n)}), V \in P^*(S^{(m)})$ is a non-anticipatory system, then any system $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$ with $f \subset g$ is non-anticipatory.*

PROOF. Let $u \in U$. We have the following possibilities:

i) u is constant. From Definition 63 we have that $\forall x \in g(u), x$ is constant, in particular $\forall x \in f(u), x$ is constant. Therefore, f is non-anticipatory;

ii) u is variable. Let $x \in f(u)$ be arbitrary. Then

ii.1) x is constant implies that f is non-anticipatory, by Definition 63 a);

ii.2) x is variable. As element of $g(u), x$ satisfies (9.1) and, by Definition 63 b), f is non-anticipatory. □

[3] Moisil presumes implicitly that his models are non-anticpatory in the sense of Definition 63

THEOREM 173. *If f is a non-anticipatory system, then its dual f^* is non-anticipatory too.*

PROOF. Let $u \in U^*$ and $x \in f^*(u)$ be arbitrary. If x is constant, then f^* is non-anticipatory, thus we can suppose that x is non-constant, implying that $\overline{x} \in f(\overline{u})$ is non-constant. By Definition 63 b), we have that $\overline{u}$ is not constant and

$$\min\{t|\overline{u}(t-0) \neq \overline{u}(t)\} = \min\{t|u(t-0) \neq u(t)\} \leq$$

$$\leq \min\{t|x(t-0) \neq x(t)\} = \min\{t|\overline{x}(t-0) \neq \overline{x}(t)\},$$

thus f^* is non-anticipatory. □

THEOREM 174. *If $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$ and $f' : U' \to P^*(S^{(n')})$, $U' \in P^*(S^{(m')})$ are non-anticipatory systems, then their Cartesian product is non-anticipatory.*

PROOF. The hypothesis states the fulfillment of the following properties

$$\forall u \in U, \forall x \in f(u), (x \textit{ is constant}) \textit{ or}$$

$$\textit{or } (u, x \textit{ are variable and } \min\{t|u(t-0) \neq u(t)\} \leq \min\{t|x(t-0) \neq x(t)\}),$$

$$\forall u' \in U', \forall x' \in f'(u'), (x' \textit{ is constant}) \textit{ or}$$

$$\textit{or } (u', x' \textit{ are variable and } \min\{t|u'(t-0) \neq u'(t)\} \leq \min\{t|x'(t-0) \neq x'(t)\}).$$

For arbitrary $u \in U$, $x \in f(u)$, $u' \in U'$, $x' \in f'(u')$, by their conjunction we get the disjunction of the following four statements:

a) x is constant and x' is constant wherefrom $x \times x' \in (f \times f')(u \times u')$ is constant;

b) x is constant, u', x' are variable and $\min\{t|u'(t-0) \neq u'(t)\} \leq \min\{t|x'(t-0) \neq x'(t)\}$ wherefrom $u \times u'$ and $x \times x' \in (f \times f')(u \times u')$ are variable and

$$\min\{t|(u \times u')(t-0) \neq (u \times u')(t)\} \leq \min\{t|u'(t-0) \neq u'(t)\} \leq$$

$$\leq \min\{t|x'(t-0) \neq x'(t)\} = \min\{t|(x \times x')(t-0) \neq (x \times x')(t)\};$$

c) u, x are variable and $\min\{t|u(t-0) \neq u(t)\} \leq \min\{t|x(t-0) \neq x(t)\}$ and x' is constant. Similarly with b);

d) u, x are variable and $\min\{t|u(t-0) \neq u(t)\} \leq \min\{t|x(t-0) \neq x(t)\}$ and u', x' are variable and $\min\{t|u'(t-0) \neq u'(t)\} \leq \min\{t|x'(t-0) \neq x'(t)\}$.

We infer that $u \times u'$ and $x \times x' \in (f \times f')(u \times u')$ are variable and

$$\min\{t|(u \times u')(t-0) \neq (u \times u')(t)\} =$$

$$= \min\{\min\{t|u(t-0) \neq u(t)\}, \min\{t|u'(t-0) \neq u'(t)\}\} \leq$$

$$\leq \min\{\min\{t|x(t-0) \neq x(t)\}, \min\{t|x'(t-0) \neq x'(t)\}\} =$$

$$= \min\{t|(x \times x')(t-0) \neq (x \times x')(t)\}.$$

We have shown that $f \times f'$ is non-anticipatory in all the four cases a),...,d). □

THEOREM 175. *If $f : U \to P^*(S^{(n)})$ and $f'_1 : U'_1 \to P^*(S^{(n')})$, $U, U'_1 \in P^*(S^{(m)})$ are non-anticipatory systems and $U \cap U'_1 \neq \emptyset$, then their parallel connection is non-anticipatory too.*

Proof. The hypothesis states the conjunction of the statements

$$\forall u \in U \cap U_1', \forall x \in f(u), (x \textit{ is constant}) \textit{ or}$$

$$\textit{or } (u, x \textit{ are variable and } \min\{t|u(t-0) \neq u(t)\} \leq \min\{t|x(t-0) \neq x(t)\}),$$

$$\forall u \in U \cap U_1', \forall x' \in f_1'(u), (x' \textit{ is constant}) \textit{ or}$$

$$\textit{or } (u, x' \textit{ are variable and } \min\{t|u(t-0) \neq u(t)\} \leq \min\{t|x'(t-0) \neq x'(t)\})$$

etc. □

Theorem 176. *Let be the systems $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$ and $h : X \to P^*(S^{(p)}), X \in P^*(S^{(n)})$ with the property that $\bigcup_{u \in U} f(u) \subset X$. If f and h are non-anticipatory systems, then their serial connection $h \circ f : U \to P^*(S^{(p)})$ is non-anticipatory.*

Proof. Let $u \in U$ be arbitrary, for which we have the following possibilities:

a) u is constant. From Definition 63 applied to f we get that $\forall x \in f(u), x$ is constant and from Definition 63 applied to h we get that $\forall y \in h(x), y$ is constant. Thus, from Definition 63 a), $h \circ f$ is non-anticipatory;

b) u is variable. We take some arbitrary $x \in f(u)$ and from Definition 63 applied to f we have the existence of two possibilities:

b.a) x is constant. At this moment we can apply Definition 63 to h, following that $\forall y \in h(x), y$ is constant wherefrom, taking into account Definition 63 a), $h \circ f$ is non-anticipatory;

b.b) x is variable satisfying $\min\{t|u(t-0) \neq u(t)\} \leq \min\{t|x(t-0) \neq x(t)\}$. We take an arbitrary $y \in h(x)$, from Definition 63 applied to h, following the existence of two possibilities:

b.b.a) y is constant. From Definition 63 a) it follows that $h \circ f$ is non-anticipatory;

b.b.b) y is variable satisfying $\min\{t|x(t-0) \neq x(t)\} \leq \min\{t|y(t-0) \neq y(t)\}$. In this case

$$\min\{t|u(t-0) \neq u(t)\} \leq \min\{t|x(t-0) \neq x(t)\} \leq \min\{t|y(t-0) \neq y(t)\},$$

hence $h \circ f$ is non-anticipatory again. □

Theorem 177. *If f is non-anticipatory and g is another system, then $f \cap g$ is non-anticipatory.*

Proof. We take into account that $f \cap g \subset f$ and Theorem 172. □

Theorem 178. *Let be the systems $f : U \to P^*(S^{(n)})$, $g : V \to P^*(S^{(n)})$, $U, V \in P^*(S^{(m)})$.*

a) If f, g are non-anticipatory, then $f \cup g$ is non-anticipatory too.

b) If $f \cup g$ is non-anticipatory, then f, g are both non-anticipatory.

Proof. a) Let $u \in U \cup V$ and $x \in (f \cup g)(u)$ be arbitrary. There are three possibilities: $u \in U \setminus V, u \in V \setminus U$ and $u \in U \cap V$ etc.

b) The inclusions $f \subset f \cup g, g \subset f \cup g$ are true and we apply Theorem 172. □

REMARK 49. *Let $f = X$ be an autonomous system. It is non-anticipatory if we are in one of the following situations: a) $\forall x \in X, x$ is constant, b) $\exists x \in X$, x is variable; then any $u \in U$ is variable and (9.1) is fulfilled.*

For the deterministic system $f : U \to S^{(n)}, U \in P^*(S^{(m)})$, the condition of non-anticipation is: $\forall u \in U$, a) $f(u)$ is constant or b) $u, x = f(u)$ are both variable and satisfy (9.1). As we have mentioned, this is the case of the ideal combinational systems.

10. Choosing 0 as initial time instant

NOTATION 22. *We use the notation*

$$S_0^{(m)} = \{u|u \in S^{(m)}, \forall t < 0, u(t) = u(0-0)\}.$$

THEOREM 179. *We state the following properties relative to some system $\widehat{f} : \widehat{U} \to P^*(S^{(n)})$, $\widehat{U} \in P^*(S^{(m)})$:*

i) $\widehat{U} \subset S_0^{(m)}$;

ii) $\forall u \in \widehat{U}, \widehat{f}(u) \subset S_0^{(n)}$;

iii) $\forall d \in \mathbf{R}, \forall u \in \widehat{U}, \forall x$,

$$(x \in \widehat{f}(u) \text{ and } u \circ \tau^d \in \widehat{U}) \Longrightarrow x \circ \tau^d \in \widehat{f}(u \circ \tau^d).$$

a) The time-invariant non-anticipatory system $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$ is given. We define the system $\widehat{f} : \widehat{U} \to P^*(S^{(n)})$ by

$$\widehat{U} = \{u|u \in U \cap S_0^{(m)} \text{ and } f(u) \cap S_0^{(n)} \neq \emptyset\},$$

$$\forall u \in \widehat{U}, \widehat{f}(u) = f(u) \cap S_0^{(n)}. \tag{10.1}$$

The system $\widehat{f}$ fulfills i), ii), iii) and is also non-anticipatory.

b) Let be the system $\widehat{f} : \widehat{U} \to P^*(S^{(n)})$ satisfying the properties i), ii), iii) and non-anticipation. The system $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$ defined by

$$U = \{u \circ \tau^d | d \in \mathbf{R}, u \in \widehat{U}\},$$

$$\forall d \in \mathbf{R}, \forall u \in \widehat{U}, f(u \circ \tau^d) = \{x \circ \tau^d | x \in \widehat{f}(u)\}$$

is time invariant and non-anticipatory.

PROOF. a) Show that $U \cap S_0^{(m)} \neq \emptyset$. Let be $u \in U$. We have the possibilities:
1) u is constant. Then $u \in S_0^{(m)}$, thus $u \in U \cap S_0^{(m)}$;
2) u is variable.

Denote $d = \min\{t|u(t-0) \neq u(t)\}$. If $d \geq 0$, then $u \in S_0^{(m)}$ and $u \in U \cap S_0^{(m)}$ are true. If $d < 0$, then for any $d' \geq -d$, $u \circ \tau^{d'} \in U$ is true because U is invariant to translations and $u \circ \tau^{d'} \in S_0^{(m)}$ holds true also, making $u \circ \tau^{d'} \in U \cap S_0^{(m)}$ true.

Show that $\widehat{U} \neq \emptyset$. Take some arbitrary $u \in U \cap S_0^{(m)}$. If $f(u) \cap S_0^{(n)} \neq \emptyset$, the property is true, otherwise let be some $x \in f(u)$. The fact that $x \notin S_0^{(n)}$ shows that it is variable and if we denote by $d = \min\{t|x(t-0) \neq x(t)\}$, we have $d < 0$. Remark that $u \circ \tau^{d'} \in U$, $u \circ \tau^{d'} \in S_0^{(m)}$, $x \circ \tau^{d'} \in f(u \circ \tau^{d'})$ and $x \circ \tau^{d'} \in S_0^{(n)}$ take place for all $d' \geq -d$. In other words $u \circ \tau^{d'} \in \widehat{U}$.

This shows that $\widehat{f}$ is well defined, in the sense that $\widehat{U} \neq \emptyset$ and $\forall u \in \widehat{U}, \widehat{f}(u) \neq \emptyset$. Moreover, i) and ii) are obviously satisfied.

Show now the truth of iii). Take $d \in \mathbf{R}$, $u \in \widehat{U}$, x arbitrary with $x \in \widehat{f}(u)$ and $u \circ \tau^d \in \widehat{U}$ true. We have the possibilities:

j) x is constant. Then $x \circ \tau^d = x$ is constant and $x \circ \tau^d \in S_0^{(n)}$;

jj) x is variable. Because $x \in f(u)$, from the non-anticipation of f we have that u is variable and

$$0 \leq \min\{t|(u \circ \tau^d)(t-0) \neq (u \circ \tau^d)(t)\} \leq \min\{t|(x \circ \tau^d)(t-0) \neq (x \circ \tau^d)(t)\},$$

showing that $x \circ \tau^d \in S_0^{(n)}$.

In both cases j), jj), $x \in \widehat{f}(u)$ has implied $x \in f(u)$ and, furthermore, $x \circ \tau^d \in f(u \circ \tau^d)$ from the time invariance of f and, eventually, $x \circ \tau^d \in \widehat{f}(u \circ \tau^d)$ $(= f(u \circ \tau^d) \cap S_0^{(n)})$.

Because $\widehat{f} \subset f$, the non-anticipation of $\widehat{f}$ is a consequence of Theorem 172.

b) Show that f is well defined in the sense that if $d, d' \in \mathbf{R}$ and there are $u, v \in \widehat{U}$ such that $u \circ \tau^d = v \circ \tau^{d'}$, we get $f(u \circ \tau^d) = f(v \circ \tau^{d'})$. Let be $x \circ \tau^d \in f(u \circ \tau^d)$. We infer that $x \in \widehat{f}(u)$ and $v = u \circ \tau^{d-d'} \in \widehat{U}$. From iii) we have that $x \circ \tau^{d-d'} \in \widehat{f}(v)$, i.e. $x \circ \tau^d = x \circ \tau^{d-d'} \circ \tau^{d'} \in f(v \circ \tau^{d'})$. We have obtained that $f(u \circ \tau^d) \subset f(v \circ \tau^{d'})$ and the inverse inclusion is shown similarly.

Show that U is invariant to translations. Let be $v \in U$. Then there are some $d \in \mathbf{R}$ and $u \in \widehat{U}$ such that $v = u \circ \tau^d$. For an arbitrary $d' \in \mathbf{R}$, as $v \circ \tau^{d'} = u \circ \tau^{d+d'}$, we infer $v \circ \tau^{d'} \in U$.

Show that f is time invariant. Let be $v \in U$ and $y \in f(v)$, meaning the existence of $u \in \widehat{U}$ and $d \in \mathbf{R}$ with $v = u \circ \tau^d$. We get $y \in f(u \circ \tau^d) = \{x \circ \tau^d | x \in \widehat{f}(u)\}$. In other words $\exists x, y = x \circ \tau^d$ and $x \in \widehat{f}(u)$. We take an arbitrary $d' \in \mathbf{R}$ for which $y \circ \tau^{d'} = x \circ \tau^{d+d'}$, $y \circ \tau^{d'} \in \{x \circ \tau^{d+d'} | x \in \widehat{f}(u)\} = f(u \circ \tau^{d+d'}) = f(v \circ \tau^{d'})$.

Show now that f is non-anticipatory. Let us take, like previously, $v \in U$ and $y \in f(v)$, for which there are $u \in \widehat{U}, x \in \widehat{f}(u)$ and $d \in \mathbf{R}$ such that $v = u \circ \tau^d$ and $y = x \circ \tau^d$. We have the possibilities:

I) y is constant. Then f is non-anticipatory;

II) y is variable. Then $x \in \widehat{f}(u)$ is variable and the hypothesis concerning the non-anticipation of $\widehat{f}$ states that u is variable and

$$\min\{t|u(t-0) \neq u(t)\} \leq \min\{t|x(t-0) \neq x(t)\}.$$

We add d to both sides of the previous inequality and we obtain

$$\min\{t|v(t-0) \neq v(t)\} = \min\{t|(u \circ \tau^d)(t-0) \neq (u \circ \tau^d)(t)\} \leq$$

$$\leq \min\{t|(x \circ \tau^d)(t-0) \neq (x \circ \tau^d)(t)\} = \min\{t|y(t-0) \neq y(t)\}.$$

□

Remark 50. *The importance of the previous theorem is that it gives the circumstances in which we can choose 0 be the initial time moment, simplifying a little the study of the asynchronous systems. In applications we often use this possibility.*

We note that this theorem represents the passage from $S^{(m)}$ to $S_0^{(m)}$ similarly with the passage in Ch. 3 from $Diff^{(m)}$ to $\widetilde{S}^{(m)}$ and the passage in Ch. 4 from $\widetilde{S}^{(m)}$ to $S^{(m)}$. On the other hand, the properties i), ii) look similar to b), c) in Definition 37 and iii) to Definition 62 of time invariance, adapted to the situation when $\widehat{U}$ is not invariant to translations (see Lemma 2, a)).

11. Non-anticipation, the second definition

DEFINITION 64. *Let the system $f : U \to P^*(S^{(n)})$ be given, $U \in P^*(S^{(m)})$. It is called **non-anticipatory** (or **dynamic**) if $\forall t \in \mathbf{R}$, $\forall u \in U$, $\forall v \in U$,*

$$u_{|(-\infty,t)} = v_{|(-\infty,t)} \Longrightarrow \{x_{|(-\infty,t]}|x \in f(u)\} = \{y_{|(-\infty,t]}|y \in f(v)\}$$

*is true and **anticipatory** otherwise.*

REMARK 51. *Definition 64 states that for any t and any u, the restrictions $x_{|(-\infty,t]}$ where $x \in f(u)$ depend on the restriction $u_{|(-\infty,t)}$ only and are independent on the values $u(t'), t' \geq t$. In other words, 'the present depends on the past and is independent on the future' or, more correctly: the history of all the possible states until the present moment, including the present depends only on the history of the input and it does not depend on the present and the future values of the input. The definition means that $\forall t \in \mathbf{R}$ a function f_t exists that associates $\forall u \in U$ to $u_{|(-\infty,t)}$ the set*

$$f_t(u_{|(-\infty,t)}) = \{x_{|(-\infty,t]}|x \in f(u)\}.$$

Definition 64 represents a perspective of non-anticipation, other than the previous one from Section 9 and the two properties are logically independent. When we shall make comparisons between the two notions of non-anticipation we shall explicitly mention to which of them we refer. When we shall use the word 'non-anticipation' only, this will implicitly refer to the notion that was defined in this section.

EXAMPLE 58. *The deterministic system $f : S^{(m)} \to S$,*

$$\forall u \in S^{(m)}, f(u) = \chi_{[0,1)} \oplus (u_1 \circ \tau^1) \cdot \chi_{[1,\infty)}$$

is non-anticipatory in the sense of Definition 64. The system f is anticipatory in the sense of Definition 63 because for $u_1 = \chi_{[2,\infty)}$, $u_2 = ... = u_m = 0$ the contradiction $\min\{t|u(t-0) \neq u(t)\} = 2 > 0 = \min\{t|x(t-0) \neq x(t)\}$ is obtained.

EXAMPLE 59. *The deterministic system $f : S \to S$,*

$$\forall u \in S, f(u) = \begin{cases} 1, if\ u = \chi_{[0,\infty)} \\ u,\ otherwise \end{cases},$$

is anticipatory in the sense of Definition 64 because for $t_1 = 1$, $u = \chi_{[0,\infty)}$, $v = \chi_{[0,2)}$ we have $u_{|(-\infty,t_1)} = v_{|(-\infty,t_1)}$ but $1_{|(-\infty,1]} \neq \chi_{[0,2)|(-\infty,1]}$. However it is non-anticipatory in the sense of Definition 63.

EXAMPLE 60. *The deterministic system $f : S \to S$*

$$\forall u \in S, f(u) = \begin{cases} 1, if\ u = \chi_{[0,\infty)} \\ u \circ \tau^{-1},\ otherwise \end{cases}$$

is anticipatory in the sense of both Definitions 63 and 64.

EXAMPLE 61. *The deterministic system*

$$Dx(t) = (x(t-0) \oplus u(t-0)) \cdot \overline{\bigcup_{\xi \in (t-d,t)} Du(\xi)}$$

$u, x \in S, d > 0$ is non-anticipatory in the sense of both Definitions 63, 64. The idea expressed by such an equation is: x switches at these time instants when u has indicated the necessity of such a switch ($x(t-0) \oplus u(t-0) = 1$) for d time units ($u_{|[t-d,t)}$ is the constant function, with null derivative in the interval $(t-d,t)$). This equation models the delay circuit.

THEOREM 180. *If f is non-anticipatory, then its initial state function ϕ_0 satisfies*

$$\forall u \in U, \forall v \in U, u(-\infty+0) = v(-\infty+0) \Longrightarrow \phi_0(u) = \phi_0(v).$$

PROOF. Let u, v be arbitrary such that $u(-\infty+0) = v(-\infty+0)$. Thus there is some t with $u_{|(-\infty,t)} = v_{|(-\infty,t)}$. From the non-anticipation of f we get $\{x_{|(-\infty,t]}|x \in f(u)\} = \{y_{|(-\infty,t]}|y \in f(v)\}$ and this implies

$$\phi_0(u) = \{x(-\infty+0)|x \in f(u)\} = \{y(-\infty+0)|y \in f(v)\} = \phi_0(v).$$

□

REMARK 52. *The previous theorem shows the existence of a partial function $\mathbf{B}^m \to P^*(\mathbf{B}^n)$ that associates with $u(-\infty+0)$ the set $\{x(-\infty+0)|x \in f(u)\}$.*

THEOREM 181. *If f is non-anticipatory, then f^* is non-anticipatory too.*

PROOF. $\forall t \in \mathbf{R}, \forall u \in U^*, \forall v \in U^*$,

$$\begin{aligned} u_{|(-\infty,t)} = v_{|(-\infty,t)} &\Longrightarrow \overline{u}_{|(-\infty,t)} = \overline{v}_{|(-\infty,t)} \\ &\Longrightarrow \{x_{|(-\infty,t]}|x \in f(\overline{u})\} = \{y_{|(-\infty,t]}|y \in f(\overline{v})\} \\ &\Longrightarrow \{\overline{x}_{|(-\infty,t]}|x \in f(\overline{u})\} = \{\overline{y}_{|(-\infty,t]}|y \in f(\overline{v})\} \\ &\Longrightarrow \{x_{|(-\infty,t]}|x \in f^*(u)\} = \{y_{|(-\infty,t]}|y \in f^*(v)\}. \end{aligned}$$

□

THEOREM 182. *Let be the systems $f : U \to P^*(S^{(n)})$, $U \in P^*(S^{(m)})$, $f' : U' \to P^*(S^{(n')})$, $U' \in P^*(S^{(m')})$. If f, f' are non-anticipatory, then $f \times f'$ is non-anticipatory too.*

PROOF. For arbitrary $t \in \mathbf{R}, u \in U, v \in U, u' \in U', v' \in U'$, the hypothesis states

$$u_{|(-\infty,t)} = v_{|(-\infty,t)} \text{ and } u'_{|(-\infty,t)} = v'_{|(-\infty,t)}.$$

From the non-anticipation of f we get

$$\{x_{|(-\infty,t]}|x \in f(u)\} = \{y_{|(-\infty,t]}|y \in f(v)\}$$

while from the non-anticipation of f' we obtain

$$\{x'_{|(-\infty,t]}|x' \in f'(u')\} = \{y'_{|(-\infty,t]}|y' \in f'(v')\}.$$

By the conjunction of the last two statements we have

$$\{(x \times x')_{|(-\infty,t]}|x \times x' \in (f \times f')(u \times u')\} = \{(y \times y')_{|(-\infty,t]}|y \times y' \in (f \times f')(v \times v')\}$$

showing the fact that $f \times f'$ is non-anticipatory. □

THEOREM 183. *Consider the systems $f : U \to P^*(S^{(n)})$, $U \in P^*(S^{(m)})$, $f'_1 : U'_1 \to P^*(S^{(n')})$, $U'_1 \in P^*(S^{(m)})$ with the property that $U \cap U'_1 \neq \emptyset$. If f, f'_1 are non-anticipatory, then the parallel connection (f, f'_1) is non-anticipatory too.*

THEOREM 184. *The systems $f : U \to P^*(S^{(n)})$, $U \in P^*(S^{(m)})$, $h : X \to P^*(S^{(p)})$, $X \in P^*(S^{(n)})$ are given with the property that $\bigcup_{u \in U} f(u) \subset X$. If f, h are non-anticipatory then the serial connection $h \circ f$ is non-anticipatory too.*

PROOF. Let $t \in \mathbf{R}, u \in U, v \in U$ be arbitrary. We have

$$u_{|(-\infty,t)} = v_{|(-\infty,t)} \Longrightarrow \{x_{|(-\infty,t]}|x \in f(u)\} = \{y_{|(-\infty,t]}|y \in f(v)\}$$

$$\Longrightarrow \{z_{|(-\infty,t]}|x \in f(u), z \in h(x)\} = \{z'_{|(-\infty,t]}|y \in f(v), z' \in h(y)\}$$

$$\Longrightarrow \{z_{|(-\infty,t]}|z \in (h \circ f)(u)\} = \{z'_{|(-\infty,t]}|z' \in (h \circ f)(v)\},$$

implying that $h \circ f$ is non-anticipatory. $\square$

THEOREM 185. *Let be the non-anticipatory systems $f : U \to P^*(S^{(n)})$, $g : V \to P^*(S^{(m)})$, $U, V \in P^*(S^{(n)})$. If $U \cap V \neq \emptyset$ and $\exists u \in U \cap V, f(u) \cap g(u) \neq \emptyset$, then $f \cap g$ is non-anticipatory.*

PROOF. Put $W = \{u|u \in U \cap V, f(u) \cap g(u) \neq \emptyset\}$ and take some arbitrary $t \in \mathbf{R}, u \in W, v \in W$ such that the hypothesis

$$u_{|(-\infty,t)} = v_{|(-\infty,t)}$$

holds true, implying that

$$\{x_{|(-\infty,t]}|x \in f(u)\} = \{y_{|(-\infty,t]}|y \in f(v)\},$$

$$\{x_{|(-\infty,t]}|x \in g(u)\} = \{y_{|(-\infty,t]}|y \in g(v)\}$$

are both true. Intersect the left hand sides and the right hand sides of these two equations to get

$$\{x_{|(-\infty,t]}|x \in (f \cap g)(u)\} = \{x_{|(-\infty,t]}|x \in f(u) \cap g(u)\} =$$

$$= \{x_{|(-\infty,t]}|x \in f(u)\} \cap \{x_{|(-\infty,t]}|x \in g(u)\} =$$

$$= \{y_{|(-\infty,t]}|y \in f(v)\} \cap \{y_{|(-\infty,t]}|y \in g(v)\} =$$

$$= \{y_{|(-\infty,t]}|y \in f(v) \cap g(v)\} = \{y_{|(-\infty,t]}|y \in (f \cap g)(v)\}.$$

$\square$

EXAMPLE 62. *Let $U = \{0, \chi_{[0,\infty)}\}$ and $f : U \to S$ be the deterministic system given by $f(0) = \chi_{[1,\infty)}$, $f(\chi_{[0,\infty)}) = \chi_{(-\infty,0)}$. It is anticipatory because $0_{|(-\infty,0)} = \chi_{[0,\infty)|(-\infty,0)}$ and $\chi_{[1,\infty)|(-\infty,0]} \neq \chi_{(-\infty,0)|(-\infty,0]}$. At the same time, f is the (disjoint) union of the systems $f_1 : \{0\} \to S$, $f_2 : \{\chi_{[0,\infty)}\} \to S$ defined by $f_1(0) = \chi_{[1,\infty)}$, $f_2(\chi_{[0,\infty)}) = \chi_{(-\infty,0)}$. The systems f_1 and f_2 are both non-anticipatory. We conclude that, in general, the union of the non-anticipatory systems is not a non-anticipatory system.*

THEOREM 186. *Any autonomous system $X \in P^*(S^{(n)})$ is non-anticipatory.*

PROOF. For any $t \in \mathbf{R}, u \in U, v \in U$ we have

$$u_{|(-\infty,t)} = v_{|(-\infty,t)} \Longrightarrow \{x_{|(-\infty,t]}|x \in X\} = \{x_{|(-\infty,t]}|x \in X\}.$$

$\square$

12. Other definitions of non-anticipation. Non-anticipation*

DEFINITION 65. *Let be the system $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$. It is called* ***non-anticipatory*** *if it satisfies one of the following conditions, called conditions of non-anticipation:*

i) $\forall t \in \mathbf{R}, \forall u \in U, \forall v \in U,$

$$u_{|(-\infty,t)} = v_{|(-\infty,t)} \Longrightarrow \{x(t)|x \in f(u)\} = \{y(t)|y \in f(v)\};$$

ii) $\forall t \in \mathbf{R}, \forall u \in U, \forall v \in U, \exists d > 0,$

$$u_{|[t-d,t)} = v_{|[t-d,t)} \Longrightarrow \{x(t)|x \in f(u)\} = \{y(t)|y \in f(v)\};$$

iii) $\forall t \in \mathbf{R}, \exists d > 0, \forall u \in U, \forall v \in U,$

$$u_{|[t-d,t)} = v_{|[t-d,t)} \Longrightarrow \{x(t)|x \in f(u)\} = \{y(t)|y \in f(v)\};$$

iv) $\exists d > 0, \forall t \in \mathbf{R}, \forall u \in U, \forall v \in U,$

$$u_{|[t-d,t)} = v_{|[t-d,t)} \Longrightarrow \{x(t)|x \in f(u)\} = \{y(t)|y \in f(v)\};$$

v) $\forall t \in \mathbf{R}, \forall u \in U, \forall v \in U,$

$$u_{|(-\infty,t]} = v_{|(-\infty,t]} \Longrightarrow \{x_{|(-\infty,t]}|x \in f(u)\} = \{y_{|(-\infty,t]}|y \in f(v)\};$$

vi) $\forall t \in \mathbf{R}, \forall u \in U, \forall v \in U$ *and*

$$u_{|(-\infty,t]} = v_{|(-\infty,t]} \Longrightarrow \{x(t)|x \in f(u)\} = \{y(t)|y \in f(v)\};$$

vii) $\forall t \in \mathbf{R}, \forall u \in U, \forall v \in U, \exists d, \exists d', 0 \leq d \leq d'$ *and*

$$u_{|[t-d',t-d]} = v_{|[t-d',t-d]} \Longrightarrow \{x(t)|x \in f(u)\} = \{y(t)|y \in f(v)\};$$

viii) $\forall t \in \mathbf{R}, \exists d, \exists d', 0 \leq d \leq d'$ *and* $\forall u \in U, \forall v \in U,$

$$u_{|[t-d',t-d]} = v_{|[t-d',t-d]} \Longrightarrow \{x(t)|x \in f(u)\} = \{y(t)|y \in f(v)\};$$

ix) $\exists d, \exists d', 0 \leq d \leq d'$ *and* $\forall t \in \mathbf{R}, \forall u \in U, \forall v \in U,$

$$u_{|[t-d',t-d]} = v_{|[t-d',t-d]} \Longrightarrow \{x(t)|x \in f(u)\} = \{y(t)|y \in f(v)\}.$$

THEOREM 187. *If $f : U \to S^{(n)}$ is a deterministic system, then Definition 65 v) and Definition 65 vi) are equivalent. We have that Definition 64 and Definition 65 i) are equivalent in this case too.*

PROOF. We prove the first statement. Because v)$\Longrightarrow$vi) is obvious, we prove vi)$\Longrightarrow$v). Let us suppose against all reason that v) is not true, i.e. $\exists t \in \mathbf{R}, \exists u \in U, \exists v \in U, u_{|(-\infty,t]} = v_{|(-\infty,t]}$ and $f(u)_{|(-\infty,t]} \neq f(v)_{|(-\infty,t]}$. This means the existence of $t_0 \leq t$ such that $u_{|(-\infty,t_0]} = v_{|(-\infty,t_0]}$ and $f(u)(t_0) \neq f(v)(t_0)$, contradiction with vi). $\square$

REMARK 53. *In Definition 65, all of i),...,ix) express the same idea like Definition 64, namely that the present depends on the past only and it is independent on the future. The implications are:*

$$\begin{array}{ccccccccc} iv) & \Longrightarrow & iii) & \Longrightarrow & ii) & \Longrightarrow & i) & \Longleftarrow & Definition 64 \\ & & & & & & \Downarrow & & \Downarrow \\ ix) & \Longrightarrow & viii) & \Longrightarrow & vii) & \Longrightarrow & vi) & \Longleftarrow & v) \end{array}$$

In ii),...,iv), vii),...,ix) the boundedness of the memory occurs: these are systems whose states do not depend on all the input segment $u_{|(-\infty,t)}$, but on the last d time units $u_{|[t-d,t)}$ only and similarly for $u_{|(-\infty,t]}$ and $u_{|[t-d',t-d]}$.

Now have a look at the non-anticipation property iv). We note that if $d > 0$ is a number for which it is fulfilled, then any number $d' \geq d$ fulfills it also: $\forall t \in \mathbf{R}, \forall u \in U, \forall v \in U$,

$$u_{|[t-d',t)} = v_{|[t-d',t)} \Longrightarrow \{x(t)|x \in f(u)\} = \{y(t)|y \in f(v)\}.$$

Our problem is whether the set of those d satisfying implication iv) is bounded from below by some $d'' > 0$, because we have a non-anticipation property

$\forall t \in \mathbf{R}, \forall u \in U, \forall v \in U$,

$$u(t-0) = v(t-0) \Longrightarrow \{x(t)|x \in f(u)\} = \{y(t)|y \in f(v)\}$$

also, like in the example

$$u(t-0) \cdot x(t) = 0$$

where $u, x \in S$. If this lower bound exists, we obtain a new shading of that concept of non-anticipation. The problem of the existence of such bounds is, in principle, the same if d is variable like in ii), iii) or if instead of one parameter d we have two parameters d, d' and two bounds, like in vii), viii), ix).

Remark that the reasoning of Theorem 187 is impossible to use if f is non-deterministic. We suppose, for this, that the system $f : S \to P^*(S)$ satisfies $f(0) = \{0, 1\}$, $f(\chi_{[2,\infty)}) = \{\chi_{(-\infty,0)}, \chi_{[0,\infty)}\}$, where $0, 1 \in S$ are the constant functions. We have $\forall t \in [0, 2)$,

$$0_{|(-\infty,t]} = \chi_{[2,\infty)|(-\infty,t]} \text{ and}$$

$$\text{and } \{0_{|(-\infty,t]}, 1_{|(-\infty,t]}\} \neq \{\chi_{(-\infty,0)|(-\infty,t]}, \chi_{[0,\infty)|(-\infty,t]}\} \text{ and}$$

$$\text{and } \{x(t)|x \in f(0)\} = \{0, 1\} = \{y(t)|y \in f(\chi_{[2,\infty)})\}.$$

EXAMPLE 63. *The system $I_d : S \to S$ defined by $\forall u \in S, x(t) = I_d(u)(t) = u(t-d)$, satisfies for $d > 0$ the non-anticipation properties i),...,ix) from Definition 65 as well as the boundedness from below property from the end of Remark 53.*

EXAMPLE 64. *Let the $S \to P^*(S)$ systems be defined by the inequalities*

$$\bigcap_{\xi \in [t-d_r,t)} u(\xi) \leq x(t),$$

$$x(t) \leq \bigcup_{\xi \in [t-d_f,t)} u(\xi),$$

$$\bigcap_{\xi \in [t-d_r,t)} u(\xi) \leq x(t) \leq \bigcup_{\xi \in [t-d_f,t)} u(\xi),$$

where $d_r > 0, d_f > 0$. The last of them represents the intersection of the first two. The three systems satisfy all the non-anticipation properties i),...,ix) from Definition 65, together with the boundedness property from Remark 53 (we note that the lower bounds are d_r, d_f and $\max\{d_r, d_f\}$).

EXAMPLE 65. *The $S \to P^*(S)$ systems described by the inequalities*

$$\bigcap_{\xi \in [t-d',t-d]} u(\xi) \leq x(t),$$

$$x(t) \leq \bigcup_{\xi \in [t-d',t-d]} u(\xi),$$

where $0 \leq d \leq d'$, as well as their intersection

$$\bigcap_{\xi\in[t-d',t-d]} u(\xi) \leq x(t) \leq \bigcup_{\xi\in[t-d',t-d]} u(\xi),$$

satisfy the non-anticipation properties v),...,ix) from Definition 65.

EXAMPLE 66. *The $S \to P^*(S)$ system defined by*

$$\int_{-\infty}^{t} u \leq x(t)$$

(see Example 30 for the definition of the integral) satisfies the non-anticipation properties v), vi) of Definition 65.

EXAMPLE 67. *Denote by $\varphi : S^{(m)} \to \mathbf{R}$ the function*

$$\forall u \in S^{(m)}, \varphi(u) = \begin{cases} 0, \textit{if } u \textit{ is constant} \\ \min\{t|u(t-0) \neq u(t)\}, \textit{if } u \textit{ is variable} \end{cases}.$$

The deterministic system

$$x(t) = \bigcap_{\xi\in[t-\varphi^4(u),t-\varphi^2(u)]} u(\xi),$$

$u, x \in S$, satisfies the non-anticipation property vii) of Definition 65.

DEFINITION 66. *The system f is called **non-anticipatory*** if it satisfies $\forall t \in \mathbf{R}, \forall u \in U, \forall v \in U$,*

$$(u_{|[t,\infty)} = v_{|[t,\infty)} \textit{ and } \{x(t)|x \in f(u)\} = \{y(t)|y \in f(v)\}) \Longrightarrow$$

$$\Longrightarrow \{x_{|[t,\infty)}|x \in f(u)\} = \{y_{|[t,\infty)}|y \in f(v)\}$$

*and **anticipatory*** otherwise.*

REMARK 54. *Unlike the non-anticipation that relates the past and the present of the input and of the states, the non-anticipation* relates their present and future. We remark that this property somehow resembles with fixing the initial conditions in a differential equation ($\{x(t)|x \in f(u)\} = \{y(t)|y \in f(v)\}$). The consequence is that the solution is unique ($\{x_{|[t,\infty)}|x \in f(u)\} = \{y_{|[t,\infty)}|y \in f(v)\}$) under an arbitrary given input ($u_{|[t,\infty)} = v_{|[t,\infty)}$).*

Here are two other non-anticipation* requirements: $\forall t \in \mathbf{R}, \forall u \in U, \forall v \in U$,

$$u_{|[t,\infty)} = v_{|[t,\infty)} \Longrightarrow \exists t' \in \mathbf{R}, \{x_{|[t',\infty)}|x \in f(u)\} = \{y_{|[t',\infty)}|y \in f(v)\}$$

and $\forall t \in \mathbf{R}, \forall u \in U, \forall v \in U$,

$$(u_{|[t,\infty)} = v_{|[t,\infty)} \textit{ and } \{x_{|(-\infty,t]}|x \in f(u)\} = \{y_{|(-\infty,t]}|y \in f(v)\}) \Longrightarrow$$

$$\Longrightarrow \exists t' \in \mathbf{R}, \{x_{|[t',\infty)}|x \in f(u)\} = \{y_{|[t',\infty)}|y \in f(v)\}.$$

The reader is invited to write other similar properties.

13. Injectivity, the first definition

DEFINITION 67. *The system* $f : U \to P^*(S^{(n)})$, $U \in P^*(S^{(m)})$ *is called* ***injective*** *(or* ***into****) if*

$$\forall u \in U, \forall v \in U, u \neq v \Longrightarrow f(u) \neq f(v).$$

REMARK 55. *This is a first perspective on injectivity, namely stating that if the inputs* u, v *are distinct, then the sets* $f(u)$, $f(v)$ *of the possible states are distinct. If* f *is deterministic, then its injectivity coincides with the usual injectivity of the (uni-valued) functions.*

EXAMPLE 68. *Consider the three systems from Example 64. The first, denoted by* $f_1 : S \to P^*(S)$, *is described by the inequality*

$$\bigcap_{\xi \in [t-d_r,t)} u(\xi) \leq x(t),$$

$d_r > 0$, $u, x \in S$ *and satisfies the property that for any* $u = \chi_{[t_0,t_1)}$ *with* $0 < t_1 - t_0 < d_r$, $\bigcap\limits_{\xi \in [t-d_r,t)} u(\xi) = 0$ *and* $f_1(u) = S$. *Thus it is not injective. Similarly, the second system* $f_2 : S \to P^*(S)$, *given by*

$$x(t) \leq \bigcup_{\xi \in [t-d_f,t)} u(\xi)$$

$d_f > 0$, *satisfies the property that for any* $u = \chi_{(-\infty,t_0)\cup[t_1,\infty)}$ *with* $0 < t_1 - t_0 < d_f$, $\bigcup\limits_{\xi \in [t-d_f,t)} u(\xi) = 1$ *and we have* $f_2(u) = S$. *Thus it is not injective either. The third system* $f_3 = f_1 \cap f_2$ *is not injective, because for any* $u = \chi_{[t_0,t_1)\cup[t_2,t_3)\cup\ldots}$, *with* $\forall k \in \mathbf{N}, t_{2k+1} - t_{2k} < d_r$ *and* $t_{2k+2} - t_{2k+1} < d_f$, *we have* $\bigcap\limits_{\xi \in [t-d_r,t)} u(\xi) = 0$, $\bigcup\limits_{\xi \in [t-d_f,t)} u(\xi) = \chi_{(t_0,\infty)}(t)$ *and* $f_3(u) = \{x | x \in S, supp\ x \subset (t_0, \infty)\}$.

EXAMPLE 69. *The autonomous systems* $f : U \to P^*(S^{(n)})$, $U \in P^*(S^{(m)})$ *with* $|U| = 1$ *are injective.*

EXAMPLE 70. *The autonomous systems* $f = X$ *with* $|U| \neq 1$ *are not injective.*

EXAMPLE 71. *The systems described by the inequalities*

$$u(t) \leq x(t),$$

$$x(t) \leq u(t),$$

$u, x \in S$ *are injective, together with their intersection that is the deterministic system* $x(t) = u(t)$.

EXAMPLE 72. *For* $d \in \mathbf{R}$ *fixed, the deterministic system* $f : S \to S, f(u) = u \circ \tau^d$ *is injective.*

EXAMPLE 73. *The system* $f : U \to P^*(S)$, $U = \{u | u \in S^{(2)}, u_1(t) \leq u_2(t)\}$ *described by the inequality*

$$u_1(t) \leq x(t) \leq u_2(t)$$

is injective. Indeed, at each time instant t *three possibilities exist:* $u_1(t) = u_2(t) = x(t) = 0$; $u_1(t) = 0, u_2(t) = 1, x(t) \in \mathbf{B}$; $u_1(t) = u_2(t) = x(t) = 1$ *and at distinct inputs distinct sets of solutions correspond.*

THEOREM 188. *The dual of an injective system is injective.*

PROOF. If f is injective, then for any $u, v \in U^*$ we have

$$u \neq v \Longrightarrow \overline{u} \neq \overline{v} \Longrightarrow f(\overline{u}) \neq f(\overline{v}) \Longrightarrow$$
$$\Longrightarrow \{\overline{x}|x \in f(\overline{u})\} \neq \{\overline{x}|x \in f(\overline{v})\} \Longrightarrow f^*(u) \neq f^*(v).$$

□

THEOREM 189. *Let $f : U \to P^*(S^{(n)})$, $U \in P^*(S^{(m)})$, $f' : U' \to P^*(S^{(n')})$, $U' \in P^*(S^{(m')})$ be injective systems. Then $f \times f'$ is injective.*

PROOF. Let us take $u \times u', v \times v' \in U \times U'$ arbitrary such that $u \times u' \neq v \times v'$, for example $u \neq v$. From the injectivity of f we infer

$$(f \times f')(u \times u') = f(u) \times f'(u') \neq f(v) \times f'(v') = (f \times f')(v \times v').$$

□

THEOREM 190. *The parallel connection of injective systems is an injective system.*

THEOREM 191. *If $F : \mathbf{B}^m \to \mathbf{B}^n$ is an injective function and $d \in \mathbf{R}$, then the system F_d is injective.*

PROOF. Let $u, v \in S^{(m)}$ such that $\exists t_0, u(t_0) \neq v(t_0)$. Then $F(u(t_0)) \neq F(v(t_0))$ wherefrom $F_d(u)(t_0 + d) \neq F_d(v)(t_0 + d)$. □

REMARK 56. *Suppose that f is symmetrical. If $m > 1$, then it is not injective because there are $\sigma \in S(\{1, ..., m\})$ and $u \in U$ such that $u \neq u_\sigma$ and $f(u) = f(u_\sigma)$.*

The spirit of Definitions 64, 65 of non-anticipation, Definition 66 of non-anticipation and Definition 67 of injectivity gives new definitions of injectivity like: $\forall t \in \mathbf{R}, \forall u \in U, \forall v \in U$,*

$$u_{|(-\infty,t)} \neq v_{|(-\infty,t)} \Longrightarrow \{x_{|(-\infty,t]}|x \in f(u)\} \neq \{y_{|(-\infty,t]}|y \in f(v)\},$$

$\forall t \in \mathbf{R}, \forall u \in U, \forall v \in U$,

$$u_{|[t,\infty)} \neq v_{|[t,\infty)} \Longrightarrow \exists t' \in \mathbf{R}, \{x_{|[t',\infty)}|x \in f(u)\} \neq \{y_{|[t',\infty)}|y \in f(v)\}$$

etc.

14. Injectivity, the second definition

DEFINITION 68. *The system $f : U \to P^*(S^{(n)})$, $U \in P^*(S^{(m)})$ is **injective** (or **into**) if it fulfills*

$$\forall u \in U, \forall v \in U, u \neq v \Longrightarrow f(u) \cap f(v) = \emptyset.$$

REMARK 57. *This second definition of injectivity states more than the previous one, namely that if two inputs of f are distinct, then the corresponding sets of possible states are not only distinct, but also disjoint. In the case of deterministic systems, it coincides with the first definition of injectivity and also with the usual injectivity. In particular Theorem 191 is true for the second definition of injectivity too.*

An injective system f creates a partition of the set $X = \bigcup_{u \in U} f(u)$ in classes of equivalence according to the relation

$$\forall x \in X, \forall y \in X, x \sim y \Longleftrightarrow \exists u \in U, x \in f(u) \text{ and } y \in f(u).$$

Moreover, in the previous definition $\exists u \in U$ is, in fact, $\exists! u \in U$ and there is a bijection between U and $X/\sim$.

EXAMPLE 74. *The system $f : S^{(m)} \to P^*(S^{(m+1)})$ defined by*

$$\forall u \in S^{(m)}, f(u) = \{u \times x' | x' \in S\}$$

is injective.

EXAMPLE 75. *The system $f : S \to P^*(S^{(2)})$,*

$$\forall u \in S, f(u) = \begin{cases} \{u \times (u \circ \tau^1), u \times (u \circ \tau^2)\}, & \text{if } u \text{ is not constant} \\ \{u \times u\}, & \text{otherwise} \end{cases}$$

is injective.

EXAMPLE 76. *The system $f : S \to P^*(S)$,*

$$\forall u \in S, f(u) = \begin{cases} \{u \circ \tau^1, u \circ \tau^2\}, & \text{if } u \text{ is not constant} \\ \{u\}, & \text{otherwise} \end{cases}$$

is injective.

THEOREM 192. *If $g : V \to P^*(S^{(n)}), V \in P^*(S^{(m)})$ is injective and $f \subset g$, then f is injective too.*

PROOF. Suppose that the domain of f is U. If $|U| = 1$, then the statement is fulfilled. Thus we can suppose that $|U| \neq 1$. Let be $u, v \in U, u \neq v$. As $g(u) \cap g(v) = \emptyset$, $f(u) \subset g(u)$, $f(v) \subset g(v)$, we infer $f(u) \cap f(v) = \emptyset$. $\square$

THEOREM 193. *If f is injective, then f^* is injective too.*

PROOF. For $u, v \in U^*$ with $u \neq v$, we infer that $f(\overline{u}) \cap f(\overline{v}) = \emptyset$. Thus $\{\overline{x} | x \in f(\overline{u})\} \cap \{\overline{x} | x \in f(\overline{v})\} = \emptyset$ and, finally, $f^*(u) \cap f^*(v) = \emptyset$. $\square$

THEOREM 194. *If the system f is injective, then f^{-1} is deterministic.*

PROOF. Denote $X = \bigcup\limits_{u \in U} f(u)$ and suppose against all reason that $\exists x \in X$ and $\exists u \in f^{-1}(x)$, $\exists v \in f^{-1}(x)$ with $u \neq v$. This fact leads to the conclusion that $x \in f(u) \cap f(v)$. Thus $f(u) \cap f(v) \neq \emptyset$, a contradiction. $\square$

THEOREM 195. *The Cartesian product of the injective systems is injective.*

THEOREM 196. *Let be the injective systems $f : U \to P^*(S^{(n)})$, $f_1' : U_1' \to P^*(S^{(n')})$, $U, U_1' \in P^*(S^{(m)})$ with $U \cap U_1' \neq \emptyset$. The parallel connection $(f, f_1') : U \cap U_1' \to P^*(S^{(n+n')})$ is injective.*

PROOF. Take $u, v \in U \cap U_1'$ as distinct (if $|U \cap U_1'| = 1$, then the property is obvious). The fact that $f(u) \cap f(v) = \emptyset$, $f_1'(u) \cap f_1'(v) = \emptyset$ implies $(f(u) \times f_1'(u)) \cap (f(v) \times f_1'(v)) = \emptyset$ i.e. $(f, f_1')(u) \cap (f, f_1')(v) = \emptyset$. $\square$

THEOREM 197. *Consider the injective systems $f : U \to P^*(S^{(n)})$ and $h : X \to P^*(S^{(p)})$, where $U \in P^*(S^{(m)})$, $X \in P^*(S^{(n)})$. We ask that the inclusion $\bigcup\limits_{u \in U} f(u) \subset X$ be fulfilled. Then $h \circ f$ is injective.*

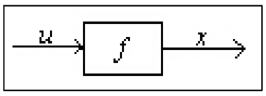

FIGURE 1

PROOF. If $|U| = 1$, then $h \circ f$ is injective. Thus we can suppose that $|U| \neq 1$. Take two arbitrary distinct inputs $u, v \in U$. In the formula

$$(h \circ f)(u) \cap (h \circ f)(v) = \bigcup_{x \in f(u)} h(x) \cap \bigcup_{x' \in f(v)} h(x') =$$

$$= \{y | \exists x \in f(u), \exists x' \in f(v), y \in h(x) \cap h(x')\}$$

$u \neq v$ implies $f(u) \cap f(v) = \emptyset$, i.e. $x \neq x'$ and eventually $h(x) \cap h(x') = \emptyset$. □

THEOREM 198. *Suppose that the systems* $f : U \to P^*(S^{(n)})$, $g : V \to P^*(S^{(n)})$, $U, V \in P^*(S^{(m)})$ *are both injective and, moreover, that the set*

$$W = \{u | u \in U \cap V, f(u) \cap g(u) \neq \emptyset\}$$

is non-empty. Then $f \cap g$ *is injective.*

PROOF. We can suppose that $|W| \neq 1$ and we take $u, v \in W$ as distinct. The relation $f(u) \cap f(v) = \emptyset$ shows that $(f(u) \cap g(u)) \cap (f(v) \cap g(v)) = \emptyset$. Thus $(f \cap g)(u) \cap (f \cap g)(v) = \emptyset$. □

THEOREM 199. *If* f *is self-dual and injective, then* $\forall u \in U, \forall x \in f(u), \overline{x} \notin f(u)$.

PROOF. We have $U = U^*$, thus $\forall u \in U$ we obtain $\overline{u} \in U$. Moreover $u \neq \overline{u}$, thus from the injectivity $f(u) \cap f(\overline{u}) = \emptyset$ and, by applying the self-duality hypothesis, we get $\emptyset = f(u) \cap f^*(\overline{u}) = f(u) \cap \{\overline{x} | x \in f(u)\}$. □

REMARK 58. *The non-anticipation of* f*:* $\forall t \in \mathbf{R}$, $\forall u \in U$, $\forall v \in U$,

$$u_{|(-\infty,t)} = v_{|(-\infty,t)} \Longrightarrow \{x_{|(-\infty,t]} | x \in f(u)\} = \{y_{|(-\infty,t]} | y \in f(v)\}$$

is related to the injectivity in the following manner. Let $u, v \in U$ *be such that* $\exists t_0$ *with* $u_{|(-\infty,t_0)} = v_{|(-\infty,t_0)}$ *and* $u(t_0) \neq v(t_0)$*. Then* $\forall x \in f(u), \forall y \in f(v)$ *we have* $x_{|(-\infty,t_0]} = y_{|(-\infty,t_0]}$ *and* $\exists t' > t_0, x(t') \neq y(t')$.

Like in Remark 56, we can give new definitions of injectivity in the spirit of Definition 68: $\forall t \in \mathbf{R}, \forall u \in U, \forall v \in U$,

$$u_{|(-\infty,t)} \neq v_{|(-\infty,t)} \Longrightarrow \{x_{|(-\infty,t]} | x \in f(u)\} \cap \{y_{|(-\infty,t]} | y \in f(v)\} = \emptyset,$$

$\forall t \in \mathbf{R}, \forall u \in U, \forall v \in U$,

$$u_{|[t,\infty)} \neq v_{|[t,\infty)} \Longrightarrow \exists t' \in \mathbf{R}, \{x_{|[t',\infty)} | x \in f(u)\} \cap \{y_{|[t',\infty)} | y \in f(v)\} = \emptyset$$

etc.

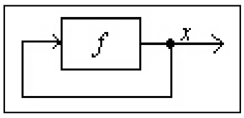

FIGURE 2

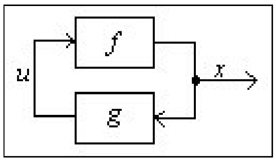

FIGURE 3

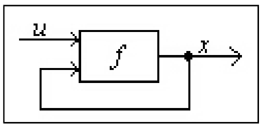

FIGURE 4

15. Huffman systems: open problem

REMARK 59. *In this section, in order to highlight its symbol f, its input u and some possible state $x \in f(u)$, a system will be drawn like in Figure 1.*

If $f : S^{(n)} \to P^*(S^{(n)})$ has the input equal to the output like in Figure 2, then it defines the autonomous system $X \subset S^{(n)}$ by

$$X = \{x|x \in f(x)\}.$$

Thus the states of the feedback system are somehow fixed points of f. Here is the first ***Open problem****: does f have fixed points? In what conditions? For the simple example*

$$\forall x \in S^{(n)}, f(x) = \begin{cases} 0, x \neq 0 \\ 1, x = 0 \end{cases}$$

the answer to the previous question is negative.

Consider two systems $f, g : S^{(n)} \to P^*(S^{(n)})$ and the autonomous system $X \subset S^{(n)}$ from Figure 3. We have

$$X = \{x|\exists u \in g(x), x \in f(u)\}.$$

Furthermore, the system $f : U \times S^{(n)} \to P^*(S^{(n)}), U \in P^*(S^{(m)})$ gives the feedback system from Figure 4. Denote it by $h : U \to P^*(S^{(n)})$. The system h is defined as

$$\forall u \in U, h(u) = \{x|x \in f(u, x)\}.$$

The idea is that of putting in the feedback loop from Figure 4 some system $g : S^{(n)} \to P^*(S^{(n)})$ like in Figure 5. The input-output association represented by this new system is defined as

$$\forall u \in U, h(u) = \{x|\exists y \in g(x), x \in f(u, y)\}.$$

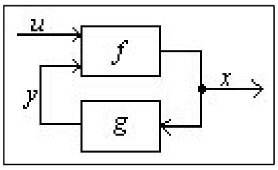

FIGURE 5

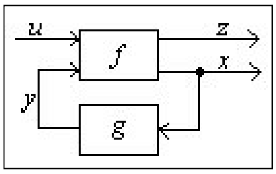

FIGURE 6

A still more general case: we have the systems $f_1 : U \times S^{(n)} \to P^*(S^{(p)})$, $f_2 : U \times S^{(n)} \to P^*(S^{(n)})$, $U \in P^*(S^{(m)})$ *and* $g : S^{(n)} \to P^*(S^{(n)})$ *and the situation from Figure 6, where by* f *we have denoted the parallel connection* (f_1, f_2)*. The system* $h : U \to P^*(S^{(p+n)})$*, giving the input-output association from Figure 6, is described by*

$$\forall u \in U, h(u) = \{(z, x) | \exists y \in g(x), z \in f_1(u, y), x \in f_2(u, y)\}. \tag{15.1}$$

There is a special case of the previous system h*, namely when* f*,* g *are combinational systems and* g *has the identical generator function: for a function* $F : \mathbf{B}^m \times \mathbf{B}^n \to \mathbf{B}^p \times \mathbf{B}^n$ *the following properties are satisfied*

$$\forall (u, y) \in U \times S^{(n)}, \exists \lim_{t\to\infty} F(u(t), y(t)) \Longrightarrow \forall (z, x) \in f(u, y), \tag{15.2}$$

$$(\lim_{t\to\infty} z(t), \lim_{t\to\infty} x(t)) = \lim_{t\to\infty} F(u(t), y(t)),$$

$$\forall x \in S^{(n)}, \exists \lim_{t\to\infty} x(t) \Longrightarrow \forall y \in g(x), \lim_{t\to\infty} y(t) = \lim_{t\to\infty} x(t), \tag{15.3}$$

i.e. f *and* g *eventually compute* F *and* $1_{\mathbf{B}^n} : \mathbf{B}^n \to \mathbf{B}^n$*. This special case is called* [**14**] *the* ***Huffman model*** *of the asynchronous circuits.*

Open problem *characterize the class* $Huff$ *of the systems satisfying:*

a) $\forall h \in Huff, h$ *has a Huffman model, i.e.* f, g *and* F *exist making (15.1), (15.2), (15.3) true;*

b) for any f, g, F *making (15.2), (15.3) true, (15.1) defines a system* $h \in Huff$*.*

The systems $h \in Huff$ *are called the* ***Huffman systems****.*

The combinational systems, i.e. the systems with the property that there is some Boolean function -called generator function- making statements like (15.2), (15.3) true have been met for the first time in Section 5 of this chapter in a special case, called 'ideal'. Such systems will occur again in Ch. 8 dedicated to stability.

We have the possibility in the definition of the Huffman systems to choose $f = F$ *i.e.* f *is an ideal combinational system with* $d = 0$*. With the notation* $F = (F_1, F_2)$*, where* $F_1 : \mathbf{B}^m \times \mathbf{B}^n \to \mathbf{B}^p$ *and* $F_2 : \mathbf{B}^m \times \mathbf{B}^n \to \mathbf{B}^n$*, we get the following*

Open problem *characterize the class* $Huff'$ *of the systems that satisfy the requirements:*

a) $\forall h \in Huff', F$ *and* g *exist such that*

$$h(u) = \{F(u,y) | y \in g(F_2(u,y))\} \tag{15.4}$$

and (15.3) are true;

b) For any F *and any* g *that makes (15.3) true, the equation (15.4) defines a system* $h \in Huff'$.

CHAPTER 6

Accesses, transitions and transfers

This chapter is devoted to the properties describing the manner in which the states of a system are accessed or transferred. Related with these, the synchronicity is treated too.

Let be $u \in U$. The **access** of the states $x \in f(u)$ to $\mu \in \mathbf{B}^n$ is the property of existence of some $t \in \mathbf{R}$ such that $x(t) = \mu$, i.e. all $x \in f(u)$ take the value μ sometime. By the **consecutive accesses** of the states x, first to $\mu' \in \mathbf{B}^n$ and then to $\mu'' \in \mathbf{B}^n$, it is understood the property

$$\forall x \in f(u), \exists t \in \mathbf{R}, x(t) = \mu' \textit{ and } \exists t' > t, x(t') = \mu''.$$

In this statement we note the existence of the restrictions $x_{|[t,t']}$ of $x \in f(u)$, called the **transitions**, with $x(t) = \mu'$ and $x(t') = \mu''$. They show how f **transfers** its states from one value to another. By the **transfer** of f, under u, from μ' to μ'' it is understood a set of such transitions.

1. Access

Remark 60. *Let be the system $f : U \to P^*(S^{(n)})$, where $U \subset S^{(m)}$ is non-empty. We state the following properties:*

$$\exists \mu \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u), \exists t \in \mathbf{R}, x(t) = \mu; \tag{1.1}$$

$$\exists \mu \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u), \exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = \mu; \tag{1.2}$$

$$\exists \mu \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u), \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = \mu; \tag{1.3}$$

$$\exists \mu \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u), \forall t_0 \in \mathbf{R}, \exists t > t_0, x(t) = \mu; \tag{1.4}$$

$$\exists \delta > 0, \exists \mu \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u), \exists t_0 \in \mathbf{R}, \forall t \in [t_0, t_0 + \delta], x(t) = \mu; \tag{1.5}$$

$$\exists \mu \in \mathbf{B}^n, \exists u \in U, \exists t \in \mathbf{R}, \forall x \in f(u), x(t) = \mu; \tag{1.6}$$

$$\exists \mu \in \mathbf{B}^n, \exists u \in U, \exists t_0 \in \mathbf{R}, \forall t < t_0, \forall x \in f(u), x(t) = \mu; \tag{1.7}$$

$$\exists \mu \in \mathbf{B}^n, \exists u \in U, \exists t_f \in \mathbf{R}, \forall t \geq t_f, \forall x \in f(u), x(t) = \mu; \tag{1.8}$$

$$\exists \mu \in \mathbf{B}^n, \exists u \in U, \forall t_0 \in \mathbf{R}, \exists t > t_0, \forall x \in f(u), x(t) = \mu; \tag{1.9}$$

$$\exists \delta > 0, \exists \mu \in \mathbf{B}^n, \exists u \in U, \exists t_0 \in \mathbf{R}, \forall t \in [t_0, t_0 + \delta], \forall x \in f(u), x(t) = \mu, \tag{1.10}$$

where the implications are

$$\begin{array}{ccccc}
(1.6) & \Longleftarrow & (1.10) & \Longleftarrow & (1.7) \\
\Downarrow & & \Downarrow & & \Downarrow \\
(1.1) & \Longleftarrow & (1.5) & \Longleftarrow & (1.2) \\
\Uparrow & & \Uparrow & & \\
(1.4) & \Longleftarrow & (1.3) & & \\
\Uparrow & & \Uparrow & & \\
(1.9) & \Longleftarrow & (1.8) & & \\
\Downarrow & & \Downarrow & & \\
(1.6) & & (1.10) & &
\end{array}$$

(1.1) is the weakest and therefore the most general property. It expresses the idea that all the possible states of f take a common value μ if we choose suitably the input. It is satisfied, for example, by the deterministic systems and so are (1.2), (1.6) and (1.7).

(1.2) and (1.3) bring in this context the ideas of race-free/constant initial/final states that, in the version from Ch. 4, (2.2), (2.3) and (2.5), (2.6) were defined by

$$\forall u \in U, \exists \mu \in \mathbf{B}^n, \forall x \in f(u), \exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = \mu,$$

$$\exists \mu \in \mathbf{B}^n, \forall u \in U, \forall x \in f(u), \exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = \mu,$$

$$\forall u \in U, \exists \mu \in \mathbf{B}^n, \forall x \in f(u), \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = \mu,$$

$$\exists \mu \in \mathbf{B}^n, \forall u \in U, \forall x \in f(u), \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = \mu.$$

A system f that satisfies (1.2) (respectively (1.3)) has two subsystems $f_2 \subset f_1 \subset f$ obtained by the restriction of the inputs to two subsets $U_2 \subset U_1 \subset U$ such that f_1 has race-free initial states and f_2 has a constant initial state (respectively f_1 has race-free final states and f_2 has a constant final state): for example, we can fix some $u^0 \in U$ making (1.2) (respectively (1.3)) true and then take $U_1 = U_2 = \{u^0\}$.

An interpretation of (1.4) is the following: there are μ and u such that $\forall x \in f(u)$, there is some sequence $(t_k) \in Seq$ with $\forall k \in \mathbf{N}$, $x(t_k) = \mu$. This statement may reflect the existence of the final state (stability) but, whenever it is true for two distinct values μ, μ'

$$\exists \mu \in \mathbf{B}^n, \exists \mu' \in \mathbf{B}^n, \mu \neq \mu', \exists u \in U, \forall x \in f(u),$$

$$\forall t_0 \in \mathbf{R}, \exists t > t_0, x(t) = \mu \text{ and } \exists t' > t_0, x(t') = \mu',$$

it shows the fact that f enters a loop under the input u (instability).

(1.5) is the requirement 'f reaches under the input u some value μ and remains there for more than $\delta > 0$ time units' and it has the variant:...$\forall t \in [t_0, t_0 + \delta), ...$ interpreted: '...remains there at least $\delta > 0$ time units'. This property should be associated not only with inertia -if so, then any $x \in \bigcup_{u \in U} f(u)$ has a 'slow' speed of variation and what 'slow' means is indicated by δ -but also with the need in modeling to get out of any region of uncertainty -if so, u is chosen in such a manner that it keeps deliberately $x \in f(u)$ at the value μ more than δ time units.

(1.6),...,(1.10) repeat (1.1),...,(1.5) under a stronger form, when all possible states of f take a common value μ synchronously, simultaneously.

In addition, if $f = X$ is an autonomous system, $X \in P^*(S^{(n)})$, then (1.1), (1.2),... take the form

$$\exists \mu \in \mathbf{B}^n, \forall x \in X, \exists t \in \mathbf{R}, x(t) = \mu,$$

$$\exists \mu \in \mathbf{B}^n, \forall x \in X, \exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = \mu,$$

...

with the interesting consequence that, in general, in the first of these two properties $\exists \mu$ *does not show the existence of a unique* μ*, but in the second property the initial value* μ *of all* $x \in X$ *is unique. Other situations of unique existence of* μ *occur also when rewriting (1.3), (1.7), (1.8) in the special case of the autonomous systems.*

Definition 69. *a) We call*

$$\Omega = \{\mu | \mu \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u), \exists t \in \mathbf{R}, x(t) = \mu\}$$

the ***set of the accessible*** *(or* ***reachable)*** ***values of*** *(****the states of****)* f*. The vectors* $\mu \in \Omega$ *are called the* ***accessible values of*** *(****the states of****)* f *or, abusively, the* ***accessible states of*** f*.*

If $\Omega \neq \emptyset$*, i.e. if (1.1) is true, we say that* f ***has accessible values of the states*** *or, abusively, that* f ***has accessible states****. We use to say that the states of* f ***take (access, reach) the values*** $\mu \in \Omega$ *or that* f ***takes the values*** $\mu \in \Omega$*.*

When (1.1) is fulfilled, we fix $\mu \in \mathbf{B}^n$ *and* $u \in U$*. The property*

$$\forall x \in f(u), \exists t \in \mathbf{R}, x(t) = \mu$$

is called the ***access of*** *(****the states of****)* f*,* ***under the input*** u*,* ***to the value*** μ*. We say that* $f(u)$ ***accesses (the value)*** μ*.*

Similar terminology and notations are given for the properties (1.2),...,(1.10) and for the following sets:

b) the ***set of the accessible initial values of*** *(****the states of****)* f*, see (1.2)*

$$\Theta'_0 = \{\mu | \mu \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u), \exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = \mu\};$$

c) the ***set of the accessible final values of*** *(****the states of****)* f*, see (1.3)*

$$\Theta'_f = \{\mu | \mu \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u), \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = \mu\};$$

d) the ***set of the accessible recurrent values of*** *(****the states of****)* f*, see (1.4)*

$$R = \{\mu | \mu \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u), \forall t_0 \in \mathbf{R}, \exists t > t_0, x(t) = \mu\};$$

e) the ***set of the accessible*** $\delta-$***persistent values of*** *(****the states of****)* f*, see (1.5)*

$$\Omega_\delta = \{\mu | \mu \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u), \exists t_0 \in \mathbf{R}, \forall t \in [t_0, t_0 + \delta], x(t) = \mu\};$$

a') the ***set of the synchronously accessible values of*** *(****the states of****)* f*, see (1.6)*

$$\Omega_s = \{\mu | \mu \in \mathbf{B}^n, \exists u \in U, \exists t \in \mathbf{R}, \forall x \in f(u), x(t) = \mu\};$$

b') the ***set of the synchronously accessible initial values of*** *(****the states of****)* f*, see (1.7)*

$$\Theta'_{0s} = \{\mu | \mu \in \mathbf{B}^n, \exists u \in U, \exists t_0 \in \mathbf{R}, \forall t < t_0, \forall x \in f(u), x(t) = \mu\};$$

c') the ***set of the synchronously accessible final values of*** *(****the states of****)* f*, see (1.8)*

$$\Theta'_{fs} = \{\mu | \mu \in \mathbf{B}^n, \exists u \in U, \exists t_f \in \mathbf{R}, \forall t \geq t_f, \forall x \in f(u), x(t) = \mu\};$$

d') the ***set of the synchronously accessible recurrent values of*** *(****the states of****)* f*, see (1.9)*

$$R_s = \{\mu | \mu \in \mathbf{B}^n, \exists u \in U, \forall t_0 \in \mathbf{R}, \exists t > t_0, \forall x \in f(u), x(t) = \mu\};$$

e') the ***set of the synchronously accessible*** $\delta-$***persistent values of*** *(****the states of****)* f*, see (1.10)*

$$\Omega_{\delta s} = \{\mu | \mu \in \mathbf{B}^n, \exists u \in U, \exists t_0 \in \mathbf{R}, \forall t \in [t_0, t_0 + \delta], \forall x \in f(u), x(t) = \mu\}.$$

REMARK 61. *The implications from Remark 60 generate the following inclusions*

$$\begin{array}{ccccc} \Omega_s & \supset & \Omega_{\delta s} & \supset & \Theta'_{0s} \\ \cap & & \cap & & \cap \\ \Omega & \supset & \Omega_\delta & \supset & \Theta'_0 \\ \cup & & \cup & & \\ R & \supset & \Theta'_f & & \\ \cup & & \cup & & \\ R_s & \supset & \Theta'_{fs} & & \\ \cap & & \cap & & \\ \Omega_s & & \Omega_{\delta s} & & \end{array}$$

for $\delta > 0$*. On the other hand, another interesting property and another remarkable set may be defined (the former implies all of (1.1),...,(1.10) and the latter is included in all of* $\Omega, ..., \Omega_{\delta s}$ *from Definition 69):*

DEFINITION 70. *If*

$$\exists \mu \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u), \forall t \in \mathbf{R}, x(t) = \mu,$$

then the set

$$Eq = \{\mu | \mu \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u), \forall t \in \mathbf{R}, x(t) = \mu\}$$

is called the ***set of the accessible equilibrium values of*** *(****the states of****)* f*. Any* $\mu \in Eq$ *is called* ***accessible equilibrium value of*** *(****the states of****)* f*, or* ***accessible equilibrium point*** *(****equilibrium state****) of* f*.*

THEOREM 200. *If* f *is deterministic, then all values of the states*

$$\{f(u)(t) | t \in \mathbf{R}, u \in U\}$$

are synchronously accessible.

PROOF. We have

$$\Omega_s = \{\mu | \mu \in \mathbf{B}^n, \exists u \in U, \exists t \in \mathbf{R}, \forall x \in f(u), x(t) = \mu\} = \{f(u)(t) | t \in \mathbf{R}, u \in U\},$$

indeed. □

REMARK 62. *In general, the set* Θ_0 *of the initial values of* f

$$\Theta_0 = \{\mu | \mu \in \mathbf{B}^n, \exists u \in U, \exists x \in f(u), \exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = \mu\}$$

does not contain only accessible values. More exactly, $\Theta'_0 \subset \Theta_0$ *and there is the possibility that* $\Theta'_0 = \emptyset$ *(*$\Theta_0 \neq \emptyset$ *is always true). We have the*

THEOREM 201. *a) If* ϕ_0 *is uni-valued, then* $\Theta'_0 \neq \emptyset$*.*
b) Suppose that f *is deterministic. Then* $\Theta'_0 \neq \emptyset$ *and* $\Theta'_0 = \Theta'_{0s} = \Theta_0$*.*

PROOF. a) The fact that ϕ_0 is uni-valued, i.e. that f has race-free initial states

$$\forall u \in U, \exists \mu \in \mathbf{B}^n, \forall x \in f(u), \exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = \mu$$

implies (1.2), i.e. $\Theta'_0 \neq \emptyset$.

b) ϕ_0 is uni-valued, thus $\Theta'_0 \neq \emptyset$ from a). The statement is obvious. □

EXAMPLE 77. *Define* $X \subset S$ *by* $X = \{0, 1\}$ *(the constant functions). The autonomous system* $f = X$ *does not have accessible values because for any* $\mu \in \mathbf{B}$ *(and for any choice of the input* $u \in U$*), one of* $0, 1$ *differs from* μ.

EXAMPLE 78. *Let be the function* $F : \mathbf{B}^m \to \mathbf{B}^n$ *and* $d \in \mathbf{R}$. *The system* F_d *has synchronously accessible values like any other deterministic system.*

EXAMPLE 79. *If the states of* f *posses the property: there is* $\delta > 0$ *such that* $\forall u \in U, \forall x \in f(u)$,

$$Dx_1(t) \cup ... \cup Dx_n(t) \leq \overline{\bigcup_{\xi \in (t, t+\delta]} (Dx_1(\xi) \cup ... \cup Dx_n(\xi))}$$

(any discontinuity is followed by continuity more than δ *time units) and if* f *has accessible values of the states, then* f *has accessible* $\delta-$*persistent values of the states.*

2. Access time

DEFINITION 71. *Suppose that* $\Omega \neq \emptyset$. *The property* (1.1) *defines the set*

$$T_{\mu,x} = \{t | t \in \mathbf{R}, x(t) = \mu\}$$

called the ***set of the access time (instants) of*** $x \in \bigcup_{u \in U} f(u)$ ***to the value*** $\mu \in \Omega$.

REMARK 63. *Let be* $\mu \in \Omega$ *and* $x \in \bigcup_{u \in U} f(u)$. *Then* $T_{\mu,x} \neq \emptyset$ *implies, by the right continuity of* x, *that we have*

$$\forall t \in T_{\mu,x}, \exists \varepsilon > 0, [t, t+\varepsilon) \subset T_{\mu,x}.$$

On the other hand, there are definitions similar to the previous one for the accesses (1.2),...,(1.10) *also, being fulfilled properties of the kind: in* (1.2)

$$\forall t \in T_{\mu,x}, (-\infty, t) \subset T_{\mu,x},$$

where $\mu \in \Theta'_0$, *in* (1.3)

$$\forall t \in T_{\mu,x}, [t, \infty) \subset T_{\mu,x},$$

where $\mu \in \Theta'_f$ *etc.*

3. Consecutive accesses

REMARK 64. *Consider the system* f. *First of all, remark the equivalence of the statements*

$$\exists \mu' \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u), \exists t' \in \mathbf{R}, x(t') = \mu'$$

$$\exists \mu' \in \mathbf{B}^n, \exists \mu'' \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u), \exists t' \in \mathbf{R}, x(t') = \mu' \text{ and } \exists t'' \in \mathbf{R}, x(t'') = \mu''$$

$$\exists \mu' \in \mathbf{B}^n, \exists \mu'' \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u), \exists t' \in \mathbf{R}, x(t') = \mu' \text{ and } \exists t'' > t', x(t'') = \mu''$$

obtained from (1.1): while in the second statement it is possible to take $\mu' = \mu''$ *and* $t' = t''$, *in the third we can take* $\mu' = \mu''$ *again and, from the right-continuity of* x, t'' *very close to* t'. *In other words, the access of the states of* f, *under some input* u, *to some value* μ' *is equivalent to the consecutive accesses of the states of* f, *under some input* u, *first to some value* μ', *and then to some value* μ''.

Whence the suggestion of mutually combining (1.1),...,(1.5) and we get the following groups of properties.

Group 1 *of properties, where we combine (1.1) with (1.1),...,(1.5):*

$$\exists \mu' \in \mathbf{B}^n, \exists \mu'' \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u), \tag{3.1}$$

$$\exists t' \in \mathbf{R}, x(t') = \mu' \text{ and } \exists t'' > t', x(t'') = \mu'',$$

$$\exists\mu'\in\mathbf{B}^n,\exists\mu''\in\mathbf{B}^n,\exists u\in U,\forall x\in f(u),$$
$$\exists t'\in\mathbf{R},x(t')=\mu' \textit{ and } \exists t>t',\forall t''<t,x(t'')=\mu'', \tag{3.2}$$

$$\exists\mu'\in\mathbf{B}^n,\exists\mu''\in\mathbf{B}^n,\exists u\in U,\forall x\in f(u),$$
$$\exists t'\in\mathbf{R},x(t')=\mu' \textit{ and } \exists t\in\mathbf{R},\forall t''\geq t,x(t'')=\mu'', \tag{3.3}$$

$$\exists\mu'\in\mathbf{B}^n,\exists\mu''\in\mathbf{B}^n,\exists u\in U,\forall x\in f(u),$$
$$\exists t'\in\mathbf{R},x(t')=\mu' \textit{ and } \forall t\in\mathbf{R},\exists t''>t,x(t'')=\mu'', \tag{3.4}$$

$$\exists\delta>0,\exists\mu'\in\mathbf{B}^n,\exists\mu''\in\mathbf{B}^n,\exists u\in U,\forall x\in f(u),$$
$$\exists t'\in\mathbf{R},x(t')=\mu' \textit{ and } \exists t>t'-\delta,\forall t''\in[t,t+\delta],x(t'')=\mu''. \tag{3.5}$$

Group 2 *of properties, where we combine (1.2) with (1.1),...,(1.5):*

$$\exists\mu'\in\mathbf{B}^n,\exists\mu''\in\mathbf{B}^n,\exists u\in U,\forall x\in f(u),$$
$$\exists t\in\mathbf{R},\forall t'<t,x(t')=\mu' \textit{ and } \exists t''\in\mathbf{R},x(t'')=\mu'', \tag{3.6}$$

$$\exists\mu'\in\mathbf{B}^n,\exists\mu''\in\mathbf{B}^n,\exists u\in U,\forall x\in f(u),$$
$$\exists t\in\mathbf{R},\forall t'<t,x(t')=\mu' \textit{ and } \exists t_1\in\mathbf{R},\forall t''<t_1,x(t'')=\mu'', \tag{3.7}$$

$$\exists\mu'\in\mathbf{B}^n,\exists\mu''\in\mathbf{B}^n,\exists u\in U,\forall x\in f(u),$$
$$\exists t\in\mathbf{R},\forall t'<t,x(t')=\mu' \textit{ and } \exists t_1\in\mathbf{R},\forall t''\geq t_1,x(t'')=\mu'', \tag{3.8}$$

$$\exists\mu'\in\mathbf{B}^n,\exists\mu''\in\mathbf{B}^n,\exists u\in U,\forall x\in f(u),$$
$$\exists t\in\mathbf{R},\forall t'<t,x(t')=\mu' \textit{ and } \forall t_1\in\mathbf{R},\exists t''>t_1,x(t'')=\mu'', \tag{3.9}$$

$$\exists\delta>0,\exists\mu'\in\mathbf{B}^n,\exists\mu''\in\mathbf{B}^n,\exists u\in U,\forall x\in f(u),$$
$$\exists t\in\mathbf{R},\forall t'<t,x(t')=\mu' \textit{ and } \exists t_1\in\mathbf{R},\forall t''\in[t_1,t_1+\delta],x(t'')=\mu''. \tag{3.10}$$

Group 3 *of properties, where we combine (1.3) with (1.1),...,(1.5):*

$$\exists\mu'\in\mathbf{B}^n,\exists\mu''\in\mathbf{B}^n,\exists u\in U,\forall x\in f(u),$$
$$\exists t\in\mathbf{R},\forall t'\geq t,x(t')=\mu' \textit{ and } \exists t''>t,x(t'')=\mu'', \tag{3.11}$$

$$\exists\mu'\in\mathbf{B}^n,\exists\mu''\in\mathbf{B}^n,\exists u\in U,\forall x\in f(u),$$
$$\exists t\in\mathbf{R},\forall t'\geq t,x(t')=\mu' \textit{ and } \exists t_1>t,\forall t''<t_1,x(t'')=\mu'', \tag{3.12}$$

$$\exists\mu'\in\mathbf{B}^n,\exists\mu''\in\mathbf{B}^n,\exists u\in U,\forall x\in f(u),$$
$$\exists t\in\mathbf{R},\forall t'\geq t,x(t')=\mu' \textit{ and } \exists t_1\in\mathbf{R},\forall t''\geq t_1,x(t'')=\mu'', \tag{3.13}$$

$$\exists\mu'\in\mathbf{B}^n,\exists\mu''\in\mathbf{B}^n,\exists u\in U,\forall x\in f(u),$$
$$\exists t\in\mathbf{R},\forall t'\geq t,x(t')=\mu' \textit{ and } \forall t_1\in\mathbf{R},\exists t''>t_1,x(t'')=\mu'', \tag{3.14}$$

$$\exists\delta>0,\exists\mu'\in\mathbf{B}^n,\exists\mu''\in\mathbf{B}^n,\exists u\in U,\forall x\in f(u),$$
$$\exists t\in\mathbf{R},\forall t'\geq t,x(t')=\mu' \textit{ and } \exists t_1>t-\delta,\forall t''\in[t_1,t_1+\delta],x(t'')=\mu''. \tag{3.15}$$

Group 4 *of properties, where we combine (1.4) with (1.1),...,(1.5):*

$$\exists\mu'\in\mathbf{B}^n,\exists\mu''\in\mathbf{B}^n,\exists u\in U,\forall x\in f(u),$$
$$\forall t\in\mathbf{R},\exists t'>t,x(t')=\mu' \textit{ and } \exists t''>t',x(t'')=\mu'', \tag{3.16}$$

$$\exists\mu'\in\mathbf{B}^n,\exists\mu''\in\mathbf{B}^n,\exists u\in U,\forall x\in f(u),$$
$$\forall t\in\mathbf{R},\exists t'>t,x(t')=\mu' \textit{ and } \exists t_1>t',\forall t''<t_1,x(t'')=\mu'', \tag{3.17}$$

$$\exists \mu' \in \mathbf{B}^n, \exists \mu'' \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u), \tag{3.18}$$

$$\forall t \in \mathbf{R}, \exists t' > t, x(t') = \mu' \textit{ and } \exists t_1 \in \mathbf{R}, \forall t'' \geq t_1, x(t'') = \mu'',$$

$$\exists \mu' \in \mathbf{B}^n, \exists \mu'' \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u), \tag{3.19}$$

$$\forall t \in \mathbf{R}, \exists t' > t, x(t') = \mu' \textit{ and } \forall t_1 \in \mathbf{R}, \exists t'' > t_1, x(t'') = \mu'',$$

$$\exists \delta > 0, \exists \mu' \in \mathbf{B}^n, \exists \mu'' \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u), \tag{3.20}$$

$$\forall t \in \mathbf{R}, \exists t' > t, x(t') = \mu' \textit{ and } \exists t_1 > t' - \delta, \forall t'' \in [t_1, t_1 + \delta], x(t'') = \mu''.$$

Group 5 *of properties, where we combine (1.5) with (1.1),...,(1.5):*

$$\exists \delta > 0, \exists \mu' \in \mathbf{B}^n, \exists \mu'' \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u), \tag{3.21}$$

$$\exists t \in \mathbf{R}, \forall t' \in [t, t + \delta], x(t') = \mu' \textit{ and } \exists t'' > t, x(t'') = \mu'',$$

$$\exists \delta > 0, \exists \mu' \in \mathbf{B}^n, \exists \mu'' \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u), \tag{3.22}$$

$$\exists t \in \mathbf{R}, \forall t' \in [t, t + \delta], x(t') = \mu' \textit{ and } \exists t_1 > t, \forall t'' < t_1, x(t'') = \mu'',$$

$$\exists \delta > 0, \exists \mu' \in \mathbf{B}^n, \exists \mu'' \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u), \tag{3.23}$$

$$\exists t \in \mathbf{R}, \forall t' \in [t, t + \delta], x(t') = \mu' \textit{ and } \exists t_1 \in \mathbf{R}, \forall t'' \geq t_1, x(t'') = \mu'',$$

$$\exists \delta > 0, \exists \mu' \in \mathbf{B}^n, \exists \mu'' \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u), \tag{3.24}$$

$$\exists t \in \mathbf{R}, \forall t' \in [t, t + \delta], x(t') = \mu' \textit{ and } \forall t_1 \in \mathbf{R}, \exists t'' > t_1, x(t'') = \mu'',$$

$$\exists \delta > 0, \exists \delta' > 0, \exists \mu' \in \mathbf{B}^n, \exists \mu'' \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u), \tag{3.25}$$

$$\exists t \in \mathbf{R}, \forall t' \in [t, t + \delta], x(t') = \mu' \textit{ and } \exists t_1 > t - \delta', \forall t'' \in [t_1, t_1 + \delta'], x(t'') = \mu''.$$

From (3.1), ...,(3.25), the weakest requirement is (3.1).

All properties (3.1), ...,(3.25) consist in the existence of two accesses and an order of reaching the accessible values, first μ' *and then* μ''*, where* μ' *and* μ'' *may be equal. The accesses take place under the same input u and we have* $\forall x \in f(u)$, $\exists t' \in T_{\mu',x}$, $\exists t'' \in T_{\mu'',x}$, $t' < t''$.

As we can see, these 25 properties do not take into account the synchronous accesses (1.6),...,(1.10). In writing them we have tried to simplify the exposure, but we must keep in mind that there are synchronous special cases of these statements. For example, (3.1) has the following special cases of synchronicity of the first access, respectively of the second access, respectively of both accesses:

$\exists \mu' \in \mathbf{B}^n, \exists \mu'' \in \mathbf{B}^n, \exists u \in U,$

$$\exists t' \in \mathbf{R}, \forall x \in f(u), x(t') = \mu' \textit{ and } \exists t'' > t', x(t'') = \mu'',$$

$\exists \mu' \in \mathbf{B}^n, \exists \mu'' \in \mathbf{B}^n, \exists u \in U,$

$$\exists t'' \in \mathbf{R}, \forall x \in f(u), \exists t' < t'', x(t') = \mu' \textit{ and } x(t'') = \mu'',$$

$\exists \mu' \in \mathbf{B}^n, \exists \mu'' \in \mathbf{B}^n, \exists u \in U,$

$$\exists t' \in \mathbf{R}, \exists t'' > t', \forall x \in f(u), x(t') = \mu' \textit{ and } x(t'') = \mu''.$$

Definition 72. *a) We call*

$$\Omega\otimes\Omega=\{(\mu',\mu'')|\mu',\mu''\in\mathbf{B}^n,\exists u\in U,$$
$$\forall x\in f(u),\exists t'\in\mathbf{R},x(t')=\mu' \text{ and } \exists t''>t',x(t'')=\mu''\}$$

the ***set of the couples of consecutive*** *(or* ***successive****)* ***accessible values of (the states of)*** *f.*

If $\Omega\otimes\Omega\neq\emptyset$, meaning that (3.1) is satisfied, we say that f ***has consecutive accessible values of the states*** *or, abusively that f* ***has consecutive accessible states****, while for $(\mu',\mu'')\in\Omega\otimes\Omega$ that there is $u\in U$ with the property that the states $x\in f(u)$* ***take (access, reach) first the value*** *μ',* ***then the value*** *μ''.*

Suppose that (3.1) is true and let us fix μ',μ'',u. We have the property

$$\forall x\in f(u),\exists t'\in\mathbf{R},x(t')=\mu' \text{ and } \exists t''>t',x(t'')=\mu'',$$

that we call the ***consecutive accesses of (the states of)*** *f,* ***under the input*** *u,* ***first to*** *μ',* ***then to*** *μ''. When it is true, we say, sometimes, that $f(u)$* ***transfers*** *μ'* ***to*** *μ''.*

The terminology and the notations are similar for the properties (3.2),...,(3.25) and for the following sets:

b) the ***set of the couples of consecutive accessible values*** *(μ',μ'')* ***of*** *f, μ''-****initial*** *(see (3.2))*

$$\Omega\otimes\Theta_0'=\{(\mu',\mu'')|\mu',\mu''\in\mathbf{B}^n,\exists u\in U,$$
$$\forall x\in f(u),\exists t'\in\mathbf{R},x(t')=\mu' \text{ and } \exists t>t',\forall t''<t,x(t'')=\mu''\};$$

...

c) the ***set of the couples of consecutive accessible values*** *(μ',μ'')* ***of*** *f, μ' $\delta-$****persistent****, μ'' $\delta'-$****persistent*** *(see (3.25))*

$$\Omega_\delta\otimes\Omega_{\delta'}=\{(\mu',\mu'')|\mu',\mu''\in\mathbf{B}^n,\exists u\in U,\forall x\in f(u),$$
$$\exists t\in\mathbf{R},\forall t'\in[t,t+\delta],x(t')=\mu' \text{ and } \exists t_1>t-\delta',\forall t''\in[t_1,t_1+\delta'],x(t'')=\mu''\}.$$

Theorem 202. *a) The property (3.1) is equivalent to (1.1).*

b) The properties (3.2), (3.6), (3.7), (3.10) and (3.22) are equivalent to (1.2).

c) The properties (3.3), (3.11), (3.13), (3.14), (3.15), (3.18) and (3.23) are equivalent to (1.3).

d) The properties (3.4), (3.16) and (3.19) are equivalent to (1.4).

e) The properties (3.5), (3.21) and (3.25) are equivalent to (1.5).

f) The properties (3.12) and (3.17) are equivalent to

$$\exists\mu\in\mathbf{B}^n,\exists u\in U,\forall x\in f(u),\forall t\in\mathbf{R},x(t)=\mu \tag{3.26}$$

i.e. the existence of a point of equilibrium.

Proof. In general, these equivalencies are easy to prove. We give the example of (3.2) $\Longleftrightarrow$ (1.2).

(3.2) $\Longrightarrow$ (1.2) From (3.2) we get

$$\exists\mu''\in\mathbf{B}^n,\exists u\in U,\forall x\in f(u),\exists t\in\mathbf{R},\forall t''<t,x(t'')=\mu'',$$

i.e. (1.2).

(1.2) $\Longrightarrow$ (3.2) From (1.2) we have

$$\exists\mu''\in\mathbf{B}^n,\exists u\in U,\forall x\in f(u),\exists t\in\mathbf{R},\exists t'<t,x(t')=\mu'' \text{ and } \forall t''<t,x(t'')=\mu'',$$

$$\exists\mu''\in\mathbf{B}^n,\exists u\in U,\forall x\in f(u),\exists t'\in\mathbf{R},x(t')=\mu'' \text{ and } \exists t>t',\forall t''<t,x(t'')=\mu''$$

μ'	μ''	*property*
accessible arbitrary value	*accessible initial value*	(3.2)
accessible initial value	*accessible arbitrary value*	(3.6)
accessible initial value	*accessible initial value*	(3.7)
accessible initial value	*accessible* $\delta-$ *persistent value*	(3.10)
accessible $\delta-$ *persistent value*	*accessible initial value*	(3.22)

FIGURE 1. The properties that are equivalent to (1.2)

μ'	μ''	*property*
accessible arbitrary value	*accessible final value*	(3.3)
accessible final value	*accessible arbitrary value*	(3.11)
accessible final value	*accessible final value*	(3.13)
accessible final value	*accessible recurrent value*	(3.14)
accessible final value	*accessible* $\delta-$ *persistent value*	(3.15)
accessible recurrent value	*accessible final value*	(3.18)
accessible $\delta-$ *persistent value*	*accessible final value*	(3.23)

FIGURE 2. The properties that are equivalent to (1.3)

and (3.2) follows. □

COROLLARY 1. *From (1.1),...,(1.5),(3.1),...,(3.25),(3.26) the following properties differ from each other.*

a) (1.1):

$$\exists \mu \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u), \exists t \in \mathbf{R}, x(t) = \mu,$$

b) (1.2):

$$\exists \mu \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u), \exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = \mu,$$

c) (1.3):

$$\exists \mu \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u), \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = \mu,$$

d) (1.4):

$$\exists \mu \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u), \forall t_0 \in \mathbf{R}, \exists t > t_0, x(t) = \mu,$$

e) (1.5):

$$\exists \delta > 0, \exists \mu \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u), \exists t_0 \in \mathbf{R}, \forall t \in [t_0, t_0 + \delta], x(t) = \mu,$$

μ'	μ''	*property*
accessible arbitrary value	*accessible recurrent value*	(3.4)
accessible recurrent value	*accessible arbitrary value*	(3.16)
accessible recurrent value	*accessible recurrent value*	(3.19)

FIGURE 3. The properties that are equivalent to (1.4)

μ'	μ''	*property*
accessible arbitrary value	*accessible $\delta -$ persistent value*	(3.5)
accessible $\delta -$ persistent value	*accessible arbitrary value*	(3.21)
accessible $\delta -$ persistent value	*accessible $\delta -$ persistent value*	(3.25)

FIGURE 4. The properties that are equivalent to (1.5)

μ'	μ''	*property*
accessible final value	*accessible initial value*	(3.12)
accessible recurrent value	*accessible initial value*	(3.17)

FIGURE 5. The properties that are equivalent to (3.26)

f) (3.26):

$$\exists \mu \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u), \forall t \in \mathbf{R}, x(t) = \mu,$$

g) (3.8):

$$\exists \mu' \in \mathbf{B}^n, \exists \mu'' \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u),$$
$$\exists t \in \mathbf{R}, \forall t' < t, x(t') = \mu' \textit{ and } \exists t_1 \in \mathbf{R}, \forall t'' \geq t_1, x(t'') = \mu'',$$

h) (3.9):

$$\exists \mu' \in \mathbf{B}^n, \exists \mu'' \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u),$$
$$\exists t \in \mathbf{R}, \forall t' < t, x(t') = \mu' \textit{ and } \forall t_1 \in \mathbf{R}, \exists t'' > t_1, x(t'') = \mu'',$$

i) (3.20):

$$\exists \delta > 0, \exists \mu' \in \mathbf{B}^n, \exists \mu'' \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u),$$
$$\forall t \in \mathbf{R}, \exists t' > t, x(t') = \mu' \textit{ and } \exists t_1 > t' - \delta, \forall t'' \in [t_1, t_1 + \delta], x(t'') = \mu'',$$

j) (3.24):

$$\exists \delta > 0, \exists \mu' \in \mathbf{B}^n, \exists \mu'' \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u),$$
$$\exists t \in \mathbf{R}, \forall t' \in [t, t + \delta], x(t') = \mu' \textit{ and } \forall t_1 \in \mathbf{R}, \exists t'' > t_1, x(t'') = \mu''.$$

REMARK 65. *In Figures 1,...,5 we have repeated the properties that are equivalent to the Corollary's items b),...,f) as stated in Theorem 202.*

Some of the implications characterizing the properties from the Corollary are the following

$$\begin{array}{ccccc} (3.8) & \Longleftarrow & (3.26) & \Longrightarrow & (3.20) \\ \Downarrow & & \Downarrow & & \\ (3.9) & \Longrightarrow & (3.24) & & \end{array}$$

4. Transition

DEFINITION 73. *Let be $x \in S^{(n)}$ and let $t_1 < t_2$ be two real numbers. The restriction $\gamma = x_{|[t_1,t_2]}$ is called the* ***transition of*** *x* ***from*** *(**the value**) $x(t_1)$* ***to*** *(**the value**) $x(t_2)$*[1]*. $[t_1, t_2]$ is called the* ***support interval*** *of the transition and the number $t_2 - t_1$ is called the* ***duration*** *of the transition.*

If x is constant on $[t_1, t_2]$, then the transition is called ***constant****, or* ***trivial*** *and if x is coordinately monotonous on the interval $[t_1, t_2]$, then the transition is called* ***monotonous****.*

DEFINITION 74. *We have the following partial law of composition of the transitions. Consider $x', x'' \in S^{(n)}$ and the numbers $t_1 < t_2 < t_3$; if $x'(t_2) = x''(t_2)$, then there is $x \in S^{(n)}$ satisfying*

$$\forall t \in [t_1, t_3], x(t) = \begin{cases} x'(t), t \in [t_1, t_2] \\ x''(t), t \in [t_2, t_3] \end{cases}.$$

Usually, the transition $\gamma = x_{|[t_1,t_3]}$ is denoted by $\gamma' \vee \gamma''$ and is called the ***union*** *of $\gamma' = x'_{|[t_1,t_2]}$ with $\gamma'' = x''_{|[t_2,t_3]}$ (in this order).*

REMARK 66. *A similar terminology is used for the restriction $\gamma_{|(-\infty,t_2]}$, called the transition of x from $x(-\infty + 0)$ to $x(t_2)$. If x', x'' satisfy $x'(t_1) = x''(t_1)$, then x exists satisfying*

$$\forall t \in (-\infty, t_2], x(t) = \begin{cases} x'(t), t \in (-\infty, t_1] \\ x''(t), t \in [t_1, t_2] \end{cases}.$$

For $\gamma' = x'_{|(-\infty,t_1]}, \gamma'' = x''_{|[t_1,t_2]}$, $x_{|(-\infty,t_2]}$ is denoted by $\gamma = \gamma' \vee \gamma''$.

Other constructions of the same kind are possible.

5. Set of support intervals

DEFINITION 75. *If $\Omega \otimes \Omega \neq \emptyset$ (see Definition 72 a)) then the property (3.1) defines a set analogue to $T_{\mu,x}$ from Definition 71, namely*

$$T_{\mu',\mu'',x} = \{[t', t''] | t' < t'', x(t') = \mu' \text{ and } x(t'') = \mu''\},$$

called the ***set of the support intervals*** *of the transitions $x_{|[t',t'']}$, where $(\mu', \mu'') \in \Omega \otimes \Omega$ and $x \in \bigcup_{u \in U} f(u)$.*

EXAMPLE 80. *The system $f : S \to P^*(S)$ defined by*

$$\bigcap_{\xi \in [t-1,t]} u(\xi) \leq x(t)$$

[1]in geometry these functions may be called curves; another terminology could be that of path

satisfies $\bigcup_{u\in S} f(u) = S$. *For* $x = \chi_{[0,1)} \oplus \chi_{[2,\infty)}, x \in S$ *we have*

$$T_{0,1,x} = \{[t',t'']|t' < t'', t' \in (-\infty,0)\cup[1,2), t'' \in [0,1)\cup[2,\infty)\},$$
$$T_{1,1,x} = \{[t',t'']|t' < t'', t', t'' \in [0,1)\cup[2,\infty)\}.$$

Remark 67. *Definitions similar to Definition 75 take place when starting with the remainder of the properties of consecutive accesses (3.1),...,(3.25). We give the example of (3.20), that defines the set* $R\otimes\Omega_\delta$, $\delta > 0$ *like this:*

$$R\otimes\Omega_\delta = \{(\mu',\mu'')|\mu',\mu'' \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u),$$
$$\forall t \in \mathbf{R}, \exists t' > t, x(t') = \mu' \text{ and } \exists t_1 > t' - \delta, \forall t'' \in [t_1, t_1+\delta], x(t'') = \mu''\}.$$

If $(\mu',\mu'') \in R\otimes\Omega_\delta$ *and* $x \in \bigcup_{u\in U} f(u)$, *then (3.20) defines the set* $T_{\mu',\mu'',x}$ *satisfying:* $(t_k), (t'_k) \in Seq$ *exist such that* $\forall k \in \mathbf{N}, t_k \in T_{\mu',x}, [t'_k, t'_k+\delta] \subset T_{\mu'',x}$, $t_k < t'_k + \delta$ *and* $[t_k, t'_k+\delta] \in T_{\mu',\mu'',x}$.

6. Transfer

Definition 76. *Fix in (3.1)* $\mu',\mu'' \in \mathbf{B}^n, u \in U$. *The consecutive accesses of the states of* f, *under* u, *to* μ' *and* μ'' *in this order*

$$\forall x \in f(u), \exists t' \in \mathbf{R}, x(t') = \mu' \text{ and } \exists t'' > t', x(t'') = \mu''$$

define the set of transitions

$$\mu' \overset{u}{\rightrightarrows} \mu'' = \{x_{|[t',t'']}|[t',t''] \in T_{\mu',\mu'',x}, x \in f(u)\}$$

called the ***full transfer of (the states of)*** f, ***under (the input)*** u, ***from (the value)*** μ' ***to (the value)*** μ'' *and its non-empty subsets*

$$\mu' \xrightarrow{u} \mu'' \subset \mu' \overset{u}{\rightrightarrows} \mu''$$

are called the ***transfers of*** f, ***under*** u, ***from*** μ' ***to*** μ''.

Remark 68. *Similarly to Definition 76, we have that*

$$\forall x \in f(u), \exists t' \in \mathbf{R}, x_{|(-\infty,t')} = \mu' \text{ and } \exists t'' > t', x(t'') = \mu''$$

defines the full transfer

$$\mu' \overset{u}{\rightrightarrows} \mu'' = \{x_{|(-\infty,t'']}|(-\infty,t''] \in T_{\mu',\mu'',x}, x \in f(u)\}.$$

Like in the case of $T_{\mu',\mu'',x}$, *when we write* $\mu' \xrightarrow{u} \mu''$ *the system that this transfer refers to is kept in mind. If necessary, in order to avoid ambiguities, either we shall explicitly mention the system, or we shall write it as a subscript:* $(\mu' \xrightarrow{u} \mu'')_f$, $(\mu' \overset{u}{\rightrightarrows} \mu'')_f$.

The inclusion, the intersection and the union of the transfers are defined by the usual inclusion, intersection and union of the sets. Another union of the transfers, denoted by $\vee$ *not by* $\cup$ *and induced by the union* $\gamma' \vee \gamma''$ *of the transitions, will be given in Definition 78.*

The dual transfer of $\mu' \xrightarrow{u} \mu''$ *is*

$$(\mu' \xrightarrow{u} \mu'')^* = \{\overline{x}_{|[t',t'']}|x_{|[t',t'']} \in \mu' \xrightarrow{u} \mu''\},$$
$$(\mu' \xrightarrow{u} \mu'')^* \subset \{\overline{x}_{|[t',t'']}|[t',t''] \in T_{\overline{\mu'},\overline{\mu''},\overline{x}}, \overline{x} \in f^*(\overline{u})\}.$$

It is a transfer of the states of f^*, *under the input* $\overline{u}$, *from the value* $\overline{\mu'}$ *to the value* $\overline{\mu''}$.

A transfer of f^{-1} *is*

$$\lambda' \overset{x}{\to} \lambda'' \subset \{u_{|[t',t'']} | [t',t''] \in T_{\lambda',\lambda'',u}, u \in f^{-1}(x)\},$$

where $\lambda', \lambda'' \in \mathbf{B}^m$ *and* $x \in \bigcup\limits_{u \in U} f(u)$.

The transfers of the Cartesian product $f \times f'$*, of the parallel connection* (f, f_1') *and of the serial connection* $h \circ f$ *are the following non-empty subsets*

$$(\mu', \widetilde{\mu}') \overset{u \times u'}{\to} (\mu'', \widetilde{\mu}'') \subset$$
$$\subset \{(x \times x')_{|[t',t'']} | [t',t''] \in T_{\mu',\mu'',x} \cap T_{\widetilde{\mu}',\widetilde{\mu}'',x'}, x \times x' \in (f \times f')(u \times u')\},$$
$$(\mu', \widetilde{\mu}') \overset{v}{\to} (\mu'', \widetilde{\mu}'') \subset$$
$$\subset \{(x \times x')_{|[t',t'']} | [t',t''] \in T_{\mu',\mu'',x} \cap T_{\widetilde{\mu}',\widetilde{\mu}'',x'}, x \times x' \in (f, f_1')(v)\},$$
$$\nu' \overset{u}{\to} \nu'' \subset \{z_{|[t',t'']} | [t',t''] \in T_{\nu',\nu'',z}, z \in (h \circ f)(u)\}$$

and the transfers of $f \cap g$, $f \cup g$ *satisfy*

$$\mu' \overset{u''}{\to} \mu'' \subset \{x_{|[t',t'']} | [t',t''] \in T_{\mu',\mu'',x}, x \in (f \cap g)(u'')\},$$

$$\mu' \overset{\widetilde{u}}{\to} \mu'' \subset \{x_{|[t',t'']} | [t',t''] \in T_{\mu',\mu'',x}, x \in (f \cup g)(\widetilde{u})\},$$

where $\mu', \mu'' \in \mathbf{B}^n$, $\widetilde{\mu}', \widetilde{\mu}'' \in \mathbf{B}^{n'}$, $\nu', \nu'' \in \mathbf{B}^p$, $u \in U$, $u' \in U'$, $v \in U \cap U_1'$, $u'' \in \{\widetilde{v} | \widetilde{v} \in U \cap V, f(\widetilde{v}) \cap g(\widetilde{v}) \neq \emptyset\}$, $\widetilde{u} \in U \cup V$; $U_1', V \in P^*(S^{(m)})$, $X \in P^*(S^{(n)})$, $U' \in P^*(S^{(m')})$ *are the domains of* f_1', g, h, f' *and it was presumed that* $(f, f_1'), h \circ f, f \cap g$ *exist.*

All these definitions are also possible by replacing (3.1) with one of (3.2),...,(3.25).

On the other hand, in the transfer $\mu' \overset{u}{\to} \mu''$*, one or both accesses may be synchronous. We have the*

DEFINITION 77. *If in (3.1) the accesses are synchronous*

$$\exists t' \in \mathbf{R}, \exists t'' > t', \forall x \in f(u), x(t') = \mu' \text{ and } x(t'') = \mu'',$$

with μ', μ'', u *fixed, then the transfer* $\mu' \overset{u}{\to} \mu''$ *is called* ***synchronous***.

DEFINITION 78. *Let* $\mu \overset{u}{\to} \mu'$, $\mu' \overset{u}{\to} \mu''$ *be two transfers,* $u \in U$ *for which we have supposed that* $(\mu, \mu'), (\mu', \mu'') \in \Omega \otimes \Omega$ *and, moreover, that the following property*

$$\forall x \in f(u), \exists t \in \mathbf{R}, x(t) = \mu \text{ and } \exists t' > t, x(t') = \mu' \text{ and } \exists t'' > t', x(t'') = \mu''$$

is satisfied. Define the partial law of composition

$$(\mu \overset{u}{\to} \mu') \vee (\mu' \overset{u}{\to} \mu'') = \{\gamma | \exists \gamma' \in \mu \overset{u}{\to} \mu', \exists \gamma'' \in \mu' \overset{u}{\to} \mu'', \gamma = \gamma' \vee \gamma''\}.$$

The transfer $(\mu \overset{u}{\to} \mu') \vee (\mu' \overset{u}{\to} \mu'')$ *is called the* ***union*** *of the transfers* $\mu \overset{u}{\to} \mu'$ *and* $\mu' \overset{u}{\to} \mu''$ *(in this order).*

EXAMPLE 81. *Let be the autonomous deterministic system* $f : S \to S$ *that satisfies* $x = f(u) = \chi_{[0,1)}$ *for any* $u \in U$ *and choose* $\mu, \mu', \mu'' \in \mathbf{B}$, $\mu = \mu'' = 0$, $\mu' = 1$. *We have*

$$T_{0,1,x} = \{[t,t'] | t \in (-\infty, 0), t' \in [0,1)\},$$
$$T_{1,0,x} = \{[t',t''] | t' \in [0,1), t'' \in [1,\infty)\},$$
$$T_{0,0,x} = \{[t,t''] | t < t'', t, t'' \in (-\infty, 0) \cup [1,\infty)\},$$

$$\begin{aligned}(0 \overset{u}{\rightrightarrows} 1) \vee (1 \overset{u}{\rightrightarrows} 0) &= \{x_{|[t,t'']} | \exists t', [t,t'] \in T_{0,1,x}, [t',t''] \in T_{1,0,x}\} = \\ &= \{\chi_{[0,1)|[t,t'']} | t \in (-\infty, 0), t'' \in [1,\infty)\},\end{aligned}$$

$$\begin{aligned}(0 \overset{u}{\rightrightarrows} 0) &= \{x_{|[t,t'']} | [t,t''] \in T_{0,0,x}\} = \\ &= \{\chi_{[0,1)|[t,t'']} | t < t'', t, t'' \in (-\infty, 0) \cup [1, \infty)\}.\end{aligned}$$

This example shows that, in general, we have $(\mu \overset{u}{\rightrightarrows} \mu') \vee (\mu' \overset{u}{\rightrightarrows} \mu'') \subset (\mu \overset{u}{\rightrightarrows} \mu'')$.

REMARK 69. $\mu' \overset{u}{\rightrightarrows} \mu''$ *gives all possibilities that* $x \in f(u)$ *reach first the accessible value* μ', *and then the accessible value* μ'' *while* $\mu' \overset{u}{\longrightarrow} \mu''$ *ignores some of these possibilities.*

The union $(\mu \overset{u}{\rightrightarrows} \mu') \vee (\mu' \overset{u}{\rightrightarrows} \mu'')$ *creates no loss; it indicates the ways that* μ, μ', μ'' *may be accessed in this order, representing just some of the ways that* μ, μ'' *may be accessed, in this order.*

7. The transfers of the non-anticipatory systems

THEOREM 203. *Let the system* f *satisfy the conditions:*
a) U *is closed under translations and under 'concatenation'*

$$\forall d \in \mathbf{R}, \forall u \in U, u \circ \tau^d \in U,$$

$$\forall t \in \mathbf{R}, \forall u \in U, \forall v \in U, u \cdot \chi_{(-\infty,t)} \oplus v \cdot \chi_{[t,\infty)} \in U;$$

b) non-anticipation $\forall t \in \mathbf{R}$, $\forall u \in U$, $\forall v \in U$,

$$u_{|(-\infty,t)} = v_{|(-\infty,t)} \Longrightarrow \{x_{|(-\infty,t]} | x \in f(u)\} = \{y_{|(-\infty,t]} | y \in f(v)\};$$

*c) non-anticipation** $\forall t \in \mathbf{R}$, $\forall u \in U$, $\forall v \in U$,

$$(u_{|[t,\infty)} = v_{|[t,\infty)} \text{ and } \{x(t) | x \in f(u)\} = \{y(t) | y \in f(v)\}) \Longrightarrow$$

$$\Longrightarrow \{x_{|[t,\infty)} | x \in f(u)\} = \{y_{|[t,\infty)} | y \in f(v)\};$$

d) time invariance

$$\forall d \in \mathbf{R}, \forall u \in U, f(u \circ \tau^d) = \{x \circ \tau^d | x \in f(u)\};$$

e) t_1, $t_2 \in \mathbf{R}$, $u^0, u^1 \in U$ *and* $\mu, \mu', \mu'' \in \mathbf{B}^n$ *are given such that*

$$\forall x \in f(u^0), \exists t_0 < t_1, x(t_0) = \mu, \tag{7.1}$$

$$\forall x \in f(u^0), x(t_1) = \mu', \tag{7.2}$$

$$\forall x' \in f(u^1), x'(t_2) = \mu', \tag{7.3}$$

$$\forall x' \in f(u^1), \exists t_3 > t_2, x'(t_3) = \mu''. \tag{7.4}$$

Put $d = t_1 - t_2$. *Then* $\widetilde{u} \in U$ *defined as*

$$\widetilde{u} = u^0 \cdot \chi_{(-\infty,t_1)} \oplus (u^1 \circ \tau^d) \cdot \chi_{[t_1,\infty)}, \tag{7.5}$$

satisfies

$$\forall \widetilde{x} \in f(\widetilde{u}), \exists t_0 < t_1, \widetilde{x}(t_0) = \mu, \tag{7.6}$$

$$\forall \widetilde{x} \in f(\widetilde{u}), \exists t_3' > t_1, \widetilde{x}(t_3') = \mu''. \tag{7.7}$$

Thus, if $f(u^0)$ *transfers* μ *to* μ', *the second access being synchronous, and if* $f(u^1)$ *transfers* μ' *to* μ'', *with the first access synchronous, then* $f(\widetilde{u})$ *transfers* μ *to* μ''.

PROOF. $\widetilde{u}$ belongs to U indeed, because of a). We remark that we have

$$\widetilde{u}_{|(-\infty,t_1)} = u^0_{|(-\infty,t_1)}. \tag{7.8}$$

From (7.8) and b) we infer

$$\{\widetilde{x}_{|(-\infty,t_1]}|\widetilde{x} \in f(\widetilde{u})\} = \{x_{|(-\infty,t_1]}|x \in f(u^0)\} \tag{7.9}$$

and if, in addition, we take into account (7.1), (7.2), then we get the truth of (7.6) and of

$$\forall \widetilde{x} \in f(\widetilde{u}), \widetilde{x}(t_1) = \mu'. \tag{7.10}$$

Let be now some arbitrary $x'' \in f(u^1 \circ \tau^d)$. From d) we obtain the existence of $x' \in f(u^1)$, such that $x'' = x' \circ \tau^d$ and we have $x''(t_1) = (x' \circ \tau^d)(t_1) = x'(t_2) = \mu'$ (we have taken into account (7.3)) thus

$$\forall x'' \in f(u^1 \circ \tau^d), x''(t_1) = \mu' \tag{7.11}$$

and, similarly,

$$\forall x'' \in f(u^1 \circ \tau^d), \exists t_3' > t_1, x''(t_3') = \mu''. \tag{7.12}$$

We see that

$$\widetilde{u}_{|[t_1,\infty)} = (u^1 \circ \tau^d)_{|[t_1,\infty)}. \tag{7.13}$$

The hypothesis of c) is fulfilled by t_1, $\widetilde{u}$ and $u^1 \circ \tau^d$, as follows from (7.10), (7.11) and (7.13). The conclusion of c) expresses the fact that

$$\{\widetilde{x}_{|[t_1,\infty)}|\widetilde{x} \in f(\widetilde{u})\} = \{x''_{|[t_1,\infty)}|x'' \in f(u^1 \circ \tau^d)\} \tag{7.14}$$

and, by (7.12), we get the truth of (7.7). □

REMARK 70. *We put the problem of defining the union of the transfers* $(\mu \xrightarrow{u} \mu') \vee (\mu' \xrightarrow{v} \mu'')$ *in a form different from the one in Definition 78, where we had* $u = v$. *In order that this fact becomes possible, we consider that the following requirements are natural: there is* $t' \in \mathbf{R}$ *such that*

$$\forall x \in f(u), x(t') = \mu', \tag{7.15}$$

$$\forall y \in f(v), y(t') = \mu', \tag{7.16}$$

$$u_{|(-\infty,t')} = v_{|(-\infty,t')}, \tag{7.17}$$

plus the non-anticipation of f *demand that, together with (7.17), imply the truth of*

$$\{x_{|(-\infty,t']}|x \in f(u)\} = \{y_{|(-\infty,t']}|y \in f(v)\}. \tag{7.18}$$

At this stage the non-anticipation requirement is crucial, since it allows us to pass from the input* u *to the input* $u \cdot \chi_{(-\infty,t')} \oplus v \cdot \chi_{[t',\infty)}$.

This is the idea from Theorem 203. Starting with the following transfers, where t_1 *and* t_2 *are fixed*

$$\mu \overset{u^0}{\rightrightarrows} \mu' = \{x_{|[t_0,t_1]}|\exists t_0 < t_1, [t_0,t_1] \in T_{\mu,\mu',x}, x \in f(u^0)\},$$

$$\mu' \overset{u^1}{\rightrightarrows} \mu'' = \{x'_{|[t_2,t_3]}|\exists t_3 > t_2, [t_2,t_3] \in T_{\mu',\mu'',x'}, x' \in f(u^1)\}$$

the theorem shows the existence of the transfer

$$(\mu \overset{u^0}{\rightrightarrows} \mu') \vee (\mu' \overset{u^1 \circ \tau^d}{\rightrightarrows} \mu'') =$$

$$= \{\widetilde{x}_{|[t_0,t_3']}|\exists t_0 < t_1, [t_0,t_1] \in T_{\mu,\mu',\widetilde{x}}, \exists t_3' > t_1, [t_1,t_3'] \in T_{\mu',\mu'',\widetilde{x}}, \widetilde{x} \in f(\widetilde{u})\}.$$

The price that we pay in order to make this construction possible is the synchronism of the access of the states of f, under the inputs u^0 and u^1 to the value μ'. In addition, because (7.15), (7.16) are fulfilled under the form: $u = u^0, t' = t_1$ in the first case, and under the form $y = x', v = u^1, t' = t_2$ in the second case, we need to translate the states x' and the input u^1, $x' \in f(u^1)$ with d time units. Thus the requirement of time invariance of f occurs too.

8. Synchronicity

REMARK 71. *Consider the system $g : V \to P^*(S^{(n)})$, $V \in P^*(S^{(m)})$ and start with the property (1.1), written for this system*

$$\exists \mu \in \mathbf{B}^n, \exists u \in V, \forall x \in g(u), \exists t \in \mathbf{R}, x(t) = \mu.$$

The subsystem $f : U \to P^*(S^{(n)})$, $U \subset V$, defined by

$$U = \{u|u \in V, \exists \mu \in \mathbf{B}^n, \forall x \in g(u), \exists t \in \mathbf{R}, x(t) = \mu\},$$

$$\forall u \in U, f(u) = g(u)$$

satisfies

$$\forall u \in U, \exists \mu \in \mathbf{B}^n, \forall x \in f(u), \exists t \in \mathbf{R}, x(t) = \mu. \tag{8.1}$$

We mention also the following stronger variants of (8.1):

$$\forall u \in U, \exists \mu \in \mathbf{B}^n, \exists t \in \mathbf{R}, \forall x \in f(u), x(t) = \mu \tag{8.2}$$

and

$$\exists t \in \mathbf{R}, \forall u \in U, \exists \mu \in \mathbf{B}^n, \forall x \in f(u), x(t) = \mu. \tag{8.3}$$

The reader is invited to reflect on these properties. Starting with the hypothesis that the system g has accessible values, we have defined its subsystem f satisfying (8.1), such that with each input $u \in U$ we can associate the set

$$\Omega_u = \{\mu|\mu \in \mathbf{B}^n, \forall x \in f(u), \exists t \in \mathbf{R}, x(t) = \mu\}$$

of the accessible values of the states of f under the input u. Of course, Ω_u and $\Omega = \bigcup_{u \in U} \Omega_u$ are the same for both systems f and g.

For arbitrary $u \in U$, in general, the set of the access time $T_{\mu,x}$ of $x \in f(u)$ to the value $\mu \in \Omega_u$ depends on the choice of x. Remark that in (8.2) we have

$$\forall u \in U, \exists \mu \in \Omega_u, \bigcap_{x \in f(u)} T_{\mu,x} \neq \emptyset$$

and in (8.3) we have

$$\exists A \in P^*(\mathbf{R}), \forall u \in U, \exists \mu \in \Omega_u, A \subset \bigcap_{x \in f(u)} T_{\mu,x}.$$

*We say that the access time of the states of f to the value μ is **unbounded** for (8.1), **bounded** for (8.2) and **fixed** for (8.3).*

When in (8.1) $u \in U$ satisfies $|\Omega_u| > 1$, two possibilities exist: the elements of Ω_u are reached by $x \in f(u)$ in an arbitrary order

$$\forall u \in U, \forall \mu \in \Omega_u, \forall x \in f(u), \exists t \in \mathbf{R}, x(t) = \mu,$$

or the order matters; then one of the following statements is true:

$$\forall u \in U, \exists k \geq 1, \exists \mu^1 \in \mathbf{B}^n, ..., \exists \mu^k \in \mathbf{B}^n, \forall x \in f(u), \exists t_1 \in \mathbf{R}, ..., \exists t_k \in \mathbf{R},$$
$$t_1 < t_2 < ... < t_k \text{ and } x(t_1) = \mu^1 \text{ and } ... \text{ and } x(t_k) = \mu^k,$$

and

$$\forall u \in U, \exists (\mu^k)_{k\geq 1} \in \mathbf{B}^n, \forall x \in f(u), \exists (t_k)_{k\geq 1} \in Seq, \forall k \geq 1, x(t_k) = \mu^k \tag{8.4}$$

respectively. The accessible values of the states must not be distinct in these formulae and in (8.4) there is the possibility $\mu^1 = \mu^2 = ...$ *meaning a cyclic behavior of the system*

$$\forall u \in U, \exists \mu \in \Omega_u, \forall x \in f(u), \exists (t_k)_{k\geq 1} \in Seq \text{ and } \forall k \geq 1, x(t_k) = \mu. \tag{8.5}$$

However, this special case is noticed also when looking at (8.1) and remarking that there we can have

$$\forall u \in U, \exists \mu \in \Omega_u, \forall x \in f(u), T_{\mu,x} \text{ is unbounded from above,}$$

i.e. μ *is an accessible recurrent value.*

Some non-exclusive special cases of (8.5) are those when

- $\mu = x(-\infty + 0)$, *the system comes back infinitely many times in the initial state,*
- $\forall k \geq 1$, $t_{k+1} = t_k + \delta$, *where* $\delta > 0$ *is a parameter (pseudo-periodicity),*
- $\mu = x(\infty - 0)$, μ *is the final state.*

We have the following stronger variants of (8.4):

$$\forall u \in U, \exists (\mu^k)_{k\geq 1} \in \mathbf{B}^n, \exists (t_k)_{k\geq 1} \in Seq, \forall x \in f(u), \forall k \geq 1, x(t_k) = \mu^k; \tag{8.6}$$

$$\exists (t_k)_{k\geq 1} \in Seq, \forall u \in U, \exists (\mu^k)_{k\geq 1} \in \mathbf{B}^n, \forall x \in f(u), \forall k \geq 1, x(t_k) = \mu^k. \tag{8.7}$$

We relate the properties (8.4), (8.6), (8.7) to those of existence of the initial state in Theorem 25 cases d), e), f), that we reproduce under the form:

$$\forall u \in U, \exists \mu^0 \in \mathbf{B}^n, \forall x \in f(u), \exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = \mu^0, \tag{8.8}$$

$$\forall u \in U, \exists \mu^0 \in \mathbf{B}^n, \exists t_0 \in \mathbf{R}, \forall x \in f(u), \forall t < t_0, x(t) = \mu^0, \tag{8.9}$$

$$\exists t_0 \in \mathbf{R}, \forall u \in U, \exists \mu^0 \in \mathbf{B}^n, \forall x \in f(u), \forall t < t_0, x(t) = \mu^0. \tag{8.10}$$

In other words, f *has race-free initial states with unbounded, bounded and fixed initial time respectively. By putting together (8.4) with (8.8), (8.6) with (8.9), (8.7) with (8.10) we get the following properties*

$$\forall u \in U, \exists (\mu^k) \in \mathbf{B}^n, \forall x \in f(u), \tag{8.11}$$
$$\exists (t_k) \in Seq, x(-\infty + 0) = \mu^0 \text{ and } \forall k \in \mathbf{N}, x(t_k) = \mu^k,$$

$$\forall u \in U, \exists (\mu^k) \in \mathbf{B}^n, \exists (t_k) \in Seq, \tag{8.12}$$
$$\forall x \in f(u), x(-\infty + 0) = \mu^0 \text{ and } \forall k \in \mathbf{N}, x(t_k) = \mu^k,$$

$$\exists (t_k) \in Seq, \forall u \in U, \tag{8.13}$$
$$\exists (\mu^k) \in \mathbf{B}^n, \forall x \in f(u), x(-\infty + 0) = \mu^0 \text{ and } \forall k \in \mathbf{N}, x(t_k) = \mu^k.$$

There (8.11),...,(8.13) represent the idea of predictability of the behavior of f *that we want to underline in this section. In (8.11), predictability is spatial only: for any* $u \in U$, *it is known that* $\mu^0, \mu^1, ...$ *are values that sometime will be reached by all states of* f, *in this order. (8.12) is an intermediary situation that has previously* *occurred (under the form (8.2), for example), when we have used the terminology of synchronous access(es). (8.13) is that situation when predictability is more complex, temporal and spacial: the discrete time instants $t_0, t_1, ...$ are known when for any u, all states of f will take the values $\mu^0, \mu^1, ...$ depending on $u \in U$.*

Remark that a new nuance of the predictability of the behavior of f is obtained by inverting u and μ in the previous properties. For example we can replace (8.1) by

$$\exists \mu \in \mathbf{B}^n, \forall u \in U, \forall x \in f(u), \exists t \in \mathbf{R}, x(t) = \mu$$

and (8.11) by

$$\exists (\mu^k) \in \mathbf{B}^n, \forall u \in U, \forall x \in f(u),$$
$$\exists (t_k) \in Seq, x(-\infty + 0) = \mu^0 \text{ and } \forall k \in \mathbf{N}, x(t_k) = \mu^k$$

respectively. These two conditions express the requirement that the system, irrespective of the choice of the input, reaches with its states certain points. For example the first property might mean the existence of a (unique) initial (final) state.

DEFINITION 79. *A system f that satisfies (8.12) is called **weakly synchronous** while if (8.13) is fulfilled, then f is said to be **strongly synchronous**.*

THEOREM 204. *If the system $g : V \to P^*(S^{(n)}), V \subset S^{(m)}$ is weakly (strongly) synchronous then any subsystem $f \subset g$ has the same property.*

THEOREM 205. *If $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$ is weakly (strongly) synchronous, then f^* has the same property.*

PROOF. For example, (8.12) implies

$$\forall u \in U^*, \exists (\mu^k) \in \mathbf{B}^n, \exists (t_k) \in Seq, \forall x \in f(\overline{u}),$$
$$\overline{x(-\infty + 0)} = \overline{\mu^0} \text{ and } \forall k \in \mathbf{N}, \overline{x(t_k)} = \overline{\mu^k}$$

wherefrom we have that f^* is weakly synchronous. □

THEOREM 206. *Let be the non-empty set $X \subset S^{(n)}$ that we identify with the autonomous system $f = X$. The system f is weakly synchronous if f is strongly synchronous if*

$$\exists (\mu^k) \in \mathbf{B}^n, \exists (t_k) \in Seq, \forall x \in X,$$
$$x(-\infty + 0) = \mu^0 \text{ and } \forall k \in \mathbf{N}, x(t_k) = \mu^k.$$

THEOREM 207. *If f is non-anticipatory: $\forall t \in \mathbf{R}, \forall u \in U, \forall v \in U,$*

$$u_{(-\infty,t)} = v_{(-\infty,t)} \Longrightarrow \{x_{|(-\infty,t]} | x \in f(u)\} = \{y_{|(-\infty,t]} | y \in f(v)\}$$

and strongly synchronous, we fix a family $(t_k) \in Seq$ that makes (8.13) true. Then for all $k \in \mathbf{N}$, the values μ^k depend on $u_{(-\infty,t_k)}$ only.

PROOF. Let be some arbitrary $k \in \mathbf{N}$, $t_k \in \mathbf{R}$ and $u, v \in U$ such that $\forall x \in f(u), x(t_k) = \mu^k$, $\forall y \in f(v), y(t_k) = \mu'^k$. If $u_{|(-\infty,t_k)} = v_{|(-\infty,t_k)}$, from the non-anticipation of f we have that $\{x_{|(-\infty,t_k]} | x \in f(u)\} = \{y_{|(-\infty,t_k]} | y \in f(v)\}$. In particular $\mu^k = \mu'^k$, where k is arbitrary, so that the property is true for any k. □

THEOREM 208. *If f is strongly synchronous and time invariant, then $\forall u \in U, \forall x \in f(u), x$ is the constant function.*

PROOF. The fact that f is strongly synchronous implies that it has race-free initial states and fixed initial time. We apply Theorem 162. □

REMARK 72. *If we join (8.7) with the existence of the initial state in Theorem 25 case i) (instead of f)) i.e. f has a constant initial state with fixed initial time, reproduced under the form*

$$\exists\mu^0 \in \mathbf{B}^n, \exists t_0 \in \mathbf{R}, \forall u \in U, \forall x \in f(u), \forall t < t_0, x(t) = \mu^0,$$

we get

$$\exists (t_k) \in Seq, \exists \mu^0 \in \mathbf{B}^n, \forall u \in U, \tag{8.14}$$

$$\exists (\mu^k)_{k\geq 1} \in \mathbf{B}^n, \forall x \in f(u), x(-\infty+0) = \mu^0 \text{ and } \forall k \in \mathbf{N}, x(t_k) = \mu^k$$

(instead of (8.13)). The only difference between (8.13) and (8.14) is that in the last property the initial state μ^0 does not depend on the input u.

DEFINITION 80. *A system f that fulfills the property (8.14) is called* ***initialized strongly synchronous****.*

CHAPTER 7

Surjectivity, controllability and accessibility

Controllability and accessibility are fundamental concepts in the systems theory. The wish to study them leads us easily to the conclusion that they do not represent a common point of view of the researchers. Our presentation is made as a continuation of the debates on the surjectivity of the systems. A comparison of our concepts of controllability and accessibility with others existing in the literature is included.

1. Surjectivity, remark

REMARK 73. *There is the temptation of considering the surjectivity of* $f : U \to P^*(S^{(n)})$, $U \in P^*(S^{(m)})$ *as defined by*

$$\forall X \in P^*(S^{(n)}), \exists u \in U, f(u) = X$$

and then of relating this concept to the first concept of injectivity from Definition 67

$$\forall u \in U, \forall v \in U, u \neq v \Longrightarrow f(u) \neq f(v).$$

From these two definitions the bijections $U \to P^*(S^{(n)})$ *should follow. The problem is that we have good reasons to believe that such bijections do not exist and, because we do not know examples for the previous property of surjectivity, we avoid it.*

2. Surjectivity, the first definition

DEFINITION 81. *The system* f *is* ***surjective*** *(or* ***onto****) if one of the following equivalent properties is true:*

a) $\forall x \in S^{(n)}, \exists u \in U, x \in f(u)$,

b) $\bigcup\limits_{u \in U} f(u) = S^{(n)}$.

REMARK 74. *The definition of surjectivity starts with the idea of referring to the states, and not to the sets of states. This is due to the fact that in that case reasoning seemed to be blocked. It states that for a surjective system any state is possible, if the input is appropriately chosen.*

Note that if f *is deterministic, then this definition of surjectivity coincides with the usual one.*

EXAMPLE 82. *The system* $f : S \to P^*(S)$,

$$\forall u \in S, f(u) = \{u, \overline{u}\}$$

is surjective. It is also self-dual.

EXAMPLE 83. *Consider the system* $f : S^{(m)} \to P^*(S^{(m)})$ *defined by*

$$\forall u \in S^{(m)}, f(u) = \{u_\sigma | \sigma \in S(\{1, ..., m\})\}.$$

It is obviously surjective and symmetrical.

EXAMPLE 84. *The system:* $f : S^{(m)} \to P^*(S^{(m)})$,

$$\forall u \in S^{(m)}, f(u) = \{u \circ \tau^d | d \in \mathbf{R}\}$$

is surjective and time invariant.

THEOREM 209. *If f is a surjective system, then its initial state function ϕ_0 satisfies*

$$\forall \mu \in \mathbf{B}^n, \exists u \in U, \mu \in \phi_0(u).$$

PROOF. For $\forall \mu \in \mathbf{B}^n$, we take some $x \in S^{(n)}$ with $x(-\infty + 0) = \mu$ for which there is some $u \in U$ such that $x \in f(u)$ and $x(-\infty + 0) \in \phi_0(u)$. □

THEOREM 210. *Suppose that f is surjective and $f \subset \dot{g}$, where $g : V \to P^*(S^{(n)}), V \in P^*(S^{(m)})$ is some system. Then g is surjective.*

PROOF. For $\forall x \in S^{(n)}$, there is some $u \in U$ such that $x \in f(u)$. Because $u \in V$ and $x \in g(u)$, the statement of the theorem follows. □

THEOREM 211. *The surjectivity of f implies the surjectivity of f^*.*

THEOREM 212. *The Cartesian product of the surjective systems $f : U \to P^*(S^{(n)})$, $U \in P^*(S^{(m)})$ and $f' : U' \to P^*(S^{(n')})$, $U' \in P^*(S^{(m')})$ is surjective.*

PROOF. Let $z \in S^{(n+n')}$ be arbitrary and we denote by x its first n coordinates and by x' its last n' coordinates. There is some $u \in U$ such that $x \in f(u)$ and there is also some $u' \in U'$ such that $x' \in f'(u')$. In other words $z \in (f \times f')(u \times u')$. □

THEOREM 213. *Let be the systems $f : U \to P^*(S^{(n)})$, $U \in P^*(S^{(m)})$ and $h : X \to P^*(S^{(p)})$, $X \in P^*(S^{(n)})$ such that the inclusion $\bigcup\limits_{u \in U} f(u) \subset X$ is true. If $h \circ f$ is surjective, then h is surjective too.*

PROOF. We have

$$\forall z \in S^{(p)}, \exists u \in U, z \in (h \circ f)(u),$$
$$\forall z \in S^{(p)}, \exists u \in U, \exists x \in f(u), z \in h(x),$$
$$\forall z \in S^{(p)}, \exists x \in X, z \in h(x).$$

□

THEOREM 214. *Consider the systems f, h with the property that h is surjective and $\bigcup\limits_{u \in U} f(u) = X$. Then $h \circ f$ is surjective. In particular the serial connection of the surjective systems is surjective.*

PROOF. Let $z \in S^{(p)}$ be arbitrary. The fact that there is $x \in X$ with $z \in h(x)$ holds true from the surjectivity of h. Then there is $u \in U$ such that $x \in f(u)$. Thus $z \in (h \circ f)(u)$. In particular, if f is surjective, then $\bigcup\limits_{u \in U} f(u) = X = S^{(n)}$ is true. □

THEOREM 215. *The union of a surjective system with an arbitrary system is surjective. In particular, the union of the surjective systems is surjective.*

PROOF. For any systems f, g we have $f \subset f \cup g$. If f is surjective then, by Theorem 210, $f \cup g$ is surjective. □

THEOREM 216. *An autonomous system $f = X$ is surjective iff $X = S^{(n)}$.*

THEOREM 217. *Let $F : \mathbf{B}^m \to \mathbf{B}^n$ be a surjective Boolean function. Then the ideal combinational system F_d is surjective for any $d \in \mathbf{R}$.*

PROOF. Take arbitrary fixed $d \in \mathbf{R}$ and $x \in S^{(n)}$. Let $(t_k) \in Seq$ be a sequence consistent with x

$$x(t) = x(t_0 - 0) \cdot \chi_{(-\infty,t_0)}(t) \oplus x(t_0) \cdot \chi_{[t_0,t_1)}(t) \oplus x(t_1) \cdot \chi_{[t_1,t_2)}(t) \oplus ...$$

The numbers $\lambda_{-1}, \lambda_0, \lambda_1, ... \in \mathbf{B}^m$ are chosen such that

$$F(\lambda_{-1}) = x(t_0 - 0),$$

$$\forall k \in \mathbf{N}, F(\lambda_k) = x(t_k),$$

their existence being assured by the surjectivity of F. We get that the function

$$u(t) = \lambda_{-1} \cdot \chi_{(-\infty,t_0-d)}(t) \oplus \lambda_0 \cdot \chi_{[t_0-d,t_1-d)}(t) \oplus \lambda_1 \cdot \chi_{[t_1-d,t_2-d)}(t) \oplus ...$$

satisfies

$$F_d(u)(t) = F(u(t-d)) =$$

$$= F(\lambda_{-1} \cdot \chi_{(-\infty,t_0-d)}(t-d) \oplus \lambda_0 \cdot \chi_{[t_0-d,t_1-d)}(t-d) \oplus \lambda_1 \cdot \chi_{[t_1-d,t_2-d)}(t-d) \oplus ...) =$$

$$= F(\lambda_{-1} \cdot \chi_{(-\infty,t_0)}(t) \oplus \lambda_0 \cdot \chi_{[t_0,t_1)}(t) \oplus \lambda_1 \cdot \chi_{[t_1,t_2)}(t) \oplus ...) =$$

$$= F(\lambda_{-1}) \cdot \chi_{(-\infty,t_0)}(t) \oplus F(\lambda_0) \cdot \chi_{[t_0,t_1)}(t) \oplus F(\lambda_1) \cdot \chi_{[t_1,t_2)}(t) \oplus ... =$$

$$= x(t_0 - 0) \cdot \chi_{(-\infty,t_0)}(t) \oplus x(t_0) \cdot \chi_{[t_0,t_1)}(t) \oplus x(t_1) \cdot \chi_{[t_1,t_2)}(t) \oplus ... = x(t).$$

□

3. Possible and necessary surjectivity

REMARK 75. *Let be the system $f : U \to P^*(S^{(n)})$, where $U \subset S^{(m)}$ is non-empty. We state the following properties:*

$$\forall \mu \in \mathbf{B}^n, \exists u \in U, \exists x \in f(u), \exists t \in \mathbf{R}, x(t) = \mu; \tag{3.1}$$

$$\forall \mu \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u), \exists t \in \mathbf{R}, x(t) = \mu; \tag{3.2}$$

$$\forall \mu \in \mathbf{B}^n, \exists u \in U, \exists t \in \mathbf{R}, \forall x \in f(u), x(t) = \mu; \tag{3.3}$$

$$\exists t \in \mathbf{R}, \forall \mu \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u), x(t) = \mu. \tag{3.4}$$

The implications:

$$(3.4) \Longrightarrow (3.3) \Longrightarrow (3.2) \Longrightarrow (3.1)$$

hold. The interpretation of (3.1),...,(3.4) is simple: after relating the surjectivity property to the sets $X \in P^*(S^{(n)})$ and to the states $x \in S^{(n)}$, we relate it to the values $\mu \in \mathbf{B}^n$. The first property states that all $\mu \in \mathbf{B}^n$ are possible values of the states of f while the last three properties, that all $\mu \in \mathbf{B}^n$ are necessary values of the states of f for suitably chosen inputs. (3.2) and (3.3) mean that all values $\mu \in \mathbf{B}^n$ are accessible, $\Omega = \mathbf{B}^n$ respectively synchronously accessible, $\Omega_s = \mathbf{B}^n$.

Note that any system f satisfying the surjectivity condition from Definition 81 satisfies also the following strong version of (3.1):

$$\forall \mu \in \mathbf{B}^n, \exists u \in U, \exists x \in f(u), \forall t \in \mathbf{R}, x(t) = \mu.$$

DEFINITION 82. *The system f is* ***possibly surjective (onto)*** *if it satisfies (3.1).*

DEFINITION 83. *The system f is called* ***necessarily surjective*** *if it satisfies either of* (3.2),...,(3.4). *If (*3.2*) is true, then f is* ***necessarily surjective with unbounded access time****. If* (3.3) *is true, then f is* ***necessarily surjective with bounded access time****. If* (3.4) *is fulfilled, then f is* ***necessarily surjective with fixed access time****. In these three situations the access time t is called* ***unbounded****,* ***bounded*** *and* ***fixed****.*

EXAMPLE 85. *The autonomous system $f : S \to S$,*

$$\forall u \in S, f(u) = S$$

is possibly surjective, but not necessarily surjective.

EXAMPLE 86. *Denote $U = \{\chi_{(-\infty,0)}, \chi_{[0,\infty)}\}$ and let $f : U \to P^*(S)$ be defined by*

$$\forall u \in U, f(u) = \{u \circ \tau^d | d \in \mathbf{R}\}.$$

The system f is necessarily surjective with unbounded access time, but not with bounded access time. We verify the property (3.2)*, for example, for $\mu = 0 : \exists u \in U$, that is $u = \chi_{(-\infty,0)}$ such that $f(u) = \{\chi_{(-\infty,d)} | d \in \mathbf{R}\}$ and $\forall x \in f(u)$, there is $t \in \mathbf{R}$, i.e. $t \geq \sup$ supp x with the property $x(t) = 0$.*

EXAMPLE 87. *$U = \{0, 1\}$ (the two $\mathbf{R} \to \mathbf{B}$ constant functions) and $f : U \to P^*(S)$ is defined as*

$$f(0) = \{x | x \in S, x(0) = 0\},$$
$$f(1) = \{x | x \in S, x(2) = 1\}.$$

The system f is necessarily surjective with bounded access time, but not with fixed access time. In order to verify that (3.3) *is fulfilled, we choose, for any $\mu \in \mathbf{B}$, the input $u = \mu$ (the equality between the constant function and the constant) and*

$$t = \begin{cases} 0, if\ \mu = 0 \\ 2, if\ \mu = 1 \end{cases}.$$

EXAMPLE 88. *We define $f : S \to P^*(S)$ by*

$$\forall u \in S, f(u) = \{x | x \in S, x(0) = u(0)\}.$$

The system f is necessarily surjective with fixed access time, that is for $t = 0$ and for any $\mu \in \mathbf{B}$ we can choose some $u \in S$ such that $u(0) = \mu$. Then $\forall x \in f(u), x(0) = \mu$. The property (3.4) *is true.*

THEOREM 218. *If f is a deterministic system, then the properties of possible surjectivity and necessary surjectivity with unbounded access time are equivalent.*

THEOREM 219. *Suppose that f is possibly surjective and that $f \subset g$, for some system $g : V \to P^*(S^{(n)})$, $V \in P^*(S^{(m)})$. Then g is possibly surjective.*

PROOF. We infer that if $\mu \in \mathbf{B}^n$ is arbitrary, then $u \in U$ exists, thus there are $u \in V$ and $x \in f(u)$, thus there are $x \in g(u)$ and $t \in \mathbf{R}$ with the property $x(t) = \mu$. □

THEOREM 220. *Suppose that the system g is necessarily surjective with unbounded (bounded, fixed) access time. Then from $f \subset g$ and $U = V$, we infer that f is necessarily surjective with unbounded (bounded, fixed) access time.*

PROOF. The properties of all states of g are, in particular, the properties of all states of f. □

THEOREM 221. *If the system f is possibly surjective (necessarily surjective with an unbounded, with a bounded, with a fixed access time) then f^* has the same surjectivity property.*

PROOF. We show this implication for the statement (3.2) :

$$\forall \mu \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u), \exists t \in \mathbf{R}, x(t) = \mu \Longleftrightarrow$$
$$\forall \mu \in \mathbf{B}^n, \exists u \in U^*, \forall x \in f(\overline{u}), \exists t \in \mathbf{R}, \overline{x}(t) = \overline{\mu} \Longleftrightarrow$$
$$\forall \mu \in \mathbf{B}^n, \exists u \in U^*, \forall x \in f^*(u), \exists t \in \mathbf{R}, x(t) = \mu.$$

□

THEOREM 222. *Consider the systems $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$ and $h : X \to P^*(S^{(p)}), X \in P^*(S^{(n)})$ with the property that the inclusion $\bigcup_{u \in U} f(u) \subset X$ holds true. If $h \circ f$ is possibly surjective (necessarily surjective with an unbounded, with a bounded, with a fixed access time), then h is possibly surjective (necessarily surjective with an unbounded, with a bounded, with a fixed access time).*

PROOF. Suppose that $h \circ f$ is possibly surjective. Then

$$\forall \nu \in \mathbf{B}^p, \exists u \in U, \exists z \in (h \circ f)(u), \exists t \in \mathbf{R}, z(t) = \nu.$$

Thus

$$\forall \nu \in \mathbf{B}^p, \exists u \in U, \exists x \in f(u), \exists z \in h(x), \exists t \in \mathbf{R}, z(t) = \nu,$$
$$\forall \nu \in \mathbf{B}^p, \exists x \in X, \exists z \in h(x), \exists t \in \mathbf{R}, z(t) = \nu$$

and h is possibly surjective.

Suppose now that $h \circ f$ is necessarily surjective with a bounded access time. We have

$$\forall \nu \in \mathbf{B}^p, \exists u \in U, \exists t \in \mathbf{R}, \forall z \in (h \circ f)(u), z(t) = \nu,$$
$$\forall \nu \in \mathbf{B}^p, \exists u \in U, \exists t \in \mathbf{R}, \forall x \in f(u), \forall z \in h(x), z(t) = \nu,$$
$$\forall \nu \in \mathbf{B}^p, \exists t \in \mathbf{R}, \exists x \in X, \forall z \in h(x), z(t) = \nu$$

i.e. h is necessarily surjective with a bounded access time. □

THEOREM 223. *If f satisfies $\bigcup_{u \in U} f(u) = X$ and h is possibly surjective, then the system $h \circ f$ is possibly surjective. In particular, if f satisfies the property of surjectivity from Definition 81 and h is possibly surjective, then $h \circ f$ is possibly surjective.*

PROOF. We have

$$\forall \nu \in \mathbf{B}^p, \exists x \in X, \exists z \in h(x), \exists t \in \mathbf{R}, z(t) = \nu,$$
$$\forall \nu \in \mathbf{B}^p, \exists u \in U, \exists x \in f(u), \exists z \in h(x), \exists t \in \mathbf{R}, z(t) = \nu,$$
$$\forall \nu \in \mathbf{B}^p, \exists u \in U, \exists z \in (h \circ f)(u), \exists t \in \mathbf{R}, z(t) = \nu.$$

The property from Definition 81 implies $\bigcup_{u \in U} f(u) = X = S^{(n)}$. □

THEOREM 224. *Let f be a necessarily surjective system with an unbounded (a bounded, a fixed) access time and g an arbitrary system with $U \subset V$ and $\forall u \in U, f(u) \cap g(u) \neq \emptyset$. Then the system $f \cap g$ is necessarily surjective with an unbounded (a bounded, a fixed) access time.*

PROOF. The support of $f \cap g$ is U. We apply Theorem 220. □

THEOREM 225. *If the system f is possibly surjective and g is an arbitrary system, then $f \cup g$ is possibly surjective. In particular, the union of the possibly surjective systems is a possibly surjective system.*

PROOF. Because $f \subset f \cup g$, the statement of the theorem follows from Theorem 219. □

4. Controllability and accessibility, points of view

REMARK 76. *The notion of controllability* [**20**] *of the linear differential equations was implicitly introduced in the works of optimal systems by L.S. Pontriagin and his colleagues under the form of certain algebraic conditions. The notion has become distinct due to the works of R.E. Kalman presented at the Conference of Differential Equations of Mexico City in 1959 and the first Congress of Automatic Control from Moscow in 1960.*

At this moment, we reproduce some points of view on controllability and accessibility. We mention that the authors to follow work with real deterministic systems and their ideas must be adapted to the present context.

Professor Toma Leonida Dragomir argues[1] *that 'controllability means the existence of a command that can bring the system in a bounded, arbitrary time interval*[2] *from an arbitrary state in a steady state*[3]*. Accessibility means the existence of a command that can bring the system from a steady state in an arbitrary state, therefore the access from the steady state to an arbitrary state, in a bounded, arbitrary time interval also. For the majority of the linear systems the two properties are equivalent, but there are linear systems for which they are not equivalent. In the case of the non-linear systems, the problem is more difficult. Historically, first appeared the concept of controllability, then the one of accessibility. Because of their equivalence in the usual linear cases, they are frequently identified'.*

Anouck Girard's opinion on controllability[4] *is that intuitively 'you can get anywhere you want in a finite amount of time'. In this approach any requirement of reaching a steady state is missing.*

F.H. Clarke et al.[5] *define the asymptotic controllability in a manner consistent, except for the asymptotic requirement, with Dragomir and remark that their definition is a 'natural generalization to control systems of the concept of uniform asymptotic stability of solutions of differential equations'.*

In his 'Kalman's Controllability Rank Condition: from Linear to Nonlinear'[6]*, Eduardo D. Sontag identifies the Dragomir's controllability and accessibility by stating that 'In principle, one wishes to study controllability from the origin'. The author considers the origin be a steady state, or, in his terminology, an 'equilibrium state' and he quotes that there is another terminology for this concept of controllability, namely that of reachability. Furthermore, 'For controllability questions from non-equilibria related results hold, except for some minor changes in definitions',*

[1] private mail

[2] this means that we do not refer to an asymptotic behavior here

[3] i.e. the final value of the state, the null vector in the linear case

[4] ME237-Control of Nonlinear Dynamic Systems, Discussion #3, Controllability and Observability of Nonlinear Systems, February 18-th, 2002

[5] F.H. Clarke, Yu.S. Ledyaev, E.D. Sontag, A.I. Subbotin, Asymptotic Controllability Implies Feedback Stabilization, IEEE Transactions on Automatic Control, Vol. XX, No. Y, 1999

[6] to appear in Mathematical System Theory: The influence of R. E. Kalman

i.e. he accepts the controllability in the sense of Girard, a simple transfer without any initial or final equilibrium. Another remark in his work is that 'often one is interested ... in controllability to zero'. By accessibility, after transposing the concept for our present needs, Sontag means the same as his 'controllability from the origin'.

We present now the definitions given by Professor Mihail Megan [**20**] *to the two concepts, translated in the language of the asynchronous systems. Note that in these definitions the author knows and makes use of the possibility of including or not the steady state requirement. To be mentioned also that, sometimes, different possibilities of translating his definitions in the asynchronous systems language exist. Let be* $t' \in \mathbf{R}$. *The system* f *is:*

a) ***exactly*** t' ***controllable*** *if for any* $\mu', \mu'' \in \mathbf{B}^n$, *we have that there are* $t'' \geq t'$ *and* $u \in U$ *such that (all) the states* $x \in f(u)$ *reach* μ', μ'' *at* t', t'';

b) ***exactly*** t' ***stable controllable*** *(originally:* ***exactly*** t' ***null controllable****) if for any* $\mu' \in \mathbf{B}^n$, *there is some steady state* $\mu'' \in \mathbf{B}^n$[7] *as well as* $t'' \geq t'$ *and* $u \in U$ *such that (all) the states* $x \in f(u)$ *reach* μ', μ'' *at* t' *and* t'';

c) ***exactly*** t' ***controllable with universal time*** *if at a)* t'' *depends on* t' *only and is independent of any of* μ', μ'', u *and* x;

d) ***exactly*** t' ***stable controllable with universal time*** *(originally:* ***exactly*** t' ***null controllable with universal time****) if at b)* t'' *depends on* t' *and is independent of the other variables.*

With such definitions Megan accepts, in fact, all the other points of view. The word 'exact' in this terminology is opposed to 'approximate', a variant that this author takes also in consideration. We do not insist in that direction because 'exactly' and 'approximately' coincide in our study. Moreover, f *is:*

e) ***exactly completely controllable*** *if a) is true for any* t';

f) ***exactly completely stable controllable*** *(originally:* ***exactly completely null controllable****) if b) is true for any* t';

g) ***uniformly exactly controllable*** *if at e)* $t'' = t' + \delta$, $\delta > 0$ *a constant;*

h) ***uniformly exactly stable controllable*** *(originally:* ***uniformly exactly null controllable****) if at f)* $t'' = t' + \delta$, $\delta > 0$ *a constant.*

We give now from the Professor Megan paper [**20**] *the definitions of accessibility as obtained after the translation in the terms of the asynchronous systems theory. Let be* $t' \in \mathbf{R}$. *The system* f *is:*

a') ***exactly*** t' ***accessible*** *if* $\forall \mu'' \in \mathbf{B}^n$, *there are some steady state* $\mu' \in \mathbf{B}^n$[8], *the time instant* $t'' \geq t'$ *and the input* $u \in U$ *such that the values* μ', μ'' *are reached by (all) the states* $x \in f(u)$ *at* t' *and respectively at* t'';

b') ***exactly*** t' ***accessible with universal time*** *if at a')* t'' *depends on* t' *only and is independent of the rest;*

c') ***exactly completely accessible*** *if a') is true for all* t';

d') ***uniformly exactly accessible*** *if at c')* $\delta > 0$ *exists such that* $t'' = t' + \delta$.

In the next sections we have used the word 'accessibility' for both of controllability and accessibility.

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[7] here it is not obvious if the translation should be $\exists \mu" \in \mathbf{B}^n$, $\mu"$ steady state or $\forall \mu" \in \mathbf{B}^n$, $\mu"$ steady state

[8] like before, there are two convenient translations here: $\exists \mu' \in \mathbf{B}^n$, μ' steady state and $\forall \mu' \in \mathbf{B}^n$, μ' steady state

5. Accessibility in the sense of having access

DEFINITION 84. *Let be the system f and the number $\delta > 0$. Consider the following statements*

$$\forall \mu \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u), \exists t \in \mathbf{R}, x(t) = \mu, \tag{5.1}$$

$$\forall \mu \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u), \exists t \in \mathbf{R}, x_{|(-\infty,t)} = \mu, \tag{5.2}$$

$$\forall \mu \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u), \exists t \in \mathbf{R}, x_{|[t,\infty)} = \mu, \tag{5.3}$$

$$\forall \mu \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u), \forall t_0 \in \mathbf{R}, \exists t > t_0, x(t) = \mu, \tag{5.4}$$

$$\forall \mu \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u), \exists t \in \mathbf{R}, x_{|[t,t+\delta]} = \mu, \tag{5.5}$$

$$\forall \mu \in \mathbf{B}^n, \exists u \in U, \exists t \in \mathbf{R}, \forall x \in f(u), x(t) = \mu, \tag{5.6}$$

$$\forall \mu \in \mathbf{B}^n, \exists u \in U, \exists t \in \mathbf{R}, \forall x \in f(u), x_{|(-\infty,t)} = \mu, \tag{5.7}$$

$$\forall \mu \in \mathbf{B}^n, \exists u \in U, \exists t \in \mathbf{R}, \forall x \in f(u), x_{|[t,\infty)} = \mu, \tag{5.8}$$

$$\forall \mu \in \mathbf{B}^n, \exists u \in U, \forall t_0 \in \mathbf{R}, \exists t > t_0, \forall x \in f(u), x(t) = \mu, \tag{5.9}$$

$$\forall \mu \in \mathbf{B}^n, \exists u \in U, \exists t \in \mathbf{R}, \forall x \in f(u), x_{|[t,t+\delta]} = \mu. \tag{5.10}$$

a) (5.1) is called the property of ***accessibility****. If $\Omega = \mathbf{B}^n$ or, equivalently, if (5.1) is true, we say that f is* ***accessible****.*

b) (5.2) is called the property of ***accessibility to the initial values of the states****. If $\Theta'_0 = \mathbf{B}^n$ or, equivalently, if (5.2) is fulfilled, we say that f is* ***accessible in the sense of the access to the initial values of the states****.*

...

c) (5.10) is the property of ***synchronous accessibility to the $\delta-$persistent values of the states****. If $\Omega_{\delta s} = \mathbf{B}^n$ or, equivalently, if (5.10) is satisfied, then we say that f is* ***accessible in the sense of the synchronous access to the $\delta-$persistent values of the states****.*

REMARK 77. *The statements (5.1),...,(5.10) are similar to (1.1),...,(1.10) from Ch. 6 where $\exists\mu$ was replaced by $\forall\mu$, i.e. instead of the existence of the access to some value μ, we have the access to any value μ. The implications between the previous properties are the same like those in Remark 60 i.e.*

$$\begin{array}{ccccc}
(5.6) & \Longleftarrow & (5.10) & \Longleftarrow & (5.7) \\
\Downarrow & & \Downarrow & & \Downarrow \\
(5.1) & \Longleftarrow & (5.5) & \Longleftarrow & (5.2) \\
\Uparrow & & \Uparrow & & \\
(5.4) & \Longleftarrow & (5.3) & & \\
\Uparrow & & \Uparrow & & \\
(5.9) & \Longleftarrow & (5.8) & & \\
\Downarrow & & \Downarrow & & \\
(5.6) & & (5.10) & &
\end{array}$$

(5.1) and (5.6) coincide with the requirements of necessary surjectivity (3.2), (3.3).

(5.6),...,(5.10) may be strengthened themselves to 'fixed time' properties, in order to give new meanings to the concept of accessibility.

(5.1),...,(5.10) are interpreted as a generalization of the Dragomir's point of view that 'controllability means the existence of a command that can bring the system in a bounded arbitrary time interval from an arbitrary state in a steady state' (see (5.3) and also (1.3) from Ch. 6) to all types of accessible values that we have used: arbitrary, initial, final, recurrent and $\delta-$persistent. The most general of these properties, (5.1) and (5.6), match also the Girard's demand that 'you can get anywhere you want in a finite amount of time'.

6. The access of the non-anticipatory systems from a final state

THEOREM 226. *Suppose that the system f is non-anticipatory in the sense of Definition 64: $\forall t \in \mathbf{R}, \forall u \in U, \forall v \in U$,*

$$u_{|(-\infty,t)} = v_{|(-\infty,t)} \Longrightarrow \{x_{|(-\infty,t]} | x \in f(u)\} = \{y_{|(-\infty,t]} | y \in f(v)\}$$

and we fix $\mu', \mu'' \in \mathbf{B}^n$, $u \in U$. The following statements are equivalent:

a) $\forall x \in f(u), \exists t' \in \mathbf{R}, x(t') = \mu'$ and $\exists t'' > t', x(t'') = \mu''$;

b) $\exists v \in U, \forall y \in f(v), \exists t' \in \mathbf{R}, u_{|(-\infty,t')} = v_{|(-\infty,t')}, y(t') = \mu'$ and $\forall x \in f(u), \exists t'' > t', x(t'') = \mu''$.

PROOF. $a) \Longrightarrow b)$ It is sufficient to take $u = v$.

$b) \Longrightarrow a)$ We fix some $v \in U$ making b) true and we denote

$$t_1 = \sup\{t' | t' \in \mathbf{R}, u_{|(-\infty,t')} = v_{|(-\infty,t')}\}.$$

If $t_1 = \infty$, then $u = v$ and a) is true. Thus we can suppose from this moment on that $t_1 < \infty$. From $u_{|(-\infty,t_1)} = v_{|(-\infty,t_1)}$ and from the non-anticipation of f, we have

$$\{x_{|(-\infty,t_1]} | x \in f(u)\} = \{y_{|(-\infty,t_1]} | y \in f(v)\}. \tag{6.1}$$

On the other hand, taking into account (6.1), b) implies

$$\forall y \in f(v), \exists t' \in \mathbf{R}, u_{|(-\infty,t')} = v_{|(-\infty,t')}, y(t') = \mu' \textit{ and}$$
$$\textit{and } \forall x \in f(u), \exists t'' > t', x(t'') = \mu'',$$
$$\forall y \in f(v), \exists t' \leq t_1, y(t') = \mu' \textit{ and } \forall x \in f(u), \exists t'' > t', x(t'') = \mu'',$$
$$\forall x \in f(u), \exists t' \in \mathbf{R}, x(t') = \mu' \textit{ and } \forall x \in f(u), \exists t'' > t', x(t'') = \mu'',$$
$$\forall x \in f(u), \exists t' \in \mathbf{R}, x(t') = \mu' \textit{ and } \exists t'' > t', x(t'') = \mu''.$$

□

REMARK 78. *The access of the states of a system to a final value and then from that final value to other values is important in the theory of the asynchronous systems. A certain triviality occurs here, in the sense that the sets (see Definition 69 and Definition 72)*

$$\Theta'_f \otimes \Omega = \{(\mu', \mu'') | \mu', \mu'' \in \mathbf{B}^n, \exists u \in U,$$
$$\forall x \in f(u), \exists t' \in \mathbf{R}, x_{|[t',\infty)} = \mu' \textit{ and } \exists t'' > t', x(t'') = \mu''\},$$
$$\Theta'_f \otimes \Theta'_f = \{(\mu', \mu'') | \mu', \mu'' \in \mathbf{B}^n, \exists u \in U,$$
$$\forall x \in f(u), \exists t' \in \mathbf{R}, x_{|[t',\infty)} = \mu' \textit{ and } \exists t'' \in \mathbf{R}, x_{|[t'',\infty)} = \mu''\},$$
$$\Theta'_f \otimes R = \{(\mu', \mu'') | \mu', \mu'' \in \mathbf{B}^n, \exists u \in U,$$
$$\forall x \in f(u), \exists t' \in \mathbf{R}, x_{|[t',\infty)} = \mu' \textit{ and } \forall t_1 \in \mathbf{R}, \exists t'' > t_1, x(t'') = \mu''\},$$

$$\Theta'_f \otimes \Omega_\delta = \{(\mu',\mu'')|\mu',\mu'' \in \mathbf{B}^n, \exists u \in U,$$
$$\forall x \in f(u), \exists t' \in \mathbf{R}, x_{|[t',\infty)} = \mu' \text{ and } \exists t'' > t'-\delta, x_{|[t'',t''+\delta]} = \mu''\}$$

$\delta > 0$, *are all equal to (see Theorem 202 c))*

$$\{(\mu,\mu)|\mu \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u), \exists t \in \mathbf{R}, x_{|[t,\infty)} = \mu\}$$

while the set

$$\Theta'_f \otimes \Theta'_0 = \{(\mu',\mu'')|\mu',\mu'' \in \mathbf{B}^n, \exists u \in U,$$
$$\forall x \in f(u), \exists t' \in \mathbf{R}, x_{|[t',\infty)} = \mu' \text{ and } \exists t'' > t', x_{|(-\infty,t'')} = \mu''\}$$

is equal to (see Theorem 202 f))

$$\{(\mu,\mu)|\mu \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u), \forall t \in \mathbf{R}, x(t) = \mu\}.$$

In other words, from an accessible final value, the only accessible value is the final value itself, with the special case when from an accessible point of equilibrium the only accessible value is the point of equilibrium itself.

The previous theorem allows us reconsidering the consecutive accesses to μ' *and then to* μ'' *in the case of the non-anticipatory systems, i.e. it gives the idea of replacing the expression from a) with the one from b), when* μ' *is the final value and* μ'' *is successively arbitrary value, initial value, final value, recurrent value and* $\delta-$*persistent value. We have*

Definition 85. *Consider the non-anticipatory system* f *in the sense of Definition 64. We call*

$$\Theta'_f \circledast \Omega = \{(\mu',\mu'')|\mu',\mu'' \in \mathbf{B}^n, \exists u \in U, \exists v \in U, \forall y \in f(v), \exists t' \in \mathbf{R},$$
$$u_{|(-\infty,t')} = v_{|(-\infty,t')}, y_{|[t',\infty)} = \mu' \text{ and } \forall x \in f(u), \exists t'' > t', x(t'') = \mu''\}$$

the ***set of the couples*** (μ',μ'') ***of consecutive accessible values of (the states of)*** f, ***with*** μ' ***final****. For* $(\mu',\mu'') \in \Theta'_f \circledast \Omega$ *we say that there is* $u \in U$ *such that the states* $x \in f(u)$ ***take (access, reach) first the final value*** μ', ***then the value*** μ''.

Let us fix μ',μ'',u*. The property*

$$\exists v \in U, \forall y \in f(v), \exists t' \in \mathbf{R},$$
$$u_{|(-\infty,t')} = v_{|(-\infty,t')}, y_{|[t',\infty)} = \mu' \text{ and } \forall x \in f(u), \exists t'' > t', x(t'') = \mu''$$

is called the ***consecutive accesses of the states of*** f, ***under the input*** u, ***first to the final value*** μ', ***then to*** μ''*. Sometimes we say that* $f(u)$ ***transfers the final value*** μ' ***to*** μ''.

The terminology and the notations are similar for the sets

$$\Theta'_f \circledast \Theta'_0 = \{(\mu',\mu'')|\mu',\mu'' \in \mathbf{B}^n, \exists u \in U, \exists v \in U, \forall y \in f(v), \exists t' \in \mathbf{R},$$
$$u_{|(-\infty,t')} = v_{|(-\infty,t')}, y_{|[t',\infty)} = \mu' \text{ and } \forall x \in f(u), \exists t'' > t', x_{|(-\infty,t'')} = \mu''\},$$
$$\Theta'_f \circledast \Theta'_f = \{(\mu',\mu'')|\mu',\mu'' \in \mathbf{B}^n, \exists u \in U, \exists v \in U, \forall y \in f(v), \exists t' \in \mathbf{R},$$
$$u_{|(-\infty,t')} = v_{|(-\infty,t')}, y_{|[t',\infty)} = \mu' \text{ and } \forall x \in f(u), \exists t'' \in \mathbf{R}, x_{|[t'',\infty)} = \mu''\},$$
$$\Theta'_f \circledast R = \{(\mu',\mu'')|\mu',\mu'' \in \mathbf{B}^n, \exists u \in U, \exists v \in U, \forall y \in f(v), \exists t' \in \mathbf{R},$$
$$u_{|(-\infty,t')} = v_{|(-\infty,t')}, y_{|[t',\infty)} = \mu' \text{ and } \forall x \in f(u), \forall t_1 \in \mathbf{R}, \exists t'' > t_1, x(t'') = \mu''\},$$
$$\Theta'_f \circledast \Omega_\delta = \{(\mu',\mu'')|\mu',\mu'' \in \mathbf{B}^n, \exists u \in U, \exists v \in U, \forall y \in f(v), \exists t' \in \mathbf{R},$$
$$u_{|(-\infty,t')} = v_{|(-\infty,t')}, y_{|[t',\infty)} = \mu' \text{ and } \forall x \in f(u), \exists t'' > t'-\delta, x_{|[t'',t''+\delta]} = \mu''\}$$

where $\delta > 0$.

REMARK 79. *There are versions of Definition 85 when one or both accesses are synchronous. We shall make use of this remark further.*

In the previous constructions μ' is not a final value under u, but under v, where u and v may differ. In other words, if $u \neq v$, then there is some time instant t' with the property that $u_{|(-\infty,t')} = v_{|(-\infty,t')}$, $\{x_{|(-\infty,t']}|x \in f(u)\} = \{y_{|(-\infty,t']}|y \in f(v)\}$, $\forall y \in f(v), y_{|[t',\infty)} = \mu'$, $u(t') \neq v(t')$ and there is the possibility that $\exists t'' > t', \exists x \in f(u), x(t'') \neq \mu'$. In particular, there is the possibility that $\forall x \in f(u), \exists t'' > t', x(t'') = \mu''$ with $\mu'' \neq \mu'$. In such circumstances, $\Theta'_f \circledast \Omega, \Theta'_f \circledast \Theta'_f, \Theta'_f \circledast R, \Theta'_f \circledast \Omega_\delta$ avoid the previously mentioned triviality and $\Theta'_f \circledast \Theta'_0$ remains trivial[9]*. Reaching this conclusion represented the purpose of the present section.*

7. Accessibility in the sense of the consecutive accesses

REMARK 80. *Let be the system f and at this moment we try to gather the previous intuition on what controllability and accessibility are. The first idea to start with is the Girard's point of view on controllability 'you can get anywhere you want (in a finite amount of time)', that in two non-synchronous variants is*

$$\exists \mu' \in \mathbf{B}^n, \forall \mu'' \in \mathbf{B}^n, \exists u \in U, \tag{7.1}$$
$$\forall x \in f(u), \exists t' \in \mathbf{R}, x(t') = \mu' \text{ and } \exists t'' > t', x(t'') = \mu''$$

and

$$\forall \mu' \in \mathbf{B}^n, \forall \mu'' \in \mathbf{B}^n, \exists u \in U, \tag{7.2}$$
$$\forall x \in f(u), \exists t' \in \mathbf{R}, x(t') = \mu' \text{ and } \exists t'' > t', x(t'') = \mu''$$

(7.2) generates the definitions a), c), e), g) of controllability of Megan from Remark 76:

$$\exists t' \in \mathbf{R}, \forall \mu' \in \mathbf{B}^n, \forall \mu'' \in \mathbf{B}^n, \exists u \in U, \exists t'' > t',$$
$$\forall x \in f(u), x(t') = \mu' \text{ and } x(t'') = \mu'';$$
$$\exists t' \in \mathbf{R}, \exists t'' > t', \forall \mu' \in \mathbf{B}^n, \forall \mu'' \in \mathbf{B}^n, \exists u \in U,$$
$$\forall x \in f(u), x(t') = \mu' \text{ and } x(t'') = \mu'';$$
$$\forall t' \in \mathbf{R}, \forall \mu' \in \mathbf{B}^n, \forall \mu'' \in \mathbf{B}^n, \exists u \in U, \exists t'' > t',$$
$$\forall x \in f(u), x(t') = \mu' \text{ and } x(t'') = \mu'';$$
$$\exists \delta > 0, \forall t' \in \mathbf{R}, \forall \mu' \in \mathbf{B}^n, \forall \mu'' \in \mathbf{B}^n, \exists u \in U,$$
$$\forall x \in f(u), x(t') = \mu' \text{ and } x(t' + \delta) = \mu''.$$

The Dragomir's point of view on controllability 'the existence of a command that can bring the system in a bounded arbitrary time interval from an arbitrary state in a steady state' means one of

$$\forall \mu' \in \mathbf{B}^n, \exists \mu'' \in \Theta'_f, \exists u \in U, \tag{7.3}$$
$$\forall x \in f(u), \exists t' \in \mathbf{R}, x(t') = \mu' \text{ and } \exists t'' > t', x_{|[t'',\infty)} = \mu'',$$

$$\forall \mu' \in \mathbf{B}^n, \forall \mu'' \in \Theta'_f, \exists u \in U, \tag{7.4}$$
$$\forall x \in f(u), \exists t' \in \mathbf{R}, x(t') = \mu' \text{ and } \exists t'' > t', x_{|[t'',\infty)} = \mu''.$$

The set of the 'steady states' Θ'_f is supposed to be non-empty and the two versions of the definition are not synchronous again. The philosophy with (7.3) and (7.4)

[9] $\Theta'_f \circledast \Theta'_0 = \Theta'_f \otimes \Theta'_0 = \{(\mu, \mu)|\mu \in \mathbf{B}^n, \exists u \in U, \forall x \in f(u), \forall t \in \mathbf{R}, x(t) = \mu\}$

generates the definitions b), d), f), h) of controllability of Megan [**20**] *(see Remark 76 again):*

$$\exists t' \in \mathbf{R}, \forall \mu' \in \mathbf{B}^n, \exists \mu'' \in \Theta'_f, \exists u \in U, \exists t'' > t',$$
$$\forall x \in f(u), x(t') = \mu' \text{ and } x_{|[t'',\infty)} = \mu'';$$
$$\exists t' \in \mathbf{R}, \forall \mu' \in \mathbf{B}^n, \forall \mu'' \in \Theta'_f, \exists u \in U, \exists t'' > t',$$
$$\forall x \in f(u), x(t') = \mu' \text{ and } x_{|[t'',\infty)} = \mu'';$$
$$\exists t' \in \mathbf{R}, \exists t'' > t', \forall \mu' \in \mathbf{B}^n, \exists \mu'' \in \Theta'_f, \exists u \in U,$$
$$\forall x \in f(u), x(t') = \mu' \text{ and } x_{|[t'',\infty)} = \mu'';$$
$$\exists t' \in \mathbf{R}, \exists t'' > t', \forall \mu' \in \mathbf{B}^n, \forall \mu'' \in \Theta'_f, \exists u \in U,$$
$$\forall x \in f(u), x(t') = \mu' \text{ and } x_{|[t'',\infty)} = \mu'';$$
$$\forall t' \in \mathbf{R}, \forall \mu' \in \mathbf{B}^n, \exists \mu'' \in \Theta'_f, \exists u \in U, \exists t'' > t',$$
$$\forall x \in f(u), x(t') = \mu' \text{ and } x_{|[t'',\infty)} = \mu'';$$
$$\forall t' \in \mathbf{R}, \forall \mu' \in \mathbf{B}^n, \forall \mu'' \in \Theta'_f, \exists u \in U, \exists t'' > t',$$
$$\forall x \in f(u), x(t') = \mu' \text{ and } x_{|[t'',\infty)} = \mu'';$$
$$\exists \delta > 0, \forall t' \in \mathbf{R}, \forall \mu' \in \mathbf{B}^n, \exists \mu'' \in \Theta'_f, \exists u \in U,$$
$$\forall x \in f(u), x(t') = \mu' \text{ and } x_{|[t'+\delta,\infty)} = \mu'';$$
$$\exists \delta > 0, \forall t' \in \mathbf{R}, \forall \mu' \in \mathbf{B}^n, \forall \mu'' \in \Theta'_f, \exists u \in U,$$
$$\forall x \in f(u), x(t') = \mu' \text{ and } x_{|[t'+\delta,\infty)} = \mu''.$$

The definition that Dragomir gives to accessibility is good for non-anticipatory systems in the sense of Definition 64 only: 'the existence of a command that can bring the system from a steady state in an arbitrary state, thus the access from the steady state to an arbitrary state in a bounded arbitrary time interval'. This means the truth of one of:

$$\exists \mu' \in \Theta'_f, \forall \mu'' \in \mathbf{B}^n, \exists u \in U, \exists v \in U, \forall y \in f(v), \exists t' \in \mathbf{R}, \tag{7.5}$$
$$u_{|(-\infty,t')} = v_{|(-\infty,t')}, y_{|[t',\infty)} = \mu' \text{ and } \forall x \in f(u), \exists t'' > t', x(t'') = \mu'',$$

$$\forall \mu' \in \Theta'_f, \forall \mu'' \in \mathbf{B}^n, \exists u \in U, \exists v \in U, \forall y \in f(v), \exists t' \in \mathbf{R}, \tag{7.6}$$
$$u_{|(-\infty,t')} = v_{|(-\infty,t')}, y_{|[t',\infty)} = \mu' \text{ and } \forall x \in f(u), \exists t'' > t', x(t'') = \mu''.$$

Here are Professor Megan's definitions of accessibility a'), b'), c'), d') from Remark 76, that have their origin in the idea expressed by (7.5):

$$\exists t' \in \mathbf{R}, \forall \mu'' \in \mathbf{B}^n, \exists \mu' \in \Theta'_f, \exists u \in U, \exists v \in U, \exists t'' > t',$$
$$\forall y \in f(v), u_{|(-\infty,t')} = v_{|(-\infty,t')}, y_{|[t',\infty)} = \mu' \text{ and } \forall x \in f(u), x(t'') = \mu'';$$
$$\exists t' \in \mathbf{R}, \exists t'' > t, \forall \mu'' \in \mathbf{B}^n, \exists \mu' \in \Theta'_f, \exists u \in U, \exists v \in U,$$
$$\forall y \in f(v), u_{|(-\infty,t')} = v_{|(-\infty,t')}, y_{|[t',\infty)} = \mu' \text{ and } \forall x \in f(u), x(t'') = \mu'';$$
$$\forall t' \in \mathbf{R}, \forall \mu'' \in \mathbf{B}^n, \exists \mu' \in \Theta'_f, \exists u \in U, \exists v \in U, \exists t'' > t',$$
$$\forall y \in f(v), u_{|(-\infty,t')} = v_{|(-\infty,t')}, y_{|[t',\infty)} = \mu' \text{ and } \forall x \in f(u), x(t'') = \mu'';$$
$$\exists \delta > 0, \forall t' \in \mathbf{R}, \forall \mu'' \in \mathbf{B}^n, \exists \mu' \in \Theta'_f, \exists u \in U, \exists v \in U,$$
$$\forall y \in f(v), u_{|(-\infty,t')} = v_{|(-\infty,t')}, y_{|[t',\infty)} = \mu' \text{ and } \forall x \in f(u), x(t' + \delta) = \mu''$$

and, similarly, for the definitions a'), b'), c'), d'), having their origin in (7.6).

Now the way of constructing accessibility properties is obvious.

CHAPTER 8

Stability

The absolutely stable systems f are those systems for which $\forall u \in U, \forall x \in f(u)$, there is the limit $\lim_{t\to\infty} x(t)$ while the relatively stable systems are defined by $\forall u \in U$, if there is $\lim_{t\to\infty} u(t)$, then $\forall x \in f(u)$, there is $\lim_{t\to\infty} x(t)$. These properties are associated with another type of expectations from the behavior of the asynchronous circuits. The stability relative to a Boolean function F generalizes the relative stability by replacing $\exists \lim_{t\to\infty} u(t)$ by $\exists \lim_{t\to\infty} F(u(t))$ and it defines the combinational systems. The importance of these properties consists in the possibility of characterizing f in discrete time. Indeed, the time instants $t \geq t_f$, when all x, respectively u and all x, respectively $F(u)$ and all x have become stable may be chosen as discrete time instants.

In this chapter these three types of stability are defined and commented. Examples are also given.

1. Absolute stability

Remark 81. *First we summarize some previously presented facts, related to the absolute stability.*

*Let be the system $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$. The system f is called **absolutely stable** and we say that it **has final (values of the) states** if*

$$\forall u \in U, \forall x \in f(u), \exists \mu \in \mathbf{B}^n, \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = \mu \tag{1.1}$$

is true. If

$$\forall u \in U, \exists \mu \in \mathbf{B}^n, \forall x \in f(u), \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = \mu, \tag{1.2}$$

*then it is called **absolutely race-free stable** and if*

$$\exists \mu \in \mathbf{B}^n, \forall u \in U, \forall x \in f(u), \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = \mu \tag{1.3}$$

*then f is called **absolutely constantly stable**. In these definitions t_f, μ are called the **final time (instant)** and respectively the **final (value of the) state**. In (1.2) the final state μ is called **race-free** and in (1.3) it is called **constant**. These notions occur in Definitions 25,...,27 stated in the more general case of the pseudo-systems while the unbounded, bounded and fixed final time instants occur in Definitions 31,...,33. The possibilities of combining different types of final states and final time instants are listed in Theorem 26.*

The notion of final value of a binary valued function is introduced together with its dual, the initial value by Definition 14. The notations for the final value of x are $\lim_{t\to\infty} x(t)$ and $x(\infty - 0)$. The final state function ϕ_f that associates with each input u the set $\{x(\infty - 0) | x \in f(u)\}$ occurs in Definition 35, together with the set of the final states Θ_f.

The concept of the system introduces an asymmetry between the existence of the initial states and the existence of the final states. This is mainly due to the fact that we use to reason in a non-anticipatory manner, from the past to the future and, in general, we must assume the existence of the initial states. The way that the pseudo-systems induce, in the case of the systems, the properties of existence of the initial/final states respectively the initial/final time instants was shown in Theorems 31 and 32.

The absolutely stable systems are those where the stabilization of the state is produced independently of the fact that the input stabilizes or not, the following implications

$$(1.3) \Longrightarrow (1.2) \Longrightarrow (1.1)$$

being true. The absolutely stable systems f have the final state function $\phi_f : U \to P^*(\mathbf{B}^n)$ defined. In the case of absolute race-free stability, $\phi_f : U \to \mathbf{B}^n$ is a uni-valued function and if f is absolutely constantly stable, then ϕ_f is the constant uni-valued function. We can identify the binary vector $\mu \in \mathbf{B}^n$ by the constant vector function $x(t) = \mu$, allowing us to define for the absolutely stable system f the system $\lim f : U \to P^*(S^{(n)})$ by $\forall u \in U, \lim f(u) = \{x(\infty - 0)|x \in f(u)\}$. If f is absolutely race-free stable, then $\lim f$ is deterministic while for f absolutely constantly stable, $\lim f$ is deterministic and autonomous.

We introduce the dual of f_μ, the restriction of the absolutely stable system f to the initial (value of the) state μ (Example 20[1]*). For an arbitrary final state $\mu \in \Theta_f$ of f, the system $f^\mu : U^\mu \to P^*(S^{(n)})$ is defined by*

$$U^\mu = \{u|u \in U, \mu \in \phi_f(u)\},$$

$$\forall u \in U^\mu, f^\mu(u) = \{x|x \in f(u), x(\infty - 0) = \mu\}.$$

*The system f^μ is a subsystem of f called the **restriction of f at the final (value of the) state μ**.*

Here are some properties.

If g has final states (race-free final states, a constant final state), then any $f \subset g$ has final states (race-free final states, a constant final state) (Theorem 36). If g has final states and $f \subset g$, we have $\forall u \in U, \phi_f(u) \subset \gamma_f(u)$ (Theorem 40). If g has a bounded final time (a fixed final time), then any $f \subset g$ has a bounded final time (a fixed final time) (Theorem 38).

The system f has final states (race-free final states, a constant final state) iff f^* has final states (race-free final states, a constant final state) (Theorem 44) and f has a bounded final time (a fixed final time) if f^* has a bounded final time (a fixed final time) (Theorem 46). Theorem 48 shows that the absolute stability of f implies $\forall u \in U^*, \phi_f^*(u) = \{\overline{\mu}|\mu \in \phi_f(\overline{u})\}$.

The systems f and f' have final states (race-free final states, constant final states) if $f \times f'$ has final states (race-free final states, a constant final state) (Theorem 58). The systems f and f' have a bounded final time (fixed final time) iff $f \times f'$ has a bounded final time (a fixed final time) (Theorem 60). If f, f' have final states, we have $\forall u \times u' \in U \times U', (\phi \times \phi')_f(u \times u') = \phi_f(u) \times \phi'_f(u')$ (Theorem 62).

Here we give without proofs some properties similar to the previous ones relative to the parallel connection (f, f_1'), that were not mentioned before. Suppose that

[1]A more appropriate construction of this dual should start from the pseudo-system f_* that in general has no initial values of the states, but has final values of the states; then take some final value $\mu \in \Theta_f$ etc. The expression 'dual of f_μ' has a certain imprecission.

$U \cap U_1' \neq \emptyset$. If f, f_1' have final states (race-free final states, constant final states) then (f, f_1') has final states (race-free final states, a constant final state). If f and f_1' have a bounded final time (a fixed final time), then (f, f_1') has a bounded final time (a fixed final time). On the other hand, we have $\forall u \in U \cap U_1', (\phi, \phi_1')_f(u) = \phi_f(u) \times \phi_{1f}'(u)$.

We continue listing the previously mentioned properties of stability. If h has final states (a constant final state) and the system $h \circ f$ is defined, then $h \circ f$ has final states (a constant final state) (Theorem 69). If h has a fixed final time and $h \circ f$ exists, then $h \circ f$ has a fixed final time (Theorem 71). Let be the systems f and h with $\bigcup\limits_{u \in U} f(u) \subset X$ fulfilled. Suppose that h has final states and we use the notations η_f, δ_f for the final state functions of $h, h \circ f$. The following formula from Theorem 73

$$\forall u \in U, \delta_f(u) = \bigcup_{x \in f(u)} \eta_f(x)$$

is true. If f has final states (race-free final states, a constant final state) and $f \cap g$ exists, then $f \cap g$ has final states (race-free final states, a constant final state) (Theorem 82). If f has a bounded final time (a fixed final time) and $f \cap g$ exists, then $f \cap g$ has a bounded final time (a fixed final time) (Theorem 84). If f, g have final states and $W = \{u | u \in U \cap V, f(u) \cap g(u) \neq \emptyset\}$ is non-empty, then by Theorem 86, we have

$$\forall u \in W, (\phi \cap \gamma)_f(u) = \phi_f(u) \cap \gamma_f(u).$$

If f, g have final states, then $f \cup g$ has final states also; if f, g have race-free final states and $\forall u \in U \cap V, f(u) \cap g(u) \neq \emptyset$, then $f \cup g$ has race-free final states. If f, g have constant final states and $\bigcup\limits_{u \in U} f(u) \cap \bigcup\limits_{u \in V} g(u) \neq \emptyset$, then, by Theorem 95, $f \cup g$ has a constant final state. If f, g have a bounded final time (a fixed final time), then $f \cup g$ has a bounded final time (a fixed final time) (Theorem 97). Suppose that f, g have final states. By Theorem 99, we have that the final state function $(\phi \cup \gamma)_f : U \cup V \to P^*(\mathbf{B}^n)$ satisfies

$$\forall u \in U \cup V, (\phi \cup \gamma)_f(u) = \begin{cases} \phi_f(u),\ u \in U \setminus V \\ \gamma_f(u),\ u \in V \setminus U \\ \phi_f(u) \cup \gamma_f(u),\ u \in U \cap V \end{cases}.$$

The constant final state function given by the existence of $k \in \{1, ..., 2^n\}$ and of $\mu^1, ..., \mu^k \in \mathbf{B}^n$ such that $\forall u \in U, \phi_f(u) = \{\mu^1, ..., \mu^k\}$ is to be treated by duality with the constant initial state function from Section 1 of Ch. 5. The meaning of $\exists \mu \in \mathbf{B}^n, \forall u \in U, \phi_f(u) = \mu$ is that of absolute constant stability.

Let be the autonomous system $f = X$. The dual of Theorem 113 states that the property of existence of the final state is given by

$$\forall x \in X, \exists \mu \in \mathbf{B}^n, \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = \mu,$$

while the existence of the race-free final states coincides with the existence of the constant final state and is given by

$$\exists \mu \in \mathbf{B}^n, \forall x \in X, \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = \mu.$$

The dual of Theorem 114 states that for the autonomous system $f = X$ the existence of the bounded final time and of the fixed final time coincide both with

$$\exists t_f \in \mathbf{R}, \forall x \in X, \exists \mu \in \mathbf{B}^n, \forall t \geq t_f, x(t) = \mu.$$

The dual of Theorem 115 states that for the absolutely stable autonomous system $f = X$ the following possibilities exist: f has final states and an unbounded final time

$$\forall x \in X, \exists \mu \in \mathbf{B}^n, \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = \mu;$$

f has final states and a bounded final time

$$\exists t_f \in \mathbf{R}, \forall x \in X, \exists \mu \in \mathbf{B}^n, \forall t \geq t_f, x(t) = \mu;$$

f has race-free final states and an unbounded final time

$$\exists \mu \in \mathbf{B}^n, \forall x \in X, \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = \mu;$$

and f has race-free final states and a bounded final time

$$\exists \mu \in \mathbf{B}^n, \exists t_f \in \mathbf{R}, \forall x \in X, \forall t \geq t_f, x(t) = \mu$$

respectively. The dual of Theorem 116 states that the final state function of the absolutely stable autonomous system f is constant and equal to the set of the final states.

If a deterministic system is absolutely stable, then it is absolutely race-free stable. If f is an absolutely stable finite system, then it has a bounded final time, the dual of Theorem 124. In particular, the absolutely stable ideal combinational systems are absolutely race-free stable and have a bounded final time.

If f is absolutely stable and self-dual, then the final state function satisfies $\phi_f = \phi_f^*$, the dual of Theorem 145.

Let f be absolutely stable and symmetrical. Then the final state function fulfills

$$\forall u \in U, \phi_f(u) = \phi_f(u_\sigma)$$

for any bijection $\sigma \in S(\{1, ..., m\})$, as follows from the dual of Theorem 153.

If the absolutely stable system f is time invariant, then, from the dual of Theorem 161, we get that its final state function fulfills

$$\forall d \in \mathbf{R}, \forall u \in U, \phi_f(u \circ \tau^d) = \phi_f(u).$$

2. Relative stability

DEFINITION 86. *a) A system f for which the property*

$$\forall u \in U \cap S_c^{(m)}, \forall x \in f(u), \exists \mu \in \mathbf{B}^n, \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = \mu$$

*is satisfied is called **relatively stable**.*

b) If the property

$$\forall u \in U \cap S_c^{(m)}, \exists \mu \in \mathbf{B}^n, \forall x \in f(u), \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = \mu$$

*is true, then f is called **relatively race-free stable**.*

*c) The system f is **relatively constantly stable** if*

$$\exists \mu \in \mathbf{B}^n, \forall u \in U \cap S_c^{(m)}, \forall x \in f(u), \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = \mu.$$

*If $U \cap S_c^{(m)} = \emptyset$, then we say that the previous stability properties are **trivially fulfilled** and if $U \cap S_c^{(m)} \neq \emptyset$, that they are **non-trivially fulfilled**.*

REMARK 82. *The relatively stable systems are those systems for which the stabilization of the input produces the stabilization of the state:* $\forall u \in U, \forall x \in f(u)$,

$$\exists \lim_{t\to\infty} u(t) \Longrightarrow \exists \lim_{t\to\infty} x(t),$$

while if $\lim_{t\to\infty} u(t)$ *does not exist,* $\lim_{t\to\infty} x(t)$ *may exist or not.*

Relative stability is analyzed similarly with the absolute stability.

3. Stability relative to a function. Combinational systems

NOTATION 23. *Let be the Boolean function* $F : \mathbf{B}^m \to \mathbf{B}^n$. *Denote*

$$S^{(m)}_{F,c} = \{u | u \in S^{(m)}, F(u) \in S^{(n)}_c\}.$$

DEFINITION 87. *a) A system* f *satisfying*

$$\forall u \in U \cap S^{(m)}_{F,c}, \forall x \in f(u), \exists \mu \in \mathbf{B}^n, \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = \mu$$

is called $F-$***relatively stable****, or* ***stable relative to the function*** F.

b) If the following property

$$\forall u \in U \cap S^{(m)}_{F,c}, \forall x \in f(u), \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = \lim_{\xi\to\infty} F(u(\xi))$$

holds true, then f *is called* $F-$***relatively race-free stable****, or* ***race-free stable relative to the function*** F. *Another terminology for* f *is that of* ***combinational system*** *and in this terminology* F *is called the* ***generator function of*** f.

c) The system f *is* $F-$***relatively constantly stable*** *if it is* $F-$*relatively race-free stable and the function* $F : U \to S^{(n)}$ *is constant,* $F = \mu$:

$$\exists \mu \in \mathbf{B}^n, \forall u \in U, \forall x \in f(u), \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = \mu.$$

If $U \cap S^{(m)}_{F,c} = \emptyset$, *the previous stability properties are trivial while if* $U \cap S^{(m)}_{F,c} \neq \emptyset$, *they are non-trivial.*

REMARK 83. *The notions of* $F-$*relative constant stability and respectively of absolute constant stability coincide. On the other hand, if* $F : \mathbf{B}^m \to \mathbf{B}^n$ *is the constant function, then* $S^{(m)}_{F,c} = S^{(m)}$ *and in Definition 87:*

a) coincides with the absolute stability;

b) coincides with c) (and with absolute constant stability).

The F*-relative race-free stability is a property of stability of 'race-free' type. Indeed,*

$$\forall u \in U \cap S^{(m)}_{F,c}, \exists \mu \in \mathbf{B}^n, \forall x \in f(u), \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = \mu,$$

where $\mu = \lim_{\xi\to\infty} F(u(\xi))$. *For example, the ideal combinational systems* F_d *are race-free stable relative to* F.

The analysis of this type of stability is similar to the one that was made at the absolute stability.

In Figure 1 we give the existing connection between the nine types of previously defined stability.

At this moment the three types of final time t_f*: unbounded, bounded and fixed*

$$\forall u \in U, \forall x \in f(u) \cap S^{(n)}_c, \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = x(t_f);$$

$$\forall u \in U, \exists t_f \in \mathbf{R}, \forall x \in f(u) \cap S^{(n)}_c, \forall t \geq t_f, x(t) = x(t_f);$$

$$\exists t_f \in \mathbf{R}, \forall u \in U, \forall x \in f(u) \cap S^{(n)}_c, \forall t \geq t_f, x(t) = x(t_f)$$

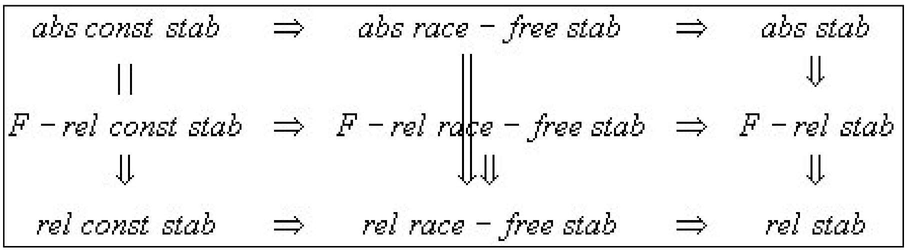

FIGURE 1. The connection between several types of stability

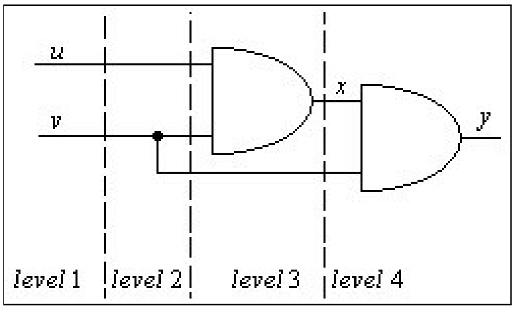

FIGURE 2. Combinational system

may be combined with the nine (non-distinct) types of stability. It follows 27 (non-distinct) possibilities, like in Theorem 26 that characterizes the pseudo-systems.

EXAMPLE 89. *Let be the circuit from Figure 2, where* $u, v, x, y \in S$*. The systems* u *and* v *are combinational, in the sense that they can be identified with the ideal combinational system* I*. Then the Cartesian product* $u \times v$ *is an ideal combinational system, representing the level 1 of analysis of the circuit.*

At level 2 we have, on one hand, the parallel connection (v, v) *that is an ideal combinational system and, on the other hand, the Cartesian product* $u \times (v, v)$ *that is an ideal combinational system too. Its generator function is* $(\lambda_1, \lambda_2) \mapsto (\lambda_1, \lambda_2, \lambda_2)$.

At the third level of analysis of the circuit we have the logical product Boolean function $(\lambda_1, \lambda_2) \mapsto \lambda_1 \cdot \lambda_2$ *and the corresponding combinational system, that is in the Cartesian product with* v*. The generator function is* $(\lambda_1, \lambda_2, \lambda_3) \mapsto (\lambda_1 \cdot \lambda_2, \lambda_3)$.

At level 4 we have a combinational system with the logical product generator function.

We conclude that the system $f : S^{(2)} \to P^*(S)$ *that models the circuit from Figure 2 is combinational, as following from serial connections, parallel connections and Cartesian products of combinational systems. Its generator function is* $(\lambda_1, \lambda_2) \mapsto \lambda_1 \cdot \lambda_2$.

4. The absolute stability of the non-anticipatory systems

THEOREM 227. *Suppose that the absolutely stable system* f *satisfies the non-anticipation property from Definition 64:* $\forall t \in \mathbf{R}$, $\forall u \in U$, $\forall v \in U$,

$$u_{|(-\infty,t)} = v_{|(-\infty,t)} \Longrightarrow \{x_{|(-\infty,t]} | x \in f(u)\} = \{y_{|(-\infty,t]} | y \in f(v)\}$$

and that it has also the fixed final time t_f. Then $\forall t \in \mathbf{R}$, $\forall u \in U$, $\forall v \in U$,

$$(u_{|(-\infty,t)} = v_{|(-\infty,t)} \textit{ and } t \geq t_f) \Longrightarrow f(u) = f(v),$$

i.e. $f(u)$ *depends on the restriction* $u_{|(-\infty,t_f)}$ *only.*

Proof. Let $t_1 \in \mathbf{R}$, $u \in U$, $v \in U$ be arbitrary such that

$$u_{|(-\infty,t_1)} = v_{|(-\infty,t_1)} \textit{ and } t_1 \geq t_f$$

be true. The non-anticipation of f gives

$$\{x_{|(-\infty,t_1]} | x \in f(u)\} = \{y_{|(-\infty,t_1]} | y \in f(v)\}.$$

From the absolute stability with a fixed final time of f (see Theorem 26, c)) we get

$$\forall x \in f(u), \exists \mu \in \mathbf{B}^n, \forall t \geq t_f, x(t) = \mu,$$

$$\forall y \in f(v), \exists \mu' \in \mathbf{B}^n, \forall t \geq t_f, y(t) = \mu'$$

and, as $t_1 \geq t_f$, we conclude that $\mu = \mu'$ and $f(u) = f(v)$. □

Theorem 228. *Let f be non-anticipatory in the sense of Definition 64.*

a) If f is absolutely stable and has the fixed final time t_f, then its final state function ϕ_f satisfies $\forall t \in \mathbf{R}$, $\forall u \in U$, $\forall v \in U$,

$$(u_{|(-\infty,t)} = v_{|(-\infty,t)} \textit{ and } t \geq t_f) \Longrightarrow \phi_f(u) = \phi_f(v),$$

i.e. $\phi_f(u)$ *depends on the restriction* $u_{|(-\infty,t_f)}$ *only.*

b) In the case that f is absolutely race-free stable and has the fixed final time t_f, we have $\forall t \in \mathbf{R}$, $\forall u \in U$, $\forall v \in U$,

$$(u_{|(-\infty,t)} = v_{|(-\infty,t)} \textit{ and } t \geq t_f) \Longrightarrow \forall x \in f(u), \forall y \in f(v), \lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t),$$

i.e. the limit $\lim_{t\to\infty} x(t)$*, that is the same for all* $x \in f(u)$*, depends on* $u_{|(-\infty,t_f)}$ *only.*

c) If f is absolutely constantly stable, then

$$\forall u \in U, \forall v \in U, \forall x \in f(u), \forall y \in f(v), \lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t).$$

In other words, $\lim_{t\to\infty} x(t)$ *is the same for all* $u \in U$ *and all* $x \in f(u)$.

Proof. a) From the previous theorem we have that $f(u) = f(v)$. Thus

$$\phi_f(u) = \{x(\infty - 0) | x \in f(u)\} = \{y(\infty - 0) | y \in f(v)\} = \phi_f(v).$$

b) This is the special case of a) when ϕ_f is uni-valued.

c) Special case of b) when ϕ_f is the constant function. □

5. Examples

Example 90. *We consider the following systems:*

$f_1 : S \to S, \forall u \in S, f_1(u) = u;$

$f_2 : S \to S, \forall u \in S, f_2(u) = u(0);$

$f_3 : S \to S, \forall u \in S, f_3(u) = 1;$

$f_4 : S \to P^*(S), \forall u \in S, f_4(u) = \{0, 1\};$

$f_5 : S \to P^*(S), \forall u \in S, f_5(u) = \left\{ \begin{array}{c} \{u, \overline{u}\}, u \in S_c \\ u, u \notin S_c \end{array} \right. ;$

$f_6 : S \to P^*(S), \forall u \in S, f_6(u) = \left\{ \begin{array}{c} u, u \in S_c \\ \{0, 1\}, u \notin S_c \end{array} \right. ;$

$$f_7 : S \to S, \forall u \in S, f_7(u) = \begin{cases} 1, u \in S_c \\ 0, u \notin S_c \end{cases};$$

$$f_8 : S \to S, \forall u \in S, f_8(u) = \begin{cases} 1, u \in S_c \\ \chi_{[0,1)\cup[2,3)\cup[4,5)\cup\ldots}, u \notin S_c \end{cases};$$

$$f_9 : S \to P^*(S), \forall u \in S, f_9(u) = \begin{cases} 1, u \in S_c \\ \{0,1\}, u \notin S_c \end{cases};$$

$$f_{10} : S \to P^*(S), \forall u \in S, f_{10}(u) = \begin{cases} \{u \circ \tau^d | d \geq 0\}, u \in S_c \\ 0, u \notin S_c \end{cases}.$$

In Table 1 we have described the behavior of these systems relative to the introduced notions of stability and relative to the final time by writing an '×' when the appropriate property is fulfilled. The function F *from the table is the identity:* $F : \mathbf{B} \to \mathbf{B}$, $\forall \lambda \in \mathbf{B}$, $F(\lambda) = \lambda$, *for which* $S_{F,c} = S_c$.

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}
absolute stability		×	×	×		×	×		×	×
absolute race − free stability		×	×				×			×
absolute constant stability			×							
relative stability	×	×	×	×	×	×	×	×	×	×
relative race − free stability	×	×	×			×	×	×	×	×
relative constant stability			×				×	×	×	
F − relative stability	×	×	×	×	×	×	×	×	×	×
F − relative race − free stability	×					×				×
unbounded final time	×	×	×	×	×	×	×	×	×	×
bounded final time	×	×	×	×	×	×	×	×	×	
fixed final time		×	×	×			×	×	×	

Table 1.

CHAPTER 9

The fundamental mode

Let be the asynchronous system $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$. The fundamental (operating) mode of f is an input $u \in U$ with the property that there is a sequence $(\mu^k)_{k\in\mathbf{N}} \in \{0,1\}^n$ such that all $x \in f(u)$ access synchronously the values $\mu^0, \mu^1, \mu^2, ...$ in this order, where μ^0 is the initial value and $\mu^1, \mu^2, ...$ are final values. If an asynchronous non-deterministic system f possesses fundamental modes, then it is interpreted to be synchronous (i.e. discrete time) and deterministic.

After introducing and studying the fundamental transfer, the fundamental mode is defined by means of this transfer. Several important properties of the fundamental modes are analyzed, including accessibility. The chapter ends with the investigation of the fundamental mode relative to a function.

1. Introduction

REMARK 84. *The concept of fundamental mode is mentioned in many papers under a non-formalized manner. First we quote* [**14**]*, where its characterization is: 'inputs are constrained to change only when all the delay elements are stable (i.e. they have the input value equal to the output value)'. 'Note that the fundamental mode excludes' the existence of 'a cycle of oscillations', that is instability. Elsewhere the author of* [**14**] *refers to the fundamental mode where 'the designer has to make sure that the circuit inputs can change only when the circuit itself is stable and ready to accept them'. Thus the fundamental mode is an input satisfying a succession of requirements of stability.*

A more restrictive opinion on our topic is expressed in [**28**] *as: 'the circuit is assumed to be in a state' (i.e. in a situation) 'where all input signals, internal signals and output signals are stable*[1]*. In such a state' (i.e. in such a situation) 'the environment is allowed to change one-input signals'. (The term 'environment' is used by some authors to express the idea of 'everything but the circuit', in particular the human being that controls, perhaps, the input and observes the output. We stress on the fact that the difference between this concept of fundamental mode and the previous one has just occurred, the allowed change of 'one-input signals' only, i.e. of one coordinate function $u_j, j \in \{1, ..., m\}$.) We quote further from* [**28**]*: 'After that, the environment is not allowed to change the input signals again until the entire circuit has stabilized'. The authors of* [**28**] *show that 'the environment' should know when the stabilization of the circuit-system took place in order to keep the input constant long enough and the name of David Huffman is indicated that has pioneered the fundamental mode in his works from the 50's.*

[1] In our context, the 'internal signals' are the states and they coincide with the 'output signals'. The statement means the existence of $u \in U, \lambda \in \mathbf{B}^m, \mu \in \mathbf{B}^n$ and t_1 so that $u(t_1) = \lambda, x(t_1) = \mu$ and $\forall t_2 > t_1$, $u_{|[t_1,t_2)} = \lambda$ implies $\forall x \in f(u), x_{|[t_1,t_2)} = \mu$.

Now the two editors of [**28**] *show how the fundamental mode of L. Lavagno is reached under the new name of 'burst mode'. 'Later work has generalized the fundamental mode approach by allowing a restricted form of multiple-input and multiple-ouput changes' (our question: did it happen that so far all $x \in f(u)$ have switched one coordinate $x_i, i \in \{1, ..., n\}$ at a time?) 'This approach is called burst mode. When in a stable state, a burst mode circuit will wait for a set of input signals to change (in arbitrary order). After such an input burst has completed the machine computes a burst of output signals and new values of the internal variables' (in our theory these two coincide). 'The environment is not allowed to produce a new input burst until the circuit has completely reacted to the previous burst - fundamental mode is assumed, but only between bursts of input changes'.*

The notion of 'burst mode' is itself a subject of controversies in literature. Therefore it is not an obvious substitute for the Lavagno's 'fundamental mode', with which it is identified by Sparso and Furber. Because at this moment the debate gets funny flavors, we reproduce the point of view from Webopedia, 'the #1 online encyclopedia dedicated to computer technology'. After quoting several techniques for implementing burst modes, the authors conclude: 'The one characteristic that all burst modes have in common is that they are temporary and unsustainable. They allow faster data transfer rates than normal, but only for a limited period of time and only under special conditions'.

We adopt the terminology of fundamental mode for an input u characterized by the fact that the states $x \in f(u)$ access synchronously an initial value $\mu^0 \in \mathbf{B}^n$ and some family $(\mu^k)_{k \geq 1} \in \mathbf{B}^n$ of final values, where the sequence $(\mu^k)_{k \in \mathbf{N}}$ does not depend on x. This makes the fundamental mode of Lavagno and the burst mode of Sparso and Furber be a special case of our fundamental mode (see Section 8 in this chapter).

It is easily seen that the just given intuition situated behind the fundamental mode supposes the existence of the non-anticipation. By the non-anticipation of $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$ in the present chapter we understand the property from Definition 64: $\forall t \in \mathbf{R}$, $\forall u \in U$, $\forall v \in U$,

$$u_{|(-\infty,t)} = v_{|(-\infty,t)} \Longrightarrow \{x_{|(-\infty,t]} | x \in f(u)\} = \{y_{|(-\infty,t]} | y \in f(v)\}$$

if no other mentions are made. The only case when this property of non-anticipation will be replaced by another one is in Section 10, where we explicitly say that we ask for non-anticipation relative to a Boolean function $F : \mathbf{B}^m \to \mathbf{B}^n$, thus no misunderstanding is possible.

2. Fundamental and hazard-free transfers

THEOREM 229. *Let be the non-anticipatory system f and fix $t_0 \in \mathbf{R}, u \in U$, $\mu \in \mathbf{B}^n$. The synchronous access of f, under u, to μ, at t_0, defined by*

$$\forall x \in f(u), x(t_0) = \mu, \tag{2.1}$$

is equivalent to the following property

$$\exists v \in U, u_{|(-\infty,t_0)} = v_{|(-\infty,t_0)}, \forall y \in f(v), y(t_0) = \mu. \tag{2.2}$$

PROOF. (2.1)$\Longrightarrow$(2.2) is obvious, with $v = u$.

(2.2)$\Longrightarrow$(2.1). From $u_{|(-\infty,t_0)} = v_{|(-\infty,t_0)}$ and the non-anticipation of f we infer that $\{x_{|(-\infty,t_0]} | x \in f(u)\} = \{y_{|(-\infty,t_0]} | y \in f(v)\}$. In particular, we have

$$\{x(t_0) | x \in f(u)\} = \{y(t_0) | y \in f(v)\} = \mu,$$

i.e. (2.1) holds true. □

REMARK 85. *When f is non-anticipatory, we propose ourselves to extend the result of Theorem 229 (by following an idea that has occurred for the first time in Theorem 226 and Definition 85) to the possible equivalencies of the synchronous initial access and the synchronous final access of f to μ, defined by*

$$\forall x \in f(u), x_{|(-\infty,t_0)} = \mu, \tag{2.3}$$

$$\forall x \in f(u), x_{|[t_0,\infty)} = \mu \tag{2.4}$$

and

$$\exists v \in U, u_{|(-\infty,t_0)} = v_{|(-\infty,t_0)}, \forall y \in f(v), y_{|(-\infty,t_0)} = \mu, \tag{2.5}$$

$$\exists v \in U, u_{|(-\infty,t_0)} = v_{|(-\infty,t_0)}, \forall y \in f(v), y_{|[t_0,\infty)} = \mu \tag{2.6}$$

respectively. We note that

- (2.3)$\Longleftrightarrow$(2.5) is true,

- (2.4)$\Longleftrightarrow$(2.6) is not true. While (2.4) shows that all $x \in f(u)$, starting with the time instant t_0, become equal to μ, (2.6) states that all $x \in f(u)$ satisfy $x(t_0) = \mu$ and, for $t > t_0$, they may keep the value μ, if, for example, $u = v$.

Fix $t_0 < t_1, u \in U$ and $\mu, \mu' \in \mathbf{B}^n$ and apply the previous remarks for the two types of synchronous consecutive accesses of further interest. Namely, the one when μ is the initial value and μ' is the final value

$$\forall x \in f(u), x_{|(-\infty,t_0)} = \mu \text{ and } x_{|[t_1,\infty)} = \mu' \tag{2.7}$$

and the one when μ, μ' are both final values

$$\forall x \in f(u), x_{|[t_0,\infty)} = \mu \text{ and } x_{|[t_1,\infty)} = \mu' \tag{2.8}$$

respectively. Let us replace in (2.7) and (2.8) the synchronous accesses of x to the final values by (2.6). After some computations that take into account the non-anticipation of f, we get the properties

$$\exists v \in U, u_{|(-\infty,t_1)} = v_{|(-\infty,t_1)}, \forall y \in f(v), y_{|(-\infty,t_0)} = \mu \text{ and } y_{|[t_1,\infty)} = \mu', \tag{2.9}$$

$$\exists v \in U, u_{|(-\infty,t_0)} = v_{|(-\infty,t_0)}, \forall y \in f(v), y_{|[t_0,\infty)} = \mu \text{ and} \tag{2.10}$$

$$\text{and } \exists v' \in U, u_{|(-\infty,t_1)} = v'_{|(-\infty,t_1)}, \forall y' \in f(v'), y'_{|[t_1,\infty)} = \mu'.$$

Each of the non-equivalent statements (2.7) and (2.9) describe the accesses of f, first to the initial value μ, then to the final value μ', with the difference that in the first case all $x \in f(u)$ stabilize at μ', while in the second case all $x \in f(u)$ may stabilize at μ', for example if $u = v$.

The non-equivalent statements (2.8) and (2.10) give two completely different ways to access synchronously first the final value μ, then the final value μ', in the sense that in (2.8) we have necessarily the triviality $\mu = \mu'$ while in (2.10) $\mu \neq \mu'$ is possible.

In the previous properties there is the possibility $\mu = \mu' =$ point of equilibrium

$$\forall x \in f(u), x = \mu, \tag{2.11}$$

with the trivialities that follow from this situation.

DEFINITION 88. *Presume that (2.9) is true and denote*

$$\mu \overset{u_{|(-\infty,t_1)}}{\rightarrow} \mu' = \{x_{|(-\infty,t_1]} | x \in f(u)\}. \tag{2.12}$$

Then $\mu \overset{u_{|(-\infty,t_1)}}{\rightarrow} \mu'$ *is called the* ***initial fundamental transfer of (the states of)*** f, ***under*** u, ***from the initial value*** μ ***to the final value*** μ'.

Conversely, stating the fact that $\mu \overset{u_{|(-\infty,t_1)}}{\rightarrow} \mu'$, *defined by (2.12), is an initial fundamental transfer means the existence of* $t_0 < t_1$ *such that (2.9) is satisfied.*

DEFINITION 89. *If (2.10) is true, denote*

$$\mu \overset{u_{|[t_0,t_1)}}{\rightarrow} \mu' = \{x_{|[t_0,t_1]} | x \in f(u)\}. \tag{2.13}$$

Then $\mu \overset{u_{|[t_0,t_1)}}{\rightarrow} \mu'$ *is called* ***non-initial fundamental transfer of (the states of)*** f, ***under the input*** u, ***from the final value*** μ ***to the final value*** μ'.

Conversely, stating the fact that $\mu \overset{u_{|[t_0,t_1)}}{\rightarrow} \mu'$, *defined by (2.13), is a non-initial fundamental transfer means the fulfillment of (2.10).*

DEFINITION 90. *If (2.11) is fulfilled, denote*

$$(\mu \overset{u}{=} \mu) = \{\mu\}. \tag{2.14}$$

Let μ *be a point of equilibrium. Then* $\mu \overset{u}{=} \mu$ *is called the* ***trivial fundamental transfer of (the states of)*** f, ***under the input*** u.

Conversely, when we say that $\mu \overset{u}{=} \mu$, *defined by (2.14), is a trivial fundamental transfer, this means the truth of (2.11).*

DEFINITION 91. *If the synchronous transfer* Γ *satisfies* $\forall \gamma \in \Gamma, \gamma$ *is coordinately monotonous, then it is called* ***hazard-free***.

REMARK 86. *In (2.9), in many situations, the synchronism of the access of the states to the initial value* μ *is not necessary. It was asked for the sake of symmetry of the exposure only.*

For the hazard-free transfers, the condition of monotony seems one of economy and normalization, the coordinates of x *do not switch more than necessary, but it has rather a functional meaning.*

The trivial fundamental transfers are hazard-free.

3. Properties of the fundamental transfers. Example

THEOREM 230. *Let be the non-anticipatory system* f *and we fix* $t_0, t_1 \in \mathbf{R}, t_0 < t_1, u \in U, \mu, \mu' \in \mathbf{B}^n$. *If (2.7) is true, then* $\mu \overset{u_{|(-\infty,t_1)}}{\rightarrow} \mu'$ *is an initial fundamental transfer, while if*

$$\exists v \in U, u_{|(-\infty,t_0)} = v_{|(-\infty,t_0)}, \forall y \in f(v), y_{|[t_0,\infty)} = \mu \text{ and } \forall x \in f(u), x_{|[t_1,\infty)} = \mu',$$

then $\mu \overset{u_{|[t_0,t_1)}}{\longrightarrow} \mu'$ *is a non-initial fundamental transfer.*

PROOF. The first hypothesis makes (2.9) true for $v = u$, while the second statement makes (2.10) true for $v' = u$. $\square$

THEOREM 231. *Let* f *be non-anticipatory and* $\mu \overset{u_{|I}}{\rightarrow} \mu'$ *be a fundamental transfer, where* $I \subset \mathbf{R}$ *is an interval of the form* $(-\infty, t_1)$ *or* $[t_0, t_1)$.

a) If $I = (-\infty, t_1)$ *and* $u' \in U$ *is arbitrary with* $u_{|(-\infty,t_1)} = u'_{|(-\infty,t_1)}$, *then* $\mu \overset{u'_{|I}}{\rightarrow} \mu'$ *is an initial fundamental transfer equal to* $\mu \overset{u_{|I}}{\rightarrow} \mu'$.

b) If $I = [t_0, t_1)$, then $\forall u' \in U$, $u_{|(-\infty,t_1)} = u'_{|(-\infty,t_1)}$ implies that $\mu \overset{u'_{|I}}{\rightarrow} \mu'$ is a non-initial fundamental transfer equal to $\mu \overset{u_{|I}}{\rightarrow} \mu'$.

PROOF. a) The transfer $\mu \overset{u'_{|(-\infty,t_1)}}{\rightarrow} \mu'$ is an initial fundamental transfer, i.e.

$$\exists t_0 < t_1, \exists v \in U, u'_{|(-\infty,t_1)} = v_{|(-\infty,t_1)},$$

$$\forall y \in f(v), y_{|(-\infty,t_0)} = \mu \text{ and } y_{|[t_1,\infty)} = \mu'$$

takes place because the hypothesis (2.9) and $u_{|(-\infty,t_1)} = u'_{|(-\infty,t_1)}$ hold. We take into account the non-anticipation of f and we get the second statement of the Theorem

$$\mu \overset{u_{|(-\infty,t_1)}}{\rightarrow} \mu' = \{x_{|(-\infty,t_1]} | x \in f(u)\} = \{x'_{|(-\infty,t_1]} | x' \in f(u')\} = \mu \overset{u'_{|(-\infty,t_1)}}{\rightarrow} \mu'.$$

b) Is proved similarly to a). □

EXAMPLE 91. *The system $f : S \to P^*(S)$ defined by the double inequality*

$$\bigcap_{\xi \in [t-1,t)} \overline{u(\xi)} \le x(t) \le \bigcup_{\xi \in [t-1,t)} \overline{u(\xi)} \tag{3.1}$$

models the computation of the logical complement of u, made with a delay of one time unit. Suppose that it is non-anticipatory and denote by $u = \chi_{[0,2)}$, $v = \chi_{[0,\infty)}$ the inputs for which the inequalities $\bigcap_{\xi \in [t-1,t)} \overline{u(\xi)} \le x(t) \le \bigcup_{\xi \in [t-1,t)} \overline{u(\xi)}$, $\bigcap_{\xi \in [t-1,t)} \overline{v(\xi)} \le y(t) \le \bigcup_{\xi \in [t-1,t)} \overline{v(\xi)}$ become

$$\chi_{(-\infty,0] \cup [3,\infty)}(t) \le x(t) \le \chi_{(-\infty,1) \cup (2,\infty)}(t), \tag{3.2}$$

$$\chi_{(-\infty,0]}(t) \le y(t) \le \chi_{(-\infty,1)}(t). \tag{3.3}$$

From (3.3) we infer that

$$\forall y \in f(v), y_{|(-\infty,0)} = 1 \text{ and } y_{|[1,\infty)} = 0$$

and, because

$$u_{|(-\infty,1)} = v_{|(-\infty,1)},$$

we have that $(1 \overset{u_{|(-\infty,1)}}{\rightarrow} 0) = (1 \overset{v_{|(-\infty,1)}}{\rightarrow} 0)$ is an initial fundamental transfer ((2.9) is true). From the inequalities (3.2), (3.3) we also infer that

$$\forall y \in f(v), y_{|[1,\infty)} = 0,$$

$$\forall x \in f(u), x_{|[3,\infty)} = 1,$$

i.e. $0 \overset{u_{|[1,3)}}{\rightarrow} 1$ is a non-initial fundamental transfer (by Theorem 230).

In general, the transitions $\gamma \in 1 \overset{u_{|(-\infty,1)}}{\rightarrow} 0$ and $\gamma' \in 0 \overset{u_{|[1,3)}}{\rightarrow} 1$ are not monotonous. We ask in what conditions, if we add the (absolute inertia) requirements[2]

$$\overline{x(t-0)} \cdot x(t) \le \bigcap_{\xi \in [t,t+\delta]} x(\xi), \tag{3.4}$$

[2] if x switches from 0 to 1, then it remains 1 more than δ time units; if x switches from 1 to 0, then it remains 0 more than δ time units

$$x(t-0)\cdot\overline{x(t)} \leq \bigcap_{\xi\in[t,t+\delta]} \overline{x(\xi)} \tag{3.5}$$

where $\delta \geq 0$, to (3.1) with $u = \chi_{[0,2)}$, i.e. to (3.2), monotony is true. Monotony means that x switches from 1 to 0 in the interval $(0,1]$ and that in this interval it cannot switch from 0 to 1 and then from 1 to 0 again. Let $0 < t_1 < t_2 < t_3 \leq 1$ be such that

$$x(t_1-0)\cdot\overline{x(t_1)} = \overline{x(t_2-0)}\cdot x(t_2) = x(t_3-0)\cdot\overline{x(t_3)} = 1.$$

Then, from the fulfillment of (3.4) and (3.5), we have $t_2 - t_1 > \delta, t_3 - t_2 > \delta$, meaning that $1 > t_3 - t_1 > 2\delta$. Thus, if $\delta \geq \frac{1}{2}$, such t_1, t_2, t_3 do not exist and any $\gamma \in 1 \overset{u_{|(-\infty,1)}}{\rightarrow} 0$ is a monotonous transition. Similarly, $\delta \geq \frac{1}{2}$ implies the fact that any $\gamma' \in 0 \overset{u_{|[1,3)}}{\rightarrow} 1$ is monotonous.

Another condition is also required here: after having switched from 1 to 0 in the interval $(0,1]$, x is also allowed to switch from 0 to 1 in the interval $(2,3]$. This gives $\delta < 3$.

The conclusion is the following: for $\delta \in [\frac{1}{2}, 3)$, the system g obtained by intersecting (3.1), (3.4), (3.5), where $u = \chi_{[0,2)}$, has the hazard-free transfers $1 \overset{u_{|(-\infty,1)}}{\rightarrow} 0$, $0 \overset{u_{|[1,3)}}{\rightarrow} 1$.

4. The composition of the fundamental transfers

Remark 87. *Theorem 232 to follow constructs the 'union' $\vee$ of the fundamental transfers $\mu \overset{u^0_{|(-\infty,t_1)}}{\rightarrow} \mu'$ and $\mu' \overset{u^1_{[t_2,t_3)}}{\rightarrow} \mu''$ in a), respectively of the fundamental transfers $\mu \overset{u^0_{|[t_0,t_1)}}{\rightarrow} \mu'$ and $\mu' \overset{u^1_{[t_2,t_3)}}{\rightarrow} \mu''$ in b) similarly to our procedure from Remark 70. The theorem represents the version of Theorem 203 when the four accesses are synchronous, μ is initial in a), final in b) and μ', μ'' are final values in a) and b).*

Theorem 232. *Let $f : U \rightarrow P^*(S^{(n)}), U \in P^*(S^{(m)})$ be a non-anticipatory system satisfying the conditions:*

i) U is closed under translations and under 'concatenation'

$$\forall d \in \mathbf{R}, \forall u \in U, u \circ \tau^d \in U,$$

$$\forall t \in \mathbf{R}, \forall u \in U, \forall v \in U, u \cdot \chi_{(-\infty,t)} \oplus v \cdot \chi_{[t,\infty)} \in U;$$

ii) non-anticipation $\forall t \in \mathbf{R}$, $\forall u \in U$, $\forall v \in U$,*

$$(u_{|[t,\infty)} = v_{|[t,\infty)} \text{ and } \{x(t)|x \in f(u)\} = \{y(t)|y \in f(v)\}) \Longrightarrow$$

$$\Longrightarrow \{x_{|[t,\infty)}|x \in f(u)\} = \{y_{|[t,\infty)}|y \in f(v)\};$$

iii) time invariance

$$\forall d \in \mathbf{R}, \forall u \in U, f(u \circ \tau^d) = \{x \circ \tau^d | x \in f(u)\}.$$

a) Suppose that $t_0 < t_1$, $t_2 < t_3$, $u^0, u^1, v^1 \in U$ and $\mu, \mu', \mu'' \in \mathbf{B}^n$ are arbitrary with

$$\forall x \in f(u^0), x_{|(-\infty,t_0)} = \mu,$$

$$\forall x \in f(u^0), x_{|[t_1,\infty)} = \mu',$$

$$u^1_{|(-\infty,t_2)} = v^1_{|(-\infty,t_2)},$$

$$\forall y' \in f(v^1), y'_{|[t_2,\infty)} = \mu',$$

$$\forall x' \in f(u^1), x'_{|[t_3,\infty)} = \mu''.$$

Denote $d = t_1 - t_2$ *and*

$$\widetilde{u}_\varepsilon = u^0 \cdot \chi_{(-\infty,t_1+\varepsilon)} \oplus (u^1 \circ \tau^{d+\varepsilon}) \cdot \chi_{[t_1+\varepsilon,\infty)}$$

for $\varepsilon \geq 0$. *We have*

$$\forall \widetilde{x} \in f(\widetilde{u}_\varepsilon), \widetilde{x}_{|(-\infty,t_0)} = \mu,$$

$$\forall \widetilde{x} \in f(\widetilde{u}_\varepsilon), \widetilde{x}_{|[t_3+d+\varepsilon,\infty)} = \mu'',$$

meaning that if $\mu \overset{u^0_{|(-\infty,t_1)}}{\rightarrow} \mu'$ *is initial fundamental and* $\mu' \overset{u^1_{|[t_2,t_3)}}{\rightarrow} \mu''$ *is non-initial fundamental, then* $\mu \overset{\widetilde{u}_{\varepsilon|(-\infty,t_3+d+\varepsilon)}}{\rightarrow} \mu''$ *is initial fundamental. In other words, if* $f(u^0)$ *transfers synchronously the initial value* μ *in the final value* μ' *and if* $f(u^1)$ *transfers synchronously the final value* μ' *in the final value* μ'', *then* $f(\widetilde{u}_\varepsilon)$ *transfers synchronously the initial value* μ *in the final value* μ''.

b) Suppose that $t_0 < t_1$, $t_2 < t_3$, $u^0, v^0, u^1, v^1 \in U$ *and* $\mu, \mu', \mu'' \in \mathbf{B}^n$ *are given such that*

$$u^0_{|(-\infty,t_0)} = v^0_{|(-\infty,t_0)}, \tag{4.1}$$

$$\forall y \in f(v^0), y_{|[t_0,\infty)} = \mu, \tag{4.2}$$

$$\forall x \in f(u^0), x_{|[t_1,\infty)} = \mu', \tag{4.3}$$

$$u^1_{|(-\infty,t_2)} = v^1_{|(-\infty,t_2)}, \tag{4.4}$$

$$\forall y' \in f(v^1), y'_{|[t_2,\infty)} = \mu', \tag{4.5}$$

$$\forall x' \in f(u^1), x'_{|[t_3,\infty)} = \mu''. \tag{4.6}$$

With the notations $d = t_1 - t_2$, $\widetilde{v} = v^0$ *and*

$$\widetilde{u}_\varepsilon = u^0 \cdot \chi_{(-\infty,t_1+\varepsilon)} \oplus (u^1 \circ \tau^{d+\varepsilon}) \cdot \chi_{[t_1+\varepsilon,\infty)}, \tag{4.7}$$

$\varepsilon \geq 0$, *we have*

$$\widetilde{u}_{\varepsilon|(-\infty,t_0)} = \widetilde{v}_{|(-\infty,t_0)}, \tag{4.8}$$

$$\forall \widetilde{y} \in f(\widetilde{v}), \widetilde{y}_{|[t_0,\infty)} = \mu, \tag{4.9}$$

$$\forall \widetilde{x} \in f(\widetilde{u}_\varepsilon), \widetilde{x}_{|[t_3+d+\varepsilon,\infty)} = \mu''. \tag{4.10}$$

This means that if $\mu \overset{u^0_{|[t_0,t_1)}}{\rightarrow} \mu'$, $\mu' \overset{u^1_{|[t_2,t_3)}}{\rightarrow} \mu''$ *are non-initial fundamental, then* $\mu \overset{\widetilde{u}_{\varepsilon|[t_0,t_3+d+\varepsilon)}}{\rightarrow} \mu''$ *is non-initial fundamental (if* $f(u^0)$ *transfers synchronously the final value* μ *in the final value* μ' *and if* $f(u^1)$ *transfers synchronously the final value* μ' *in the final value* μ'', *then* $f(\widetilde{u}_\varepsilon)$ *transfers synchronously the final value* μ *in the final value* μ'').

PROOF. b) First of all remark that, by i), $\widetilde{u}_\varepsilon$ given by (4.7) belongs to U.

The equality (4.8) is satisfied because for any $\varepsilon \geq 0$, we have $t_1 + \varepsilon \geq t_1 > t_0$, and from the definition of $\widetilde{v}$, we have

$$\widetilde{u}_{\varepsilon|(-\infty,t_0)} \overset{(4.7)}{=} u^0_{|(-\infty,t_0)} \overset{(4.1)}{=} v^0_{|(-\infty,t_0)} = \widetilde{v}_{|(-\infty,t_0)}.$$

The relation (4.9) is true because it coincides with the hypothesis (4.2).

Prove (4.10). From (4.4) and from the non-anticipation of f we infer

$$\{y'_{|(-\infty,t_2]}|y' \in f(v^1)\} = \{x'_{|(-\infty,t_2]}|x' \in f(u^1)\}$$

and taking into account (4.5), we can see that

$$\{y'(t_2)|y' \in f(v^1)\} = \{x'(t_2)|x' \in f(u^1)\} = \mu'. \tag{4.11}$$

The time invariance of f implies that

$$\{x''|x'' \in f(u^1 \circ \tau^{d+\varepsilon})\} = \{x' \circ \tau^{d+\varepsilon}|x' \in f(u^1)\},$$

thus

$$\{x'(t_2)|x' \in f(u^1)\} = \{x''(t_1+\varepsilon)|x'' \in f(u^1 \circ \tau^{d+\varepsilon})\}. \tag{4.12}$$

From

$$\widetilde{u}_{\varepsilon|(-\infty,t_1+\varepsilon)} = u^0_{|(-\infty,t_1+\varepsilon)}$$

and from the non-anticipation we get

$$\{\widetilde{x}_{|(-\infty,t_1+\varepsilon]}|\widetilde{x} \in f(\widetilde{u}_\varepsilon)\} = \{x_{|(-\infty,t_1+\varepsilon]}|x \in f(u^0)\}.$$

In particular, we have

$$\{\widetilde{x}(t_1+\varepsilon)|\widetilde{x} \in f(\widetilde{u}_\varepsilon)\} = \{x(t_1+\varepsilon)|x \in f(u^0)\}. \tag{4.13}$$

Then

$$\begin{aligned} \{\widetilde{x}(t_1+\varepsilon)|\widetilde{x} \in f(\widetilde{u}_\varepsilon)\} &\overset{(4.13)}{=} \{x(t_1+\varepsilon)|x \in f(u^0)\} \overset{(4.3)}{=} \mu' = \\ &\overset{(4.5)}{=} \{y'(t_2)|y' \in f(v^1)\} \overset{(4.11)}{=} \{x'(t_2)|x' \in f(u^1)\} = \\ &\overset{(4.12)}{=} \{x''(t_1+\varepsilon)|x'' \in f(u^1 \circ \tau^{d+\varepsilon})\}. \end{aligned} \tag{4.14}$$

Because

$$\widetilde{u}_{\varepsilon|[t_1+\varepsilon,\infty)} = (u^1 \circ \tau^{d+\varepsilon})_{|[t_1+\varepsilon,\infty)}, \tag{4.15}$$

(4.14), (4.15) and the non-anticipation* of f show that

$$\{\widetilde{x}_{|[t_1+\varepsilon,\infty)}|\widetilde{x} \in f(\widetilde{u}_\varepsilon)\} = \{x''_{|[t_1+\varepsilon,\infty)}|x'' \in f(u^1 \circ \tau^{d+\varepsilon})\}. \tag{4.16}$$

But the fact that $t_3 + d + \varepsilon > t_1 + \varepsilon$ and

$$\{x''_{|[t_1+\varepsilon,\infty)}|x'' \in f(u^1 \circ \tau^{d+\varepsilon})\} = \{(x' \circ \tau^{d+\varepsilon})_{|[t_1+\varepsilon,\infty)}|x' \in f(u^1)\} \tag{4.17}$$

indicate the truth of

$$\begin{aligned} \{\widetilde{x}_{|[t_3+d+\varepsilon,\infty)}|\widetilde{x} \in f(\widetilde{u}_\varepsilon)\} &\overset{(4.16)}{=} \{x''_{|[t_3+d+\varepsilon,\infty)}|x'' \in f(u^1 \circ \tau^{d+\varepsilon})\} = \\ \overset{(4.17)}{=} \{(x' \circ \tau^{d+\varepsilon})_{|[t_3+d+\varepsilon,\infty)}|x' \in f(u^1)\} &= \{x'_{|[t_3,\infty)}|x' \in f(u^1)\} \overset{(4.6)}{=} \mu''. \end{aligned}$$

So, (4.10) is proved. ∎

DEFINITION 92. *We use the notations from the previous theorem and we suppose that the requirements stated there are fulfilled. We have the following partial law of composition of the fundamental transfers*

$$(\mu \overset{u^0_{|(-\infty,t_1)}}{\to} \mu') \vee (\mu' \overset{u^1_{|[t_2,t_3)}}{\to} \mu'') = \mu \overset{\widetilde{u}_{\varepsilon|(-\infty,t_3+d+\varepsilon)}}{\to} \mu'',$$

$$(\mu \overset{u^0_{|[t_0,t_1)}}{\to} \mu') \vee (\mu' \overset{u^1_{|[t_2,t_3)}}{\to} \mu'') = \mu \overset{\widetilde{u}_{\varepsilon|[t_0,t_3+d+\varepsilon)}}{\to} \mu''.$$

5. A special case of the composition of the fundamental transfers

THEOREM 233. *Assume that the system f is non-anticipatory. The following statements are true:*

a) for any $t_1 < t_2,\ u \in U$ and $\mu, \mu', \mu'' \in \mathbf{B}^n$, such that the transfers $\mu \overset{u_{|(-\infty,t_1)}}{\to} \mu'$, $\mu' \overset{u_{|[t_1,t_2)}}{\to} \mu''$ are fundamental, the transfer $\mu \overset{u_{|(-\infty,t_2)}}{\to} \mu''$ is fundamental;

b) suppose that $t_1 < t_2 < t_3,\ u \in U$ and $\mu, \mu', \mu'' \in \mathbf{B}^n$ are arbitrary and satisfy the property that the transfers $\mu \overset{u_{|[t_1,t_2)}}{\to} \mu'$, $\mu' \overset{u_{|[t_2,t_3)}}{\to} \mu''$ are fundamental. Then the transfer $\mu \overset{u_{|[t_1,t_3)}}{\to} \mu''$ is fundamental.

PROOF. a) By hypothesis there are $t_0 < t_1, v \in U$ and $v' \in U$, such that

$$u_{|(-\infty,t_1)} = v_{|(-\infty,t_1)}, \forall y \in f(v), y_{|(-\infty,t_0)} = \mu \text{ and } y_{|[t_1,\infty)} = \mu',$$

$$u_{|(-\infty,t_2)} = v'_{|(-\infty,t_2)}, \forall y' \in f(v'), y'_{|[t_2,\infty)} = \mu''.$$

Because $v_{|(-\infty,t_0)} = v'_{|(-\infty,t_0)}$, from the non-anticipation of f we have

$$\{y_{|(-\infty,t_0)} | y \in f(v)\} = \{y'_{|(-\infty,t_0)} | y' \in f(v')\} = \mu,$$

thus

$$u_{|(-\infty,t_2)} = v'_{|(-\infty,t_2)}, \forall y' \in f(v'), y'_{|(-\infty,t_0)} = \mu \text{ and } y'_{|[t_2,\infty)} = \mu''$$

is true, meaning that the transfer $\mu \overset{u_{|(-\infty,t_2)}}{\to} \mu''$ is fundamental.

b) is made similarly with a). □

REMARK 88. *In the conditions of and with the notations from the previous theorem, we have the following partial law of composition of the fundamental transfers:*

$$(\mu \overset{u_{|(-\infty,t_1)}}{\to} \mu') \vee (\mu' \overset{u_{|[t_1,t_2)}}{\to} \mu'') = \mu \overset{u_{|(-\infty,t_2)}}{\to} \mu'';$$

$$(\mu \overset{u_{|[t_1,t_2)}}{\to} \mu') \vee (\mu' \overset{u_{|[t_2,t_3)}}{\to} \mu'') = \mu \overset{u_{|[t_1,t_3)}}{\to} \mu'',$$

representing a special case of Definition 92. Theorem 233 restates the results from Theorem 232 under a simplified form. For example in Theorem 232, a) we have $u^0 = u^1$. For this reason the requirements of closure of U under the concatenation of the inputs and of non-anticipation are removed. Because $u^0 = u^1$ and $t_1 = t_2$ the requirements of closure of U under translations and of time invariance disappear too.*

6. The fundamental mode

THEOREM 234. *Consider the system f supposed to be non-anticipatory and let $u \in U$ be a fixed input. The following statements are equivalent:*

a) there are $(t_k) \in Seq$, $(u^k) \in U$ and $(\mu^k) \in \mathbf{B}^n$ such that

$$\forall x \in f(u^0), x_{|(-\infty,t_0)} = \mu^0 \textit{ and } x_{|[t_1,\infty)} = \mu^1,$$

$$u_{|(-\infty,t_1)} = u^0_{|(-\infty,t_1)},\ u_{|(-\infty,t_2)} = u^1_{|(-\infty,t_2)},\ u_{|(-\infty,t_3)} = u^2_{|(-\infty,t_3)}, \ldots$$

$$\forall x \in f(u^1), x_{|[t_2,\infty)} = \mu^2,\ \forall x \in f(u^2), x_{|[t_3,\infty)} = \mu^3,\ \forall x \in f(u^3), x_{|[t_4,\infty)} = \mu^4, \ldots;$$

b) there are $(t_k) \in Seq$ and $(\mu^k) \in \mathbf{B}^n$ such that the transfers $\mu^0 \xrightarrow{u_{|(-\infty,t_1)}} \mu^1$, $\mu^1 \xrightarrow{u_{|[t_1,t_2)}} \mu^2$, $\mu^2 \xrightarrow{u_{|[t_2,t_3)}} \mu^3, \ldots$ are fundamental;

c) there are $(t_k) \in Seq$ and $(\mu^k) \in \mathbf{B}^n$ such that the transfers $\mu^0 \xrightarrow{u_{|(-\infty,t_1)}} \mu^1$, $\mu^0 \xrightarrow{u_{|(-\infty,t_2)}} \mu^2$, $\mu^0 \xrightarrow{u_{|(-\infty,t_3)}} \mu^3, \ldots$ are initial fundamental.

PROOF. *a)* $\Longrightarrow$ *b)* Let (t_k), (u^k) and (μ^k) be like in *a)*. Because

$$u_{|(-\infty,t_1)} = u^0_{|(-\infty,t_1)},\ \forall x \in f(u^0), x_{|(-\infty,t_0)} = \mu^0 \textit{ and } x_{|[t_1,\infty)} = \mu^1 \tag{6.1}$$

is true, $\mu^0 \xrightarrow{u_{|(-\infty,t_1)}} \mu^1$ is an initial fundamental transfer. The fact that

$$u_{|(-\infty,t_1)} = u^0_{|(-\infty,t_1)},\ \forall x \in f(u^0), x_{|[t_1,\infty)} = \mu^1, \tag{6.2}$$

$$u_{|(-\infty,t_2)} = u^1_{|(-\infty,t_2)},\ \forall x \in f(u^1), x_{|[t_2,\infty)} = \mu^2 \tag{6.3}$$

implies that $\mu^1 \xrightarrow{u_{|[t_1,t_2)}} \mu^2$ is non-initial fundamental etc.

b) $\Longrightarrow$ *c)* There are (t_k) and (μ^k) such that $\mu^0 \xrightarrow{u_{|(-\infty,t_1)}} \mu^1$, $\mu^1 \xrightarrow{u_{|[t_1,t_2)}} \mu^2$, $\mu^2 \xrightarrow{u_{|[t_2,t_3)}} \mu^3, \ldots$ are fundamental. Like in Theorem 233 and Definition 88,

$$\mu^0 \xrightarrow{u_{|(-\infty,t_2)}} \mu^2 = (\mu^0 \xrightarrow{u_{|(-\infty,t_1)}} \mu^1) \vee (\mu^1 \xrightarrow{u_{|[t_1,t_2)}} \mu^2),$$

$$\mu^0 \xrightarrow{u_{|(-\infty,t_3)}} \mu^3 = (\mu \xrightarrow{u_{|(-\infty_1,t_2)}} \mu^2) \vee (\mu^2 \xrightarrow{u_{|[t_2,t_3)}} \mu^3),$$

$$\ldots$$

are initial fundamental.

c) $\Longrightarrow$ *a)* Consider the sequences (t_k) and (μ^k) like in *c)*. The fact that $\mu^0 \xrightarrow{u_{|(-\infty,t_1)}} \mu^1$ is initial fundamental shows the existence of $u^0 \in U$ such that (6.1) holds true and because $\mu^0 \xrightarrow{u_{|(-\infty,t_2)}} \mu^2$ is initial fundamental, we obtain the existence of $u^1 \in U$, with (6.3) true etc. The statement from a) is true. $\square$

DEFINITION 93. *If one of the previous properties a), b), c) from Theorem 234 is satisfied, the input u is called a* ***fundamental (operating) mode (of*** *f).*

THEOREM 235. *If f is non-anticipatory and $t_0 < t_1, u \in U, \mu, \mu' \in \mathbf{B}^n$ are fixed, then the fact that*

$$\forall x \in f(u), x_{|(-\infty,t_0)} = \mu \textit{ and } x_{|[t_1,\infty)} = \mu'$$

implies that u is a fundamental mode of f.

PROOF. There are the sequences $(t'_k) \in Seq$ and $(\mu^k) \in \mathbf{B}^n$ satisfying

$$t'_0 = t_0, t'_1 = t_1, t'_k, k \geq 2 \text{ arbitrary},$$

$$\mu^0 = \mu, \mu^1 = \mu^2 = ... = \mu'.$$

We note that $\mu \overset{u_{|(-\infty,t'_1)}}{\longrightarrow} \mu', \mu \overset{u_{|(-\infty,t'_2)}}{\longrightarrow} \mu', \mu \overset{u_{|(-\infty,t'_3)}}{\longrightarrow} \mu', ...$ are initial fundamental transfers. $\square$

REMARK 89. *The evolution of* f *under the fundamental mode* u *may be interpreted as a discrete time symbolic evolution of a deterministic system of the form*

$$\mu^0 = x(0) \overset{u^0}{\to} \mu^1 = x(1) \overset{u^1}{\to} ... \overset{u^k}{\to} \mu^{k+1} = x(k+1) \overset{u^{k+1}}{\to} ...,$$

where the initial fundamental transfer $\mu^0 \overset{u^0_{|(-\infty,t_1)}}{\longrightarrow} \mu^1$ *is identified to the symbolic transfer* $x(0) \overset{u^0}{\to} x(1)$ *and a non-initial fundamental transfer of rank* $k \geq 1$, $\mu^k \overset{u^k_{|[t_k,t_{k+1})}}{\longrightarrow} \mu^{k+1}$ *is identified with the symbolic transfer* $x(k) \overset{u^k}{\to} x(k+1)$.

In the hypothesis of the previous Theorem, the symbolic evolution may be considered to be given by a finite sequence

$$\mu^0 = x(0) \overset{u^0}{\to} \mu^1 = x(1) \overset{u^1}{\to} ... \overset{u^k}{\to} \mu^{k+1} = x(k+1)$$

where k *can be* 0.

In addition: if the input $v \in U$ *is a fundamental mode of* f, *then a set* $V \subset U$ *exists with the property that* $v \in V$ *(for example* $V = \{v\}$*) and* $f_{|V}$ *is strongly synchronous, i.e. it satisfies*

$$\exists (t_k) \in Seq, \forall u \in V,$$

$$\exists (\mu^k) \in \mathbf{B}^n, \forall x \in f(u), x_{|(-\infty,t_0)} = \mu^0 \text{ and } \forall k \geq 1, x(t_k) = \mu^k.$$

EXAMPLE 92. *In Example 91 the inputs* u *and* v *are fundamental modes of both systems* f, g.

EXAMPLE 93. *The deterministic system* $f : S \to S$,

$$\forall u \in S,\ f(u) = \begin{cases} 1, u = \chi_{[0,1)\cup[2,3)\cup[4,5)\cup...} \\ 0, otherwise \end{cases}$$

satisfies the following properties: there are $u = \chi_{[0,1)\cup[2,3)\cup[4,5)\cup...}$, *the unbounded sequence* $0 < 2 < 4 < ...$ *of real numbers, the family*

$$u^0 = \chi_{[0,1)},\ u^1 = \chi_{[0,1)\cup[2,3)},\ u^2 = \chi_{[0,1)\cup[2,3)\cup[4,5)}, ...$$

of inputs and the binary null sequence $0_k \in \mathbf{B}, k \in \mathbf{N}$ *such that*

$$f(u^0)_{|(-\infty,0)} = 0 \text{ and } f(u^0)_{|[2,\infty)} = 0,$$

$$u_{|(-\infty,2)} = u^0_{|(-\infty,2)},\ u_{|(-\infty,4)} = u^1_{|(-\infty,4)}, ...$$

$$f(u^1)_{|[4,\infty)} = 0,\ f(u^2)_{|[6,\infty)} = 0, ...$$

The statements

$$f(u)_{|(-\infty,2]} = f(u^0)_{|(-\infty,2]},\ f(u)_{|(-\infty,4]} = f(u^1)_{|(-\infty,4]}, ...$$

are false, since f *is anticipatory. Therefore* u *is not a fundamental mode of* f.

THEOREM 236. *Let u be a fundamental mode of the non-anticipatory system f. Then there are the families $(t_k) \in Seq$ and $(u^k) \in U$ such that*

$$\forall k \in \mathbf{N}, u_{|(-\infty,t_{k+1})} = u^k_{|(-\infty,t_{k+1})}$$

and all $u^k, k \in \mathbf{N}$ are fundamental modes of f.

PROOF. From Theorem 234, c) there are $(t_k) \in Seq$ and $(\mu^k) \in \mathbf{B}^n$ such that the transfers $\mu^0 \overset{u_{|(-\infty,t_1)}}{\longrightarrow} \mu^1$, $\mu^0 \overset{u_{|(-\infty,t_2)}}{\longrightarrow} \mu^2$, $\mu^0 \overset{u_{|(-\infty,t_3)}}{\longrightarrow} \mu^3, ...$ are initial fundamental, i.e. there is the sequence $(u^k) \in U$ with

$$u_{|(-\infty,t_1)} = u^0_{|(-\infty,t_1)}, \ \forall x \in f(u^0), x_{|(-\infty,t_0)} = \mu^0 \ and \ x_{|[t_1,\infty)} = \mu^1,$$
$$u_{|(-\infty,t_2)} = u^1_{|(-\infty,t_2)}, \ \forall x \in f(u^1), x_{|(-\infty,t_0)} = \mu^0 \ and \ x_{|[t_2,\infty)} = \mu^2,$$
$$u_{|(-\infty,t_3)} = u^2_{|(-\infty,t_3)}, \ \forall x \in f(u^2), x_{|(-\infty,t_0)} = \mu^0 \ and \ x_{|[t_3,\infty)} = \mu^3,$$
$$...$$

Thus $\mu^0 \overset{u^0_{|(-\infty,t_1)}}{\longrightarrow} \mu^1$, $\mu^0 \overset{u^1_{|(-\infty,t_2)}}{\longrightarrow} \mu^2$, $\mu^0 \overset{u^2_{|(-\infty,t_3)}}{\longrightarrow} \mu^3, ...$ are initial fundamental and by Theorem 235, we obtain that all $u^k, k \in \mathbf{N}$ are fundamental modes of f. □

THEOREM 237. *Let be the non-anticipatory system f and the sequences $(t_k) \in Seq$, $(u^k) \in U$, such that the following properties to be fulfilled:*

a) for any $v \in S^{(n)}$ and any sequences $(\xi_k) \in Seq$, $(v^k) \in U$, from

$$\forall k \in \mathbf{N}, v_{|(-\infty,\xi_{k+1})} = v^k_{|(-\infty,\xi_{k+1})}$$

we infer that $v \in U$;

b) $u^k, k \in \mathbf{N}$ are all fundamental modes of f;

c) we have

$$\forall k \in \mathbf{N}, u^k_{|(-\infty,t_{k+1})} = u^{k+1}_{|(-\infty,t_{k+1})}. \tag{6.4}$$

Then u defined by

$$\forall k \in \mathbf{N}, u_{|(-\infty,t_{k+1})} = u^k_{|(-\infty,t_{k+1})} \tag{6.5}$$

is a fundamental mode of f.

PROOF. Remark that (6.4) allows us writing (6.5) and that, by hypothesis a), this last relation defines a unique u, that belongs to U . Prove that u is a fundamental mode of f.

The fact that all u^k are fundamental modes of f shows the existence of $(t^k_{k'})_{k'} \in Seq, (u^{kk'})_{k'} \in U$ and $(\mu^{kk'})_{k'} \in \mathbf{B}^n$ such that

$$\forall x \in f(u^{k0}), x_{|(-\infty,t^k_0)} = \mu^{k0} \text{ and } x_{|[t^k_1,\infty)} = \mu^{k1},$$
$$u^k_{|(-\infty,t^k_1)} = u^{k0}_{|(-\infty,t^k_1)}, u^k_{|(-\infty,t^k_2)} = u^{k1}_{|(-\infty,t^k_2)}, u^k_{|(-\infty,t^k_3)} = u^{k2}_{|(-\infty,t^k_3)}, ...$$
$$\forall x \in f(u^{k1}), x_{|[t^k_2,\infty)} = \mu^{k2}, \forall x \in f(u^{k2}), x_{|[t^k_3,\infty)} = \mu^{k3},$$
$$\forall x \in f(u^{k3}), x_{|[t^k_4,\infty)} = \mu^{k4}, ...$$

Define the sequence $(t'_k) \in Seq$ by

$$(t'_k) = \{t^{k_1}_{k_2} | k_1 \in \mathbf{N}, k_2 \in \mathbf{N}\}$$

and we fix the association $\mathbf{N} \ni k \mapsto (k_1, k_2) \in \mathbf{N} \times \mathbf{N}$ characterized by

$$\forall k \in \mathbf{N}, t'_k = t^{k_1}_{k_2}.$$

This association, together with the requirement of non-anticipation of f give us the possibility to define

$$\forall k \in \mathbf{N}, u'^k = u^{k_1 k_2},$$
$$\forall k \in \mathbf{N}, \mu'^k = \mu^{k_1 k_2}.$$

The sequences $(t'_k), (u'^k), (\mu'^k)$ fulfill the requirement a) of Theorem 234. □

7. A property of existence

THEOREM 238. *Let be the non-anticipatory system f. Suppose that the following requirements are fulfilled:*

a) for any $(t_k) \in Seq$ and any sequence $(u^k) \in U$ of inputs, we have $u^0 \cdot \chi_{(-\infty,t_0)} \oplus u^1 \cdot \chi_{[t_0,t_1)} \oplus u^2 \cdot \chi_{[t_1,t_2)} \oplus ... \in U$[3]*;*

b) f satisfies the following property of race-free initialization with bounded initial time:

$$\forall u \in U, \exists \mu \in \mathbf{B}^n, \exists t_0 \in \mathbf{R}, \forall x \in f(u), x_{|(-\infty,t_0)} = \mu;$$

c) f is absolutely race-free stable with a bounded final time, i.e.

$$\forall u \in U, \exists \mu' \in \mathbf{B}^n, \exists t_1 \in \mathbf{R}, \forall x \in f(u), x_{|[t_1,\infty)} = \mu'.$$

Then, for any sequence $(u^k) \in U$ of inputs, there are the time instants $(t_k) \in Seq$ such that the input

$$\widetilde{u} = u^0 \cdot \chi_{(-\infty,t_1)} \oplus u^1 \cdot \chi_{[t_1,t_2)} \oplus ... \oplus u^k \cdot \chi_{[t_k,t_{k+1})} \oplus ...$$

is a fundamental mode of f.

PROOF. Consider some real number $\delta > 0$ and the arbitrary sequence $(u^k) \in U$ of inputs. From b) we infer the existence of $\mu^0 \in \mathbf{B}^n$ and $t_0 \in \mathbf{R}$, such that

$$\forall x \in f(u^0), x_{|(-\infty,t_0)} = \mu^0,$$

while from c) we have the existence of $\mu^1 \in \mathbf{B}^n$ and $t_1 \in \mathbf{R}$ with $t_1 > t_0 + \delta$ and

$$\forall x \in f(u^0), x_{|[t_1,\infty)} = \mu^1.$$

Furthermore, from a) we have that $u^0 \cdot \chi_{(-\infty,t_1)} \oplus u^1 \cdot \chi_{[t_1,\infty)} \in U$, while from c) the existence of $\mu^2 \in \mathbf{B}^n$ and $t_2 \in \mathbf{R}$, such that $t_2 > t_1 + \delta$ and

$$\forall x \in f(u^0 \cdot \chi_{(-\infty,t_1)} \oplus u^1 \cdot \chi_{[t_1,\infty)}), x_{|[t_2,\infty)} = \mu^2$$

is inferred. The construction of (t_k) and the fact that $(t_k) \in Seq$ are obvious. On the other hand, by taking in consideration a), $\widetilde{u}$ obtained in this way belongs to U. The statement that $\widetilde{u}$ is a fundamental mode of f is inferred from the equalities

$$\widetilde{u}_{|(-\infty,t_1)} = u^0_{|(-\infty,t_1)}, \widetilde{u}_{|(-\infty,t_2)} = (u^0 \cdot \chi_{(-\infty,t_1)} \oplus u^1 \cdot \chi_{[t_1,\infty)})_{|(-\infty,t_2)}, ...$$

□

REMARK 90. *The previous theorem shows that in certain circumstances on f, for any sequence $(u^k) \in U$, if t_1 is large enough, the system stabilizes with all its states from t_1 and $(x(-\infty+0) \overset{\widetilde{u}_{|(-\infty,t_1)}}{\longrightarrow} x(t_1)) = (x(-\infty+0) \overset{u^0_{|(-\infty,t_1)}}{\longrightarrow} x(t_1))$ is initial fundamental and does not depend on the choice of $x \in f(\widetilde{u})$. Moreover, for*

[3]The properties of closure of U from Theorem 237, a) and Theorem 238, a) are equivalent and they are called **safety**.

any k and any time instant t_k, if we let u^k participate for a sufficiently long time at the construction of $\widetilde{u}$,

$$\widetilde{u}_{|[t_k,t_{k+1})} = u^k_{|[t_k,t_{k+1})},$$

i.e. if t_{k+1} is large enough, then the system stabilizes with all its states from t_{k+1} and the transfer $(x(t_k) \overset{\widetilde{u}_{|[t_k,t_{k+1})}}{\longrightarrow} x(t_{k+1})) = (x(t_k) \overset{u^k_{|[t_k,t_{k+1})}}{\longrightarrow} x(t_{k+1}))$ is non-initial fundamental and is independent on the choice of $x \in f(\widetilde{u})$.

EXAMPLE 94. *Here are a few examples of sets U that satisfy the requirement a) of Theorem 238.*

i) $U = \{\lambda\}$, where $\lambda \in \mathbf{B}^m$. We identify again the constant with the constant function. The set U has that property because for any unbounded sequence $t_0 < t_1 < t_2 < ...$ and any $(u^k) \in U$, we have $u^0 \cdot \chi_{(-\infty,t_0)} \oplus u^1 \cdot \chi_{[t_0,t_1)} \oplus u^2 \cdot \chi_{[t_1,t_2)} \oplus ... = \lambda$.

ii) The previous example is generalized in the following way. Let $H \subset \mathbf{B}^m$ be a non-empty set. The set $U \subset S^{(m)}$, defined by

$$U = \{u |\text{ there are } (\lambda^k) \in H \text{ and } (t_k) \in Seq$$
$$\text{such that } u = \lambda^0 \cdot \chi_{(-\infty,t_0)} \oplus \lambda^1 \cdot \chi_{[t_0,t_1)} \oplus \lambda^2 \cdot \chi_{[t_1,t_2)} \oplus ...\}$$

has the required property. In particular, for $H = \mathbf{B}^m$, we obtain the set $U = S^{(m)}$.

iii) Take some non-empty set $V \subset S^{(m)}$, for which

$$U = \{u |\text{ there are } (u^k) \in V \text{ and } (t_k) \in Seq$$
$$\text{such that } u = u^0 \cdot \chi_{(-\infty,t_0)} \oplus u^1 \cdot \chi_{[t_0,t_1)} \oplus u^2 \cdot \chi_{[t_1,t_2)} \oplus ...\}.$$

The set U has the property a) of Theorem 238.

THEOREM 239. *If the non-anticipatory system f satisfies the following properties:*

a) race-free initialization with a bounded initial time

$$\forall u \in U, \exists \mu \in \mathbf{B}^n, \exists t_0 \in \mathbf{R}, \forall x \in f(u), x_{|(-\infty,t_0)} = \mu,$$

b) absolute race-free stability with a bounded final time

$$\forall u \in U, \exists \mu' \in \mathbf{B}^n, \exists t_1 \in \mathbf{R}, \forall x \in f(u), x_{|[t_1,\infty)} = \mu'$$

then

$$\forall u \in U, \exists \mu \in \mathbf{B}^n, \exists \mu' \in \mathbf{B}^n, \exists t_0 \in \mathbf{R}, \exists t_1 > t_0,$$
$$\forall x \in f(u), x_{|(-\infty,t_0)} = \mu \text{ and } x_{|[t_1,\infty)} = \mu',$$

i.e. for any u, there are some μ, μ' and $t_0 < t_1$ such that $\mu \overset{u_{|(-\infty,t_1)}}{\rightarrow} \mu'$ is initial fundamental.

PROOF. From the first part of the proof of Theorem 238, where $u^0 = u$. □

THEOREM 240. *Suppose that the non-anticipatory system f is absolutely race-free stable with a bounded final time, i.e.*

$$\forall u \in U, \exists \mu \in \mathbf{B}^n, \exists t \in \mathbf{R}, \forall x \in f(u), x_{|[t,\infty)} = \mu.$$

Then $\forall u \in U$, there are the vectors $\mu, \mu' \in \mathbf{B}^n$ and the numbers $t_0 < t_1$ such that the transfer $\mu \overset{u_{|[t_0,t_1)}}{\longrightarrow} \mu'$ is non-initial fundamental.

PROOF. It is sufficient to consider the property: for any $u \in U$, there are μ and t_0 such that $\forall x \in f(u), x_{|[t_0,\infty)} = \mu$; then $\mu' = \mu$ and $t_1 > t_0$ arbitrary make that the conclusion of the theorem be fulfilled. □

8. Fundamental mode, special case

DEFINITION 94. *For any* $t_1 \in \mathbf{R}$, *the* ***prefix*** *of* $u \in S^{(m)}$ *is the function* $u_{t_1} \in S^{(m)}$ *given by*

$$u_{t_1}(t) = \begin{cases} u(t), t < t_1 \\ u(t_1 - 0), t \geq t_1 \end{cases}.$$

THEOREM 241. *Let* f *be a non-anticipatory system and let be the input* $u \in U$. *For any* $(t_k) \in Seq$ *and* $(\mu^k) \in \mathbf{B}^n$ *such that* $u_{t_1}, u_{t_2}, u_{t_3}, ... \in U$ *and*

$$\forall x \in f(u_{t_1}), x_{|(-\infty,t_0)} = \mu^0 \text{ and } x_{|[t_1,\infty)} = \mu^1$$

$\forall x \in f(u_{t_2}), x_{|[t_2,\infty)} = \mu^2$, $\forall x \in f(u_{t_3}), x_{|[t_3,\infty)} = \mu^3$, $\forall x \in f(u_{t_4}), x_{|[t_4,\infty)} = \mu^4, ...$
u is a fundamental mode of f.

PROOF. Define the sequence $(u^k) \in U$ by $u^k = u_{t_{k+1}}, k \in \mathbf{N}$. Because, for any $k \geq 0$, we have $u_{|(-\infty,t_{k+1})} = u^k_{|(-\infty,t_{k+1})}$, the statement of Theorem 234, a) is true. □

COROLLARY 2. *Suppose that the non-anticipatory system f and the input* $u \in U$ *are given. If the sequences* $(t_k) \in Seq$, $(\mu^k) \in \mathbf{B}^n$ *and* $(\lambda^k) \in \mathbf{B}^m$ *satisfy*

$$u(t) = \lambda^0 \cdot \chi_{(-\infty,t_1)}(t) \oplus \lambda^1 \cdot \chi_{[t_1,t_2)}(t) \oplus \lambda^2 \cdot \chi_{[t_2,t_3)}(t) \oplus ...$$

$\lambda^0, \lambda^0 \cdot \chi_{(-\infty,t_1)} \oplus \lambda^1 \cdot \chi_{[t_1,\infty)}, \lambda^0 \cdot \chi_{(-\infty,t_1)} \oplus \lambda^1 \cdot \chi_{[t_1,t_2)} \oplus \lambda^2 \cdot \chi_{[t_2,\infty)}, ... \in U$ *and*

$$\forall x \in f(\lambda^0), x_{|(-\infty,t_0)} = \mu^0 \text{ and } x_{|[t_1,\infty)} = \mu^1,$$

$$\forall x \in f(\lambda^0 \cdot \chi_{(-\infty,t_1)} \oplus \lambda^1 \cdot \chi_{[t_1,\infty)}), x_{|[t_2,\infty)} = \mu^2,$$

$$\forall x \in f(\lambda^0 \cdot \chi_{(-\infty,t_1)} \oplus \lambda^1 \cdot \chi_{[t_1,t_2)} \oplus \lambda^2 \cdot \chi_{[t_2,\infty)}), x_{|[t_3,\infty)} = \mu^3,$$

$$...$$

then u is a fundamental mode of f.

PROOF. This is a special case of the previous theorem when $u_{t_1} = \lambda^0$, $u_{t_2} = \lambda^0 \cdot \chi_{(-\infty,t_1)} \oplus \lambda^1 \cdot \chi_{[t_1,\infty)}$, $u_{t_3} = \lambda^0 \cdot \chi_{(-\infty,t_1)} \oplus \lambda^1 \cdot \chi_{[t_1,t_2)} \oplus \lambda^2 \cdot \chi_{[t_2,\infty)}, ...$ □

REMARK 91. *Theorem 241 gives a new perspective on the fundamental mode: when* $\forall k \geq 1$, *the stabilization of x to the value* $x(t_k)$ *is a direct consequence of the fact that, before* t_k, *u has stabilized to the value* $u(t_k - 0)$. *Thus, at the time instants* $t_1, t_2, t_3, ...$, *u and all* $x \in f(u)$ *are in equilibrium*

$$\forall k \geq 1, \forall t \geq t_k, u_{t_k}(t) = u(t_k - 0) \text{ and } \forall x \in f(u_{t_k}), x(t) = x(t_k)$$

and we consider the equilibrium be true at the time instant t_0 *also under the form*

$$\forall t < t_0, u(t) = u(t_0 - 0) \text{ and } \forall x \in f(u_{t_1}), x(t) = x(t_0 - 0)$$

by a suitable choice of t_0.

The situation described in Theorem 241 includes the possibilities $\exists k \geq 1, u_{t_k} = u_{t_{k+1}}$ *and* $\exists k \geq 1, u = u_{t_k}$ *respectively.*

Corollary 2 represents that special case of Theorem 241, when u is constant in the intervals $(-\infty, t_1)$, $[t_1, t_2)$, $[t_2, t_3), ...$

The following theorem is an adaptation of Theorem 238 to the present context.

THEOREM 242. *Let the non-anticipatory system f be given and let $H\subset \mathbf{B}^m$ be a non-empty set. If*

a) $U=\{\lambda^0\cdot\chi_{(-\infty,t_1)}\oplus\lambda^1\cdot\chi_{[t_1,t_2)}\oplus\lambda^2\cdot\chi_{[t_2,t_3)}\oplus ...|(\lambda^k)\in H,(t_k)\in Seq\}$;

b) f has race-free initial states with a bounded initial time

$$\forall u\in U,\exists\mu\in\mathbf{B}^n,\exists t_0\in\mathbf{R},\forall x\in f(u),x_{|(-\infty,t_0)}=\mu;$$

c) f is relatively race-free stable with a bounded final time

$$\forall u\in U\cap S_c^{(m)},\exists\mu'\in\mathbf{B}^n,\exists t_1\in\mathbf{R},\forall x\in f(u),x_{|[t_1,\infty)}=\mu',$$

then for any $(\lambda^k)\in H$, there are the time instants $(t_k)\in Seq$ such that the input

$$u=\lambda^0\cdot\chi_{(-\infty,t_1)}\oplus\lambda^1\cdot\chi_{[t_1,t_2)}\oplus\lambda^2\cdot\chi_{[t_2,t_3)}\oplus ...$$

is a fundamental mode of f.

PROOF. We just remark that the closure property of Theorem 238, a) is fulfilled and that $\lambda^0,\lambda^0\cdot\chi_{(-\infty,t_1)}\oplus\lambda^1\cdot\chi_{[t_1,\infty)},\lambda^0\cdot\chi_{(-\infty,t_1)}\oplus\lambda^1\cdot\chi_{[t_1,t_2)}\oplus\lambda^2\cdot\chi_{[t_2,\infty)},...\in U\cap S_c^{(m)}$, for any $(\lambda^k)\in H$ and any $(t_k)\in Seq$. The proof is similar with that of Theorem 238. □

9. Accessibility vs fundamental mode

THEOREM 243. *Let be the non-anticipatory system $f:U\rightarrow P^*(S^{(n)}),U\in P^*(S^{(m)})$ and we suppose that the following requirements are fulfilled:*

a) for any $(t_k)\in Seq$ and any $(u^k)\in U$, we have $u^0\cdot\chi_{(-\infty,t_0)}\oplus u^1\cdot\chi_{[t_0,t_1)}\oplus u^2\cdot\chi_{[t_1,t_2)}\oplus ...\in U$;

b) f has race-free initial states and a bounded initial time, i.e.

$$\forall u\in U,\exists\mu\in\mathbf{B}^n,\exists t\in\mathbf{R},\forall x\in f(u),x_{|(-\infty,t)}=\mu;$$

c) any vector from $\mathbf{B}^n$ is a final state under an input having arbitrary initial segment

$$\forall\mu\in\mathbf{B}^n,\forall u\in U,\forall t\in\mathbf{R},\exists v\in U,\exists t'>t,$$
$$u_{|(-\infty,t)}=v_{|(-\infty,t)}\text{ and }\forall y\in f(v),y_{|[t',\infty)}=\mu.$$

Then there is some $\mu^0\in\mathbf{B}^n$ such that for any sequence $\mu^k\in\mathbf{B}^n,k\geq 1$ of binary vectors, there are a sequence $(t_k)\in Seq$ and an input $\widetilde{u}\in U$ such that $\mu^0\overset{\widetilde{u}_{|(-\infty,t_1)}}{\longrightarrow}\mu^1$, $\mu^1\overset{\widetilde{u}_{|[t_1,t_2)}}{\longrightarrow}\mu^2$, $\mu^2\overset{\widetilde{u}_{|[t_2,t_3)}}{\longrightarrow}\mu^3$, ... are fundamental transfers.

PROOF. Let $v^0\in U$ be an arbitrary input. From b) we get the existence of $\mu^0\in\mathbf{B}^n$ and $t_0\in\mathbf{R}$ depending on v^0, such that

$$\forall x\in f(v^0),x_{|(-\infty,t_0)}=\mu^0. \tag{9.1}$$

Fix the sequence $\mu^k\in\mathbf{B}^n,k\geq 1$ and an arbitrary number $\delta>0$. At this moment the property c) implies the existence of $u^0\in U$ and $t_1>t_0+\delta$ such that

$$v^0_{|(-\infty,t_0)}=u^0_{|(-\infty,t_0)}\text{ and }\forall x\in f(u^0),x_{|[t_1,\infty)}=\mu^1,$$

of $u^1\in U$ and $t_2>t_1+\delta$ such that

$$u^0_{|(-\infty,t_1)}=u^1_{|(-\infty,t_1)}\text{ and }\forall x\in f(u^1),x_{|[t_2,\infty)}=\mu^2,$$

of $u^2\in U$ and $t_3>t_2+\delta$ such that

$$u^1_{|(-\infty,t_2)}=u^2_{|(-\infty,t_2)}\text{ and }\forall x\in f(u^2),x_{|[t_3,\infty)}=\mu^3,$$

...

Obviously, the transfers $\mu^0 \xrightarrow{u^0_{|(-\infty,t_1)}} \mu^1$, $\mu^1 \xrightarrow{u^1_{|[t_1,t_2)}} \mu^2$, $\mu^2 \xrightarrow{u^2_{|[t_2,t_3)}} \mu^3, ...$ are fundamental.

The way that (t_k) was constructed guarantees the fact that this sequence belongs to Seq. Thus, by a), the input $\widetilde{u}$ defined as

$$\widetilde{u} = u^0 \cdot \chi_{(-\infty,t_1)} \oplus u^1 \cdot \chi_{[t_1,t_2)} \oplus u^2 \cdot \chi_{[t_2,t_3)} \oplus ...$$

belongs to U. We have

$$\widetilde{u}_{|(-\infty,t_1)} = u^0_{|(-\infty,t_1)},\ \widetilde{u}_{|(-\infty,t_2)} = u^1_{|(-\infty,t_2)},\ \widetilde{u}_{|(-\infty,t_3)} = u^2_{|(-\infty,t_3)}, ...$$

wherefrom we infer that the transfers $\mu^0 \xrightarrow{\widetilde{u}_{|(-\infty,t_1)}} \mu^1$, $\mu^1 \xrightarrow{\widetilde{u}_{|[t_1,t_2)}} \mu^2$, $\mu^2 \xrightarrow{\widetilde{u}_{|[t_2,t_3)}} \mu^3, ...$ equal to $\mu^0 \xrightarrow{u^0_{|(-\infty,t_1)}} \mu^1$, $\mu^1 \xrightarrow{u^1_{|[t_1,t_2)}} \mu^2$, $\mu^2 \xrightarrow{u^2_{|[t_2,t_3)}} \mu^3, ...$ (see Theorem 231) are fundamental. $\square$

THEOREM 244. *Let the non-anticipatory system $f : U \to P^*(S^{(n)})$ be given and suppose that the conditions:*

a) for any $(t_k) \in Seq$ and any sequence $(u^k) \in U$ of inputs, we have $u^0 \cdot \chi_{(-\infty,t_0)} \oplus u^1 \cdot \chi_{[t_0,t_1)} \oplus u^2 \cdot \chi_{[t_1,t_2)} \oplus ... \in U$;

b) f has race-free initial states and a bounded initial time

$$\forall u \in U, \exists \mu \in \mathbf{B}^n, \exists t \in \mathbf{R}, \forall x \in f(u), x_{|(-\infty,t)} = \mu;$$

c) the vectors from $\mathbf{B}^n$ are accessible final states in the following manner

$$\forall \mu \in \mathbf{B}^n, \forall u \in U, \forall t \in \mathbf{R}, \exists \lambda \in U, \exists t' > t,$$

$$\forall y \in f(u \cdot \chi_{(-\infty,t)} \oplus \lambda \cdot \chi_{[t,\infty)}), y_{|[t',\infty)} = \mu$$

are fulfilled (we have identified $\lambda \in \mathbf{B}^m$ with the constant input $\lambda \in U$). Then there is $\mu^0 \in \mathbf{B}^n$ such that for any sequence $\mu^k \in \mathbf{B}^n, k \geq 1$ of binary vectors, there are the time instants $(t_k) \in Seq$ and the constants $(\lambda^k) \in \mathbf{B}^m$ with the property that $\mu^0 \xrightarrow{\widetilde{u}_{|(-\infty,t_1)}} \mu^1$, $\mu^1 \xrightarrow{\widetilde{u}_{|[t_1,t_2)}} \mu^2$, $\mu^2 \xrightarrow{\widetilde{u}_{|[t_2,t_3)}} \mu^3, ...$ are fundamental transfers; we have denoted

$$\widetilde{u} = \lambda^0 \cdot \chi_{(-\infty,t_1)} \oplus \lambda^1 \cdot \chi_{[t_1,t_2)} \oplus \lambda^2 \cdot \chi_{[t_2,t_3)} \oplus ...$$

PROOF. Special case of Theorem 243. $\square$

THEOREM 245. *Suppose that the non-anticipatory system f is given, such that*

a) for any $(t_k) \in Seq$ and any $(u^k) \in U$, we have $u^0 \cdot \chi_{(-\infty,t_0)} \oplus u^1 \cdot \chi_{[t_0,t_1)} \oplus u^2 \cdot \chi_{[t_1,t_2)} \oplus ... \in U$;

b) f has race-free initial states and a bounded initial time

$$\forall u \in U, \exists \mu \in \mathbf{B}^n, \exists t \in \mathbf{R}, \forall x \in f(u), x_{|(-\infty,t)} = \mu;$$

c) f has accessible final states and a bounded time under the form

$$\exists \delta > 0, \forall \mu \in \mathbf{B}^n, \forall u \in U, \forall t \in \mathbf{R}, \exists v \in U, \exists t' \in (t, t+\delta),$$

$$u_{|(-\infty,t)} = v_{|(-\infty,t)} \text{ and } \forall y \in f(v), y_{|[t',\infty)} = \mu.$$

Then there are $\delta > 0$ and $\mu^0 \in \mathbf{B}^n$ such that for any sequence $\mu^k \in \mathbf{B}^n, k \geq 1$, we have the existence of $t_0 \in \mathbf{R}$ and $\widetilde{u} \in U$ with the property that $\mu^0 \xrightarrow{\widetilde{u}_{|(-\infty,t_0+\delta)}} \mu^1$, $\mu^1 \xrightarrow{\widetilde{u}_{|[t_0+\delta,t_0+2\delta)}} \mu^2$, $\mu^2 \xrightarrow{\widetilde{u}_{|[t_0+2\delta,t_0+3\delta)}} \mu^3, ...$ are fundamental transfers.

PROOF. Similar to that of Theorem 243. □

THEOREM 246. *The non-anticipatory system f satisfies the requirements:*
a) f has race-free initial states and a bounded initial time

$$\forall u \in U, \exists \mu \in \mathbf{B}^n, \exists t \in \mathbf{R}, \forall x \in f(u), x_{|(-\infty,t)} = \mu;$$

b) the vectors from $\mathbf{B}^n$ are accessible final states

$$\forall \mu \in \mathbf{B}^n, \exists u \in U, \exists t \in \mathbf{R}, \forall x \in f(u), x_{|[t,\infty)} = \mu.$$

Then

$$\forall \mu' \in \mathbf{B}^n, \exists \mu \in \mathbf{B}^n, \exists u \in U, \exists t_0 \in \mathbf{R}, \exists t_1 > t_0,$$
$$\forall x \in f(u), x_{|(-\infty,t_0)} = \mu \text{ and } x_{|[t_1,\infty)} = \mu'$$

i.e. for any μ', we have the existence of μ, u, t_0 and $t_1 > t_0$, such that $\mu \overset{u_{|(-\infty,t_1)}}{\longrightarrow} \mu'$ is initial fundamental.

PROOF. Let $\mu' \in \mathbf{B}^n$ be arbitrary and fixed. Item *b*) shows the existence of $u \in U$ and $t_1 \in \mathbf{R}$, such that

$$\forall x \in f(u), x_{|[t_1,\infty)} = \mu'.$$

Because of item a), we infer the existence of $\mu \in \mathbf{B}^n$ and $t_0 \in \mathbf{R}$ that can be chosen $< t_1$ with

$$\forall x \in f(u), x_{|(-\infty,t_0)} = \mu.$$

□

REMARK 92. *In the theorems of this section occurred the following accessibility properties:*

a) any vector from $\mathbf{B}^n$ is a final state under an input having an arbitrary initial segment

$$\forall \mu \in \mathbf{B}^n, \forall u \in U, \forall t \in \mathbf{R}, \exists v \in U, \exists t' > t,$$
$$u_{|(-\infty,t)} = v_{|(-\infty,t)} \text{ and } \forall y \in f(v), y_{|[t',\infty)} = \mu;$$

b) version of a), where the access in a final state is made under a constant input

$$\forall \mu \in \mathbf{B}^n, \forall u \in U, \forall t \in \mathbf{R}, \exists \lambda \in U, \exists t' > t,$$
$$\forall y \in f(u \cdot \chi_{(-\infty,t)} \oplus \lambda \cdot \chi_{[t,\infty)}), y_{|[t',\infty)} = \mu;$$

c) version of a), where the access in a final state is made in the following way

$$\exists \delta > 0, \forall \mu \in \mathbf{B}^n, \forall u \in U, \forall t \in \mathbf{R}, \exists v \in U, \exists t' \in (t, t+\delta),$$
$$u_{|(-\infty,t)} = v_{|(-\infty,t)} \text{ and } \forall y \in f(v), y_{|[t',\infty)} = \mu;$$

d) version of a), where the inputs under which the vectors from $\mathbf{B}^n$ are final states do not have an arbitrary initial segment

$$\forall \mu \in \mathbf{B}^n, \exists u \in U, \exists t \in \mathbf{R}, \forall x \in f(u), x_{|[t,\infty)} = \mu.$$

We have the implications

$$\begin{array}{ccccc} b) & \Longrightarrow & a) & \Longrightarrow & d) \\ & & \Uparrow & & \\ & & c) & & \end{array}$$

10. The fundamental mode relative to a function

DEFINITION 95. *Let be the system* $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$ *and the Boolean function* $F : \mathbf{B}^m \to \mathbf{B}^n$. *If the following property is fulfilled:* $\forall t \in \mathbf{R}$, $\forall u \in U$, $\forall v \in U$,

$$\forall \xi < t, F(u(\xi)) = F(v(\xi)) \Longrightarrow \{x_{|(-\infty,t]} | x \in f(u)\} = \{y_{|(-\infty,t]} | y \in f(v)\}$$

we say that f *is* ***non-anticipatory relative to the function*** F.

THEOREM 247. *Let be* f, F *and consider the following statements:*
i) f *is non-anticipatory (in the sense of Definition 64);*
ii) f *is non-anticipatory relative to* F;
iii) f *is an autonomous system.*
Then:
a) the implications iii)$\Longrightarrow$*ii)*$\Longrightarrow$*i) are true;*
b) if F *is injective, then i)*$\Longrightarrow$*ii) takes place;*
c) if $F = \mu$ *is the constant function,* $\mu \in \mathbf{B}^n$, *then we have ii)*$\Longrightarrow$*iii).*

PROOF. a) Suppose that $f = X$ is autonomous, $X \in P^*(S^{(n)})$. We get that $\forall t \in \mathbf{R}, \forall u \in U, \forall v \in U$,

$$\{x_{|(-\infty,t]} | x \in f(u)\} = \{x'_{|(-\infty,t]} | x' \in X\} = \{y_{|(-\infty,t]} | y \in f(v)\}$$

represents the conclusion of implication ii), thus iii)$\Longrightarrow$ii). Furthermore, we fix t, u, v arbitrarily, such that

$$u_{|(-\infty,t)} = v_{|(-\infty,t)}, \tag{10.1}$$

$$F(u(\cdot))_{|(-\infty,t)} = F(v(\cdot))_{|(-\infty,t)} \Longrightarrow \tag{10.2}$$
$$\Longrightarrow \{x_{|(-\infty,t]} | x \in f(u)\} = \{y_{|(-\infty,t]} | y \in f(v)\}.$$

From (10.1) we infer

$$F(u(\cdot))_{|(-\infty,t)} = F(v(\cdot))_{|(-\infty,t)}, \tag{10.3}$$

thus, from (10.2), we get

$$\{x_{|(-\infty,t]} | x \in f(u)\} = \{y_{|(-\infty,t]} | y \in f(v)\}. \tag{10.4}$$

In this way ii)$\Longrightarrow$i) is proved.

b) Let $t \in \mathbf{R}, u \in U, v \in U$ be arbitrary with the property that

$$u_{|(-\infty,t)} = v_{|(-\infty,t)} \Longrightarrow \{x_{|(-\infty,t]} | x \in f(u)\} = \{y_{|(-\infty,t]} | y \in f(v)\} \tag{10.5}$$

and (10.3) are true. Suppose also that F is injective. From (10.3) and from the injectivity of F, (10.1) is inferred and, by taking into account (10.5), we get that (10.4) is also true. i) has implied ii).

c) Let $t \in \mathbf{R}, u \in U, v \in U$ be arbitrary such that (10.2) holds true. Then the equation (10.3) is also fulfilled, from the fact that F is constant. The conclusion of (10.2) is true, thus (10.4) holds true. As t, u, v were arbitrary, we have obtained that $\forall u \in U, \forall v \in U, f(u) = f(v)$, thus f is autonomous. □

DEFINITION 96. *Suppose that the system* f *is non-anticipatory relative to the function* F *and let be* $u \in U$ *such that there are* $(t_k) \in Seq$, $(u^k) \in U$ *and* $\mu^0 \in \mathbf{B}^n$ *satisfying the properties*

$$\forall x \in f(u^0), x_{|(-\infty,t_0)} = \mu^0 \text{ and } x_{|[t_1,\infty)} = F(u(t_1 - 0)),$$

$$\forall k \in \mathbf{N}, \forall \xi \in \mathbf{R}, F(u^k(\xi)) = \begin{cases} F(u(\xi)), \xi < t_{k+1} \\ F(u(t_{k+1}-0)), \xi \geq t_{k+1} \end{cases},$$

$$\forall k \geq 1, \forall x \in f(u^k), x_{|[t_{k+1},\infty)} = F(u(t_{k+1}-0)).$$

Then we say that the input u is a ***fundamental (operating) mode (of f) relative to*** *F.*

REMARK 93. *Let be the function $u \in U$ and the number $t \in \mathbf{R}$. We note that the functions $v \in U$ having the property that*

$$\forall \xi \in \mathbf{R}, F(v(\xi)) = \begin{cases} F(u(\xi)), \xi < t \\ F(u(t-0)), \xi \geq t \end{cases}$$

act here as prefixes of u. In other words, v is the prefix of u relative to F. Definition 96 follows the idea from Theorem 241, where $\mu^k = F(u(t_k - 0)), k \geq 1$ We note that those u^k equal to $u_{t_{k+1}}, k \in \mathbf{N}$ become prefixes of u relative to F here.

In general, the fact that u is a fundamental mode of f relative to F consists in the non-anticipation of f relative to F and in the existence of $(t_k) \in Seq, \mu^0 \in \mathbf{B}^n$, such that $\mu^0 \overset{u_{|(-\infty,t_1)}}{\longrightarrow} F(u(t_1 - 0))$, $F(u(t_1 - 0)) \overset{u_{|[t_1,t_2)}}{\longrightarrow} F(u(t_2 - 0))$, $F(u(t_2 - 0)) \overset{u_{|[t_2,t_3)}}{\longrightarrow} F(u(t_3 - 0))$,... are fundamental transfers.

We have the special case when F is injective and (Theorem 247) the non-anticipation of f relative to F coincides with the non-anticipation of f.

Another special case is the one when $F = \mu$ is the constant function; then (Theorem 247) f is autonomous and there are $(t_k) \in Seq, \mu^0 \in \mathbf{B}^n$ such that $\mu^0 \overset{u_{|(-\infty,t_1)}}{\longrightarrow} \mu$, $\mu \overset{u_{|[t_1,t_2)}}{\longrightarrow} \mu$, $\mu \overset{u_{|[t_2,t_3)}}{\longrightarrow} \mu$,... are fundamental transfers for any $u \in U$.

We state the version of Theorem 238 valid in this context.

THEOREM 248. *Let be the function F and the system f non-anticipatory relative to F. Suppose that the following properties are fulfilled:*

a) for any $(t_k) \in Seq$ and any sequence $(u^k) \in U$ of inputs, we have $u^0 \cdot \chi_{(-\infty,t_0)} \oplus u^1 \cdot \chi_{[t_0,t_1)} \oplus u^2 \cdot \chi_{[t_1,t_2)} \oplus ... \in U$;

b) race-free initialization with a bounded initial time

$$\forall u \in U, \exists \mu \in \mathbf{B}^n, \exists t \in \mathbf{R}, \forall x \in f(u), x_{|(-\infty,t)} = \mu;$$

c) $F-$relative race-free stability with a bounded final time

$$\forall u \in U \cap S_{F,c}^{(m)}, \exists t \in \mathbf{R}, \forall x \in f(u), x_{|[t,\infty)} = \lim_{\xi \to \infty} F(u(\xi)).$$

Then for any sequence $(u^k) \in U \cap S_{F,c}^{(m)}$ of inputs, there are the time instants $(t_k) \in Seq$ such that the input

$$\widetilde{u} = u^0 \cdot \chi_{(-\infty,t_1)} \oplus u^1 \cdot \chi_{[t_1,t_2)} \oplus ... \oplus u^k \cdot \chi_{[t_k,t_{k+1})} \oplus ...$$

is a fundamental mode of f relative to F.

Part 2

Delay theory

CHAPTER 10

Delays

The delays are the asynchronous systems $f : S \to P^*(S)$ having the property of race-free stability relative to the identity $1_{\mathbf{B}} : \mathbf{B} \to \mathbf{B}$. They represent the mathematical models of the delay circuits. Delay theory is the mathematical theory that considers the delay circuit as the fundamental circuit in digital electronics. The modeling is made by using delays and Boolean functions. While the asynchronous systems theory gives models at a synthetical level, with functional blocks, here the most detailed level of modeling is considered, starting with the delays that occur in gates and wires. The chapter surveys the traditional description of delays, then some particular delays and examples.

1. Introduction. The delay circuit

REMARK 94. *Because in delay theory the fundamental circuit of digital electrical engineering is considered the delay circuit, also called the **delay buffer**, the purpose of this introduction is that of giving some informal explanations on that circuit.*

*The delay circuit is the circuit that computes the identity $1_{\mathbf{B}} : \mathbf{B} \to \mathbf{B}$. Let $f : S \to P^*(S)$ be a system that models it. With the common identification of the model and the modeled circuit, $u \in S$ is called the **input** of the circuit and any $x \in f(u)$ is called its (**possible**) **state**.*

Note that any 1-dimensional signal is an admissible input for the delay buffer and this is why the attribute 'admissible' given to the input is generally omitted. The fact that all signals are admissible inputs does not mean necessarily that the engineer that handles the circuit has the possibility to use any signal as input, but rather that the circuit has the capability to respond to any signal when applied to its input.

Given the real parameters $0 < d_{r,\min} \leq d_{r,\max}$, $0 < d_{f,\min} \leq d_{f,\max}$, the meaning of the indexes 'r' and 'f' is that of 'raise' (switch of a signal from 0 to 1) and 'fall' (switch of a signal from 1 to 0).

Our analysis starts with the statement that

$$\forall x \in f(0), x = 0,$$

i.e. 0 is a point of equilibrium of f under the null input.

The input satisfying $u_{|(-\infty,t_0)} = 0$ switches at the time instant t_0 from 0 to 1. Let be $t_1 > t_0$ with the property

$$\forall \xi \in [t_0, t_1), u(\xi) = 1.$$

The states of the system do not switch simultaneously with the input

$$\forall x \in f(u), x(t_0) = 0.$$

We have a look at the values that the functions $x \in f(u)$ take in t_1.

a) $0 < t_1 - t_0 < d_{r,\min}$; *at* t_1, *the state of the circuit is necessarily null:*

$$t_0 < t_1 < t_0 + d_{r,\min} \text{ and } \forall x \in f(u), x(t_1) = 0.$$

The interpretation is that the circuit's inertia did not allow such a fast switch of the state from 0 *to* 1 *to occur.*

b) $d_{r,\min} \leq t_1 - t_0 < d_{r,\max}$*; at* t_1, *the state of the circuit may be* 0 *or* 1 :

$$t_0 + d_{r,\min} \leq t_1 < t_0 + d_{r,\max} \text{ and } \exists x \in f(u), x(t_1) = 0 \text{ and } \exists x \in f(u), x(t_1) = 1.$$

An uncertainty occurs here (generated by several causes).

c) $t_1 - t_0 \geq d_{r,\max}$*; at* t_1, *the delay circuit has necessarily the state value equal to* 1 :

$$t_1 \geq t_0 + d_{r,\max} \text{ and } \forall x \in f(u), x(t_1) = 1.$$

The cause (the input equal to 1*) was sufficiently persistent in order to have an effect (the state of the circuit is necessarily* 1*).*

The intuitive description of the circuit continues by asking that the statements dual to the previous ones take place, as follows by the replacement of 'r',0,1 by 'f',1,0.

The circuit computes the identity on $\mathbf{B}$ *because if* u *is constant for a sufficiently long time,* x *becomes constant also and equal to* u.

A manner of describing the previous facts is given by the following system

$$\bigcap_{\xi \in [t-d_{r,\max},t)} u(\xi) \leq x(t) \leq \bigcup_{\xi \in [t-d_{f,\max},t)} u(\xi),$$

$$\overline{x(t-0)} \cdot x(t) \leq \bigcap_{\xi \in [t-d_{r,\min},t)} u(\xi),$$

$$x(t-0) \cdot \overline{x(t)} \leq \bigcap_{\xi \in [t-d_{f,\min},t)} \overline{u(\xi)}$$

even if this is not quite obvious for the moment.

Remark that several aspects of non-anticipation and time invariance have accompanied the previous modeling of the delay circuit.

From a historical point of view, equations and inequalities written with $\mathbf{R} \to \mathbf{B}$ *functions that model the asynchronous circuits are dated the 80's. The paper* [**36**] *anticipated the birth of delay theory*[1] *and* [**40**] *is the work where this theory was exposed for the first time.*

2. An overview of delays: informal definitions

REMARK 95. *In this section we present some aspects of the intuition situated behind delay theory.*

By the word 'delay' two things are understood: a real non-negative number and a logical condition (in the sense of model). Usually in definitions they occur together, since a complete separation is very difficult. Our present purpose is to

[1] The facts presented there were written under the impression of what the author has called 'the inertia's paradox', i.e. the non-formalized theories dedicated to the models of the asynchronus circuits lead to the possibility that two inertial delay buffers connected in series are not an inertial delay buffer. The 'paradox' was solved later when we have outlined the existence of several types of inertia and only part of them (such as the bounded delays) have the property of closure under serial connection.

show how the real numbers are specified by some informal rules of modeling. We have the

Classification of delays*. In other words, the terminology is the following. As numbers, the delays are:*

i) transport delays [**13**], [**28**], *or transmission delays for transitions* [**15**], [**16**]*;*

ii) inertial delays [**13**], *or thresholds true for cancellation* [**15**], [**16**].

The classification of the delays as logical conditions follows. These, the so-called delay conditions, define models and idioms like 'fixed delay' will often be used as a short form for 'fixed delay condition' or 'fixed delay property' or 'fixed delay model'. Thus, from the timing properties point of view, we have:

j) unbounded delays;

jj) bounded delays;

jjj) fixed delays,

while from the memory properties point of view, the delays are:

1) pure delays, i.e. delays without memory;

2) inertial delays, i.e. delays with memory.

A special attention will be given to avoiding the confusions due to the terminological abuse remarked at ii) and 2).

In the following chapters by delay we shall (also) mean an asynchronous system $f : S \to P^*(S)$ *that is race-free stable relative to the identical function* $1_{\mathbf{B}}$, *the model of the delay circuit. The classification of those systems will obey the classification of the logical conditions j), jj), jjj); 1), 2).*

Now we give the informal definitions of these concepts.

DEFINITION 97. *(informal) The* ***transport delays*** [**13**], [**28**] *represent the 'time interval*[2] *between a transition in an input to the gate and a corresponding output transition. If the output transition occurs from 0 to 1, the delay is rising, otherwise falling'.*

DEFINITION 98. *(informal) The* ***inertial delays*** *represent* [**13**] *the 'minimum amount of time during which an input signal must persist to affect a change at the output'.*

In [**17**] *the same notion is called the* ***latency delay*** *while in* [**28**], *by making use of the VHDL hardware description language, the* ***reject time****. In* [**8**] *the terminology is that of* ***threshold period****.*

REMARK 96. *We have the following* ***Convention****: the distinct numbers transport delay and inertial delay are generally taken to be equal* [**17**] *when the last exists, i.e. in the presence of inertia. We quote the following opinion* [**15**], [**16**]*: 'A common form of implementation of the inertial delay model' (here we refer to 2)) 'is the one in which the transmission delay for transitions d is the same as the threshold for cancellation. In other words, when a transition appears at the input' (the statement means that* $\overline{u(t-0)} \cdot u(t) = 1$ *or* $u(t-0) \cdot \overline{u(t)} = 1$*), 'the transition will appear at the output after d' (determinism plus* $\overline{x(t+d-0)} \cdot x(t+d) = 1$ *or* $x(t+d-0) \cdot \overline{x(t+d)} = 1$*) 'unless a second transition occurs within that period' (i.e. only if* $\forall \xi \in (t, t+d), Du(\xi) = 0$*).*

DEFINITION 99. *(informal) The unbounded delay is defined by:*

a) [**8**]*: 'a delay may take on any finite value';*

[2] The definition does not refer to a time interval, but to its length.

b) [**14**]*: 'no bound on the magnitude is known a priori, except that it is positive and finite'.*

REMARK 97. *The quotations from the previous definition refer to the delay-number in the sense i) of transport delay from our classification.*

*We mention the existence of a similar idea of unboundedness, called the **finitary weak-fairness** that is defined*[3] *in discrete time by: 'for every run of the system*[4]*, there is an unknown bound k such that no enabled transition is postponed for more than k consecutive times.'*

The unbounded delay model is generally evaluated [**6**] *to be 'robust to delay variations', but 'unrealistically conservative'. In other words, it is too general.*

DEFINITION 100. *(informal) The **bounded delay** (in* [**28**] *called the **min-max delay**) is defined as:*

a) [**8**]*: 'a delay may have any value in a given time interval';*

b) [**14**]*: a delay is bounded 'if an upper and lower bound on its magnitude are known before the synthesis or analysis of the circuit is begun';*

c) [**6**]*: 'every component is assumed to have an uncertain delay, that lies between given upper and lower bounds. The delay bounds take into account potential delay variations due to statistical fluctuations in the fabrication process, variations in ambient temperature, power supply etc.';*

d) [**13**]*: 'In practice, manufactured circuits of the same design may have different gate delays due to manufacturing fluctuations in delay related parameters such as capacitance, resistivity and transistors sizes. To be practical, we need to provide an analysis for not just a manufactured instance of a design but the entire family of manufactured circuits of the same design. To model manufacturing uncertainties, we assume the gate delays to be variable within closed intervals. Therefore a complete delay analysis determines the delays of circuits with variable gate delays...'*

REMARK 98. *The bounded delay model is considered to be the most realistic of the three: unbounded, bounded and fixed.*

On the other hand, non-conflicting differences occur in the approaches when defining and using the bounded delays in different variants: from having no lower bounds or upper bounds, the poorest case of the bounded delay model, to having four such bounds $d_{r,\min} \leq d_{r,\max}$, $d_{f,\min} \leq d_{f,\max}$, the richest case of bounded delay that makes use of the distinction between the rising and the falling delays. The more detailed the model is, the more difficult is its handling and the more realistic are its results.

DEFINITION 101. *(informal) The **fixed delay** is a special case of bounded delay, when it 'is assumed to have a fixed value'* [**8**] *and the lower bounds of the delays are equal to the upper bounds of the delays making the delay be fixed, known.*

REMARK 99. *The fixed delay model is considered to be very unrealistic, in the sense that small variations of the delays due to variations in the ambient temperature, power supply,... cause great, unacceptable differences between the model and the modeled circuit.* [**6**]*: 'Since it is almost impossible to obtain a precise delay of a component in a chip, this is not a realistic model for timing verification purpose'.*

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[3] Rajeev Alur, Thomas Henzinger, Finitary Fairness, Proc. of the Ninth Annual IEEE Symposium on Logic in Computer Science, LICS 1994, pp 52-61

[4] something like $\forall x \in f(u)$

DEFINITION 102. *(informal) The **pure delay**, or **ideal delay** is defined like this:*

a) [**14**]*: a delay is considered to be pure 'if it transmits each event on its input to its output*[5]*, i.e. it corresponds to a pure translation in time of the input waveform';*

b) [**6**]*: 'a pure delay simply shifts a waveform in time without altering its shape'.*

The same idea is found in [**13**]*, where the pure delay timed Boolean functions are defined.*

REMARK 100. [**16**] *refers to the pure delays, by considering that 'This model is unrealistic in the sense that practical gates will not transit a pulse caused by two transitions very close together whereas the model guarantees that every transition will be at the output irrespective of the proximity of the successive pulses'.*

DEFINITION 103. *(informal) The **inertial delay** (or **latency delay**) has generated the most controversies, see also* [**6**], [**13**]*. The following opinions are generally accepted:*

a) the inertial delays [**15**], [**16**] *'model the fact that the practical circuits will not respond to two transitions which are very close together. The inertial delay model is one in which input transitions are replicated at the output after some period of time unless two transitions occur at the input within some defined period, in which case neither transition is transmitted';*

b) [**14**]*: 'pulses shorter than or equal to the delay magnitude are not transmitted', see also the Convention in Remark 96, since here by delay magnitude it is understood the transport delay equal to the inertial delay;*

c) [**8**]*: 'an inertial delay has a threshold period d. Pulses of duration less than d are filtered out'. Here the inertial delay is the inertial delay model and the Convention is not necessarily true.*

DEFINITION 104. *(alternative) In* [**1**], [**17**] *the authors show intuitively what inertia is and then two variants of fixed and bounded inertial delays respectively are mentioned. We reproduce only the second variant from* [**1**]*, called there the non-deterministic inertial delay, for the reason of making the exposure as simple as possible. For the same reason, we have changed the language and the notations. The state $x^0 \in \mathbf{B}$ and the real numbers $0 \leq d_{\min} \leq d_{\max}$ are given and the requirements are*

i) $\forall t \in [0, d_{\min}), x(t) = x^0$ *(initialization);*

ii) $\forall t \geq d_{\min}, Dx(t) = 1 \Longrightarrow \exists t' \in suppDu \cap [t - d_{\max}, t - d_{\min}]$ *such that* $x(t) = u(t')$ *and* $(t', t) \cap suppDu = \emptyset$;

iii) $\forall t \in suppDu, (t, t + d_{\max}] \cap suppDu \neq \emptyset$ *or* $[t + d_{\min}, t + d_{\max}] \cap suppDx \neq \emptyset$.

REMARK 101. *In* [**1**] *we can find the following remark relative to Definition 104: 'one could assume that changes should persist for at least l_1 time units but propagated after $l_2, l_2 > l_1$ time'. In other words, for the sake of accuracy one could abandon the 'common form of implementation of the inertial delay model' from the Convention in Remark 96.*

DEFINITION 105. *(alternative) In the approach from* [**5**]*, two variants namely of fixed and of bounded inertial delays are given too. From them we reproduce the second one under the form:*

i) $Dx(t) = 1 \Longrightarrow \forall \xi \in [t - d_{\min}, t), u(\xi) = x(t)$;

[5] the 'events' on the input are $\overline{u(t-0)} \cdot u(t) = 1, u(t-0) \cdot \overline{u(t)} = 1$ and the 'events' on the output are $\overline{x(t-0)} \cdot x(t) = 1, x(t-0) \cdot \overline{x(t)} = 1$

ii) $\forall \lambda \in \mathbf{B}, ((\forall \xi \in [t, t+d_{\max}), u(\xi) = \lambda) \Longrightarrow (\exists \delta \in [t, t+d_{\max}), \forall \xi \in [\delta, t+d_{\max}), x(\xi) = \lambda))$.

REMARK 102. *We make brief comments and a comparison between Definitions 104, 105:*

- in the second definition, the initialization is missing. If we start by definition from null initial conditions, then the initialization is not necessary, while if we reason for any possible initial value, then the initialization is also missing. The possibility that the initialization is missing in Definition 104 is given by the value $d_{\min} = 0$*, too;*

- Definition 104, ii) and Definition 105, i) essentially express the same idea, even if they are obviously not equivalent. Similarly, Definition 104, iii) expresses the idea from Definition 105, ii).

A subtle problem arising here is if in our classification of the delays from Remark 95, any of i), ii) is consistent with any of j), jj), jjj) and with any of 1), 2). At the present level of the debates, we just confirm that both i), ii) are consistent with 2), as many authors show.

3. The universal delay

NOTATION 24. *For* $\lambda \in \mathbf{B}$*, we make the notation*

$$S_c(\lambda) = \{x | x \in S, \lim_{t \to \infty} x(t) = \lambda\}.$$

DEFINITION 106. *The system* $f_{UD} : S \to P^*(S)$ *defined by*

$$\forall u \in S, f_{UD}(u) = \begin{cases} S_c(0), if\ u \in S_c(0) \\ S_c(1), if\ u \in S_c(1) \\ S, if\ u \in S \setminus S_c \end{cases}$$

is called the ***universal delay*** *(**condition**, or **property**, or **model**).*

REMARK 103. *The system* f_{UD} *models the delay circuits, i.e. the circuits that compute the identity* $1_{\mathbf{B}} : \mathbf{B} \to \mathbf{B}$ *and gives the minimal information about these circuits.*

THEOREM 249. *The initial state function* $\phi_0 : S \to P^*(\mathbf{B})$ *of* f_{UD} *is constant:*

$$\forall u \in S, \phi_0(u) = \{0, 1\}.$$

THEOREM 250. *The system* f_{UD} *is self-dual.*

PROOF. We note that $S = S^*$. We have

$$\forall u \in S, f^*_{UD}(u) = \{\overline{x} | x \in f_{UD}(\overline{u})\} = \begin{cases} S_c(0)^*, if\ \overline{u} \in S_c(0) \\ S_c(1)^*, if\ \overline{u} \in S_c(1) \\ S^*, if\ \overline{u} \in S \setminus S_c \end{cases} =$$

$$= \begin{cases} S_c(1), if\ u \in S_c(1) \\ S_c(0), if\ u \in S_c(0) \\ S, if\ u \in S \setminus S_c \end{cases} = f_{UD}(u).$$

□

REMARK 104. *The inverse of* f_{UD} *is the system defined as:* $f^{-1}_{UD} : S \to P^*(S)$,

$$\forall x \in S, f^{-1}_{UD}(x) = \begin{cases} S \setminus S_c(1), if\ x \in S_c(0) \\ S \setminus S_c(0), if\ x \in S_c(1) \\ S \setminus S_c, if\ x \in S \setminus S_c \end{cases}.$$

THEOREM 251. $f_{UD} \circ f_{UD} = f_{UD}$.

PROOF. We have two possibilities.

a) If $u \in S_c(\lambda)$, $\lambda \in \mathbf{B}$, then

$$(f_{UD} \circ f_{UD})(u) = \bigcup_{y \in f_{UD}(u)} f_{UD}(y) = \bigcup_{y \in S_c(\lambda)} f_{UD}(y) =$$

$$= \bigcup_{y \in S_c(\lambda)} S_c(\lambda) = S_c(\lambda) = f_{UD}(u).$$

b) If $u \in S \setminus S_c$, then

$$(f_{UD} \circ f_{UD})(u) = \bigcup_{y \in f_{UD}(u)} f_{UD}(y) = \bigcup_{y \in S} f_{UD}(y) \supset \bigcup_{y \in S \setminus S_c} f_{UD}(y) = \bigcup_{y \in S \setminus S_c} S = S.$$

The inclusion $\forall u \in S \setminus S_c, (f_{UD} \circ f_{UD})(u) \subset S$ is obvious, thus $(f_{UD} \circ f_{UD})(u) = S$. Because

$$\forall u \in S \setminus S_c, f_{UD}(u) = S,$$

in this case the statement of the theorem is also true. □

THEOREM 252. *The system f_{UD} is time invariant.*

PROOF. Let $u \in S$ and $d \in \mathbf{R}$ be arbitrary. We can write

$$f_{UD}(u \circ \tau^d) = \begin{cases} S_c(0), if\ u \circ \tau^d \in S_c(0) \\ S_c(1), if\ u \circ \tau^d \in S_c(1) \\ S, if\ u \circ \tau^d \in S \setminus S_c \end{cases} = \begin{cases} S_c(0), if\ u \in S_c(0) \\ S_c(1), if\ u \in S_c(1) \\ S, if\ u \in S \setminus S_c \end{cases} =$$

$$= \{x \circ \tau^d | x \in \begin{cases} S_c(0), if\ u \in S_c(0) \\ S_c(1), if\ u \in S_c(1) \\ S, if\ u \in S \setminus S_c \end{cases} \} = \{x \circ \tau^d | x \in f_{UD}(u)\}.$$

□

THEOREM 253. *The system f_{UD} is non-anticipatory in the sense of Definition 64 and in the sense of all concepts of non-anticipation contained in Definition 65.*

PROOF. We show the first statement, because the others are similar. Let $u, v \in S$ and $t \in \mathbf{R}$ be arbitrary while the hypothesis states $u_{|(-\infty,t)} = v_{|(-\infty,t)}$. We have

$$\{x_{|(-\infty,t]} | x \in f_{UD}(u)\} = \{x_{|(-\infty,t]} | x \in S\} = \{y_{|(-\infty,t]} | y \in f_{UD}(v)\}.$$

□

THEOREM 254. *The system f_{UD} is non-anticipatory*.*

PROOF. Let $u, v \in S$ and $t \in \mathbf{R}$ be arbitrary, such that $u_{|[t,\infty)} = v_{|[t,\infty)}$. Because $\{x(t) | x \in f_{UD}(u)\} = \{0, 1\} = \{y(t) | y \in f_{UD}(v)\}$ is true, we have the possibilities:

a) $\exists \lambda \in \mathbf{B}, u, v \in S_c(\lambda)$. Then

$$\{x_{|[t,\infty)} | x \in f_{UD}(u)\} = \{x_{|[t,\infty)} | x \in S_c(\lambda)\} = \{y_{|[t,\infty)} | y \in f_{UD}(v)\};$$

b) $u, v \in S \setminus S_c$ and

$$\{x_{|[t,\infty)} | x \in f_{UD}(u)\} = \{x_{|[t,\infty)} | x \in S\} = \{y_{|[t,\infty)} | y \in f_{UD}(v)\}.$$

□

THEOREM 255. *The system f_{UD} fulfills the following surjectivity properties:*

$$\forall x \in S, \exists u \in S, x \in f_{UD}(u);$$

$$\forall \mu \in \mathbf{B}, \exists u \in S, \forall x \in f_{UD}(u), x(\infty - 0) = \mu.$$

THEOREM 256. *The system f_{UD} is race-free relatively stable and race-free stable relative to the identity $1_{\mathbf{B}} : \mathbf{B} \to \mathbf{B}$.*

PROOF. We note that $S_c = S_{1_{\mathbf{B}},c}$ and

$$\forall u \in S_c, \forall x \in f_{UD}(u), x(\infty - 0) = u(\infty - 0),$$

thus the two statements are true. □

4. Delays

THEOREM 257. *Let be the system $f : S \to P^*(S)$. The following statements are equivalent:*

a) $f \subset f_{UD}$;

b) f is race-free stable relative to $1_{\mathbf{B}}$.

PROOF. *a)* $\Longrightarrow$ *b)* The system f_{UD} is race-free stable relative to $1_{\mathbf{B}}$ (Theorem 256) and any of its subsystems with the support set S satisfies the same property.

b) $\Longrightarrow$ *a)* Suppose that f is race-free stable relative to $1_{\mathbf{B}}$, meaning that

$$\forall \lambda \in \mathbf{B}, \forall u \in S_c(\lambda), \forall x \in f(u), x(\infty - 0) = \lambda.$$

If $u \in S_c(0)$, then $f(u) \subset S_c(0)$; if $u \in S_c(1)$, then $f(u) \subset S_c(1)$ and if $u \in S \setminus S_c$, then $f(u) \subset S$, i.e. a) is true. □

DEFINITION 107. *If a system $f : S \to P^*(S)$ satisfies one of the previous equivalent conditions a), b), then it is called a **delay condition** or shortly a **delay**.*

REMARK 105. *The delays model the delay circuits, giving in general some more information about their behavior than f_{UD}.*

The concept of delay should be identified with that of unbounded delay from Definition 99, in spite of the fact that there is no need of positive transport delay here. This means that we agree rather with item a) than with item b) from that definition.

THEOREM 258. *Any subsystem $f : S \to P^*(S)$ of a delay g is a delay.*

DEFINITION 108. *Let be the delays f, g with $f \subset g$. Then f is called a **subdelay** of g.*

REMARK 106. *The system f_{UD} is the greatest delay relative to the inclusion $\subset$ of the delays.*

THEOREM 259. *In the presence of the axiom of choice, any delay has a deterministic subdelay.*

PROOF. See Theorem 127 and its proof. □

THEOREM 260. *In the inclusion $f \subset g$ of delays, if g is deterministic, then f is deterministic and $f = g$.*

PROOF. See Theorem 126. □

THEOREM 261. *The dual of a delay f is a delay too.*

PROOF. The set S is invariant to complements: $\forall u, u \in S \Longrightarrow \overline{u} \in S$. On the other hand, let $\lambda \in \mathbf{B}$ and $u \in S_c(\lambda)$ be arbitrary. The statement follows from the fact that

$$f^*(u) = \{\overline{x} | x \in f(\overline{u})\} \subset \{\overline{x} | x \in S_c(\overline{\lambda})\} = S_c(\lambda).$$

$\square$

THEOREM 262. *The serial connection of two delays is a delay too.*

PROOF. Given the delays $f, g : S \to P^*(S)$, the requirement for the existence of their serial connection $g \circ f : S \to P^*(S)$, $\forall u \in S, (g \circ f)(u) = \{y | \exists x \in f(u), y \in g(x)\}$ is that $\bigcup\limits_{u \in S} f(u) \subset S$ and it is fulfilled. Furthermore, for any $u \in S$, we have:

$$\begin{aligned} (g \circ f)(u) &\subset (g \circ f_{UD})(u) && \text{(Theorem 74 a))} \\ &\subset (f_{UD} \circ f_{UD})(u) && \text{(Theorem 74 b))} \\ &= f_{UD}(u). && \text{(Theorem 251)} \end{aligned}$$

$\square$

THEOREM 263. *The intersection of the delays $f, g : S \to P^*(S)$ with $\forall u \in S, f(u) \cap g(u) \neq \emptyset$ is a delay.*

PROOF. The intersection of the delays f, g as systems was defined by $f \cap g : W \to P^*(S)$, $\forall u \in W$, $(f \cap g)(u) = f(u) \cap g(u)$ where it was supposed that the set $W = \{u | u \in S \cap S, f(u) \cap g(u) \neq \emptyset\}$ is non-empty. We can see from the hypothesis that $W = S$. $f \cap g$ is a subdelay of f from Theorem 258. $\square$

REMARK 107. *The previous theorem is easily generalized in the following manner. Given $f, g : S \to P^*(S)$, where f is an arbitrary system and g is a delay such that $\forall u \in S, f(u) \cap g(u) \neq \emptyset$, the intersection $f \cap g$ is a delay. We have here the special case when $g = f_{UD}$; the delay h given by*

$$\forall u \in S, h(u) = f(u) \cap f_{UD}(u)$$

*is called the **delay defined** (or that is **induced**) **by the system** f.*

THEOREM 264. *The union of two delays is a delay too.*

PROOF. Consider the delays f, g and their union $f \cup g$. We can see that the domain of $f \cup g$ is $S \cup S = S$ and we take some arbitrary $\lambda \in \mathbf{B}$ and $u \in S_c(\lambda)$. We have $f(u) \cup g(u) \subset S_c(\lambda)$. $\square$

THEOREM 265. *Let be the deterministic delays $f_1, f_2 : S \to S$ satisfying*

$$\forall u \in S, \forall t \in \mathbf{R}, f_1(u)(t) \leq f_2(u)(t)$$

and the system $f : S \to P^*(S)$ defined by

$$\forall u \in S, f(u) = \{x | \forall t \in \mathbf{R}, f_1(u)(t) \leq x(t) \leq f_2(u)(t)\}.$$

Then f is a delay.

PROOF. Let $\lambda \in \mathbf{B}$ and $u \in S_c(\lambda)$ be arbitrary. There are $t_1, t_2 \in \mathbf{R}$ such that $f_1(u)_{|[t_1,\infty)} = f_2(u)_{|[t_2,\infty)} = \lambda$, thus $f(u) \subset S_c(\lambda)$. $\square$

REMARK 108. *There are not autonomous delays. On the other hand, the only Boolean function for which F_d is a delay, $d \in \mathbf{R}$ is $F = 1_{\mathbf{B}}$.*

5. Examples of delays

Example 95. *The deterministic delay $I_d : S \to S$ is defined for $d \in \mathbf{R}$ as:*

$$\forall u \in S, I_d(u)(t) = u(t-d).$$

Instead of I_0 we use the notation I. Remark that I is the neuter element of the serial connection of the delays: for any delay f we have

$$\forall u \in S, (f \circ I)(u) = \{y | \exists x, x = u, y \in f(x)\} = f(u),$$

$$\forall u \in S, (I \circ f)(u) = \{y | \exists x, x \in f(u), y = x\} = f(u).$$

The delay I_d has all the properties of the ideal combinational systems. Extra properties will be presented later.

Example 96. *Let be $d_1, d_2, ..., d_k \in \mathbf{R}$ and f the delay resulted by the union $I_{d_1} \cup I_{d_2} \cup ... \cup I_{d_k}$. If $d_1, d_2, ..., d_k$ are distinct, we have*

$$\forall u \in S, f(u) = \{u \circ \tau^{d_1}, u \circ \tau^{d_2}, ..., u \circ \tau^{d_k}\}.$$

Example 97. *We define the delay f by*

$$\forall u \in S, f(u)(t) = \{x | u(t-d) \le x(t) \le u(t-d) \cup u(t-d')\},$$

where $d, d' \in \mathbf{R}$, see Theorem 265.

Example 98. *We define $f : S \to S$,*

$$\forall u \in S, f(u)(t) = \lim_{\xi \to \infty} \bigcap_{\omega \in [\xi, \infty)} u(\omega).$$

First of all we note that the function in ξ : $\bigcap_{\omega \in [\xi, \infty)} u(\omega)$ is monotonous increasing, implying that the limit exists. Second, if $u \in S_c(\lambda)$ for $\lambda \in \mathbf{B}$, then $f(u) = \lambda$, thus f is a delay, indeed. We have

$$\forall u \in S, f(u)(t) = \begin{cases} 1, if \ u \in S_c(1) \\ 0, \ else \end{cases}.$$

Similarly the deterministic system $f : S \to S$,

$$\forall u \in S, f(u)(t) = \lim_{\xi \to \infty} \bigcup_{\omega \in [\xi, \infty)} u(\omega) = \begin{cases} 0, if \ u \in S_c(0) \\ 1, \ else \end{cases}$$

is a delay.

Remark 109. *Inspired by the previous example and making use of Theorem 265, we define the non-deterministic delay $f : S \to P^*(S)$ by*

$$\forall u \in S, f(u)(t) = \{x | x \in S, \lim_{\xi \to \infty} \bigcap_{\omega \in [\xi, \infty)} u(\omega) \le x(t) \le \lim_{\xi \to \infty} \bigcup_{\omega \in [\xi, \infty)} u(\omega)\}.$$

We note that f coincides with f_{UD}.

Example 99. *The deterministic system $f : S \to S$ defined by*

$$\forall u \in S, f(u)(t) = \bigcap_{\xi \in [t, \infty)} u(\xi)$$

is a delay. Indeed, for all $u \in S$ we have $f(u) \in S$ and this is proved similarly with Theorem 23. On the other hand, if $\lambda \in \mathbf{B}$ exists such that $u \in S_c(\lambda)$, then there is $t_1 \in \mathbf{R}$ such that $u_{|[t_1, \infty)} = \lambda$ and $f(u)_{|[t_1, \infty)} = \lambda$, thus $f(u) \in S_c(\lambda)$.

The deterministic system $f : S \to S$,

$$\forall u \in S, f(u)(t) = \bigcup_{\xi \in [t,\infty)} u(\xi)$$

is a delay too.

EXAMPLE 100. *From the previous example and from Theorem 265 we get that the system*

$$\forall u \in S, f(u)(t) = \{x| \bigcap_{\xi \in [t,\infty)} u(\xi) \le x(t) \le \bigcup_{\xi \in [t,\infty)} u(\xi)\}$$

is a delay.

EXAMPLE 101. *For* $0 \le m \le d$, *by Theorem 23, the function*

$$x(t) = \bigcap_{\xi \in [t-d,t-d+m]} u(\xi)$$

is a signal, thus $x(t) = f(u)(t)$ *defines a deterministic delay. Three other delays are defined by*[6]*:*

$$x(t) = \bigcap_{\xi \in [t,t+d]} u(\xi);$$

$$x(t) = \bigcup_{\xi \in [t-d,t-d+m]} u(\xi);$$

$$x(t) = \bigcup_{\xi \in [t,t+d]} u(\xi).$$

EXAMPLE 102. *The following inequalities define non-deterministic delays:*

$$\bigcap_{\xi \in [t-d,t]} u(\xi) \le x(t) \le u(t);$$

$$\bigcap_{\xi \in [t-d,t]} u(\xi) \le x(t) \le u(t-d) \cup u(t);$$

$$\bigcap_{\xi \in [t-d,t]} u(\xi) \le x(t) \le \bigcup_{\xi \in [t,t+d]} u(\xi)$$

for any $d > 0$. *The possibility* $d = 0$ *exists, when the three delays coincide with the deterministic delay* I.

EXAMPLE 103. *The system*

$$\overline{x(t-0)} \cdot x(t) = \overline{x(t-\delta_f-0)} \cdot \bigcap_{\xi \in [t-\delta_f,t)} \overline{x(\xi)} \cdot u(t), \tag{5.1}$$

$$x(t-0) \cdot \overline{x(t)} = x(t-\delta_r-0) \cdot \bigcap_{\xi \in [t-\delta_r,t)} x(\xi) \cdot \overline{u(t)}, \tag{5.2}$$

$\delta_r \ge 0, \delta_f \ge 0$, *defines a delay iff* $\delta_r = \delta_f = 0$. *In this situation it coincides with the identity* I.

---

[6]The first and the third of these four delays, in their discrete time version, are called by Moisil slow acting relays: slow closing relay and slow releasing relay.

PROOF. Suppose that $\delta_r > 0$ and let be the input $u = \chi_{[0,\delta)}$, where $\delta \in (0, \delta_r)$. We solve the system formed by the equations (5.1), (5.2).

$t < 0$. We have

$$\overline{x(t-0)} \cdot x(t) = 0, \tag{5.3}$$

$$x(t-0) \cdot \overline{x(t)} = x(t-\delta_r-0) \cdot \bigcap_{\xi \in [t-\delta_r,t)} x(\xi), \tag{5.4}$$

for which the only solution satisfies $x(t) = 0$. Indeed, the assumption that $t_0 \leq 0$ exists such that $x_{|(-\infty,t_0)} = 1$ makes (5.4) be false for $t < t_0$. The assumption that $t < 0$ exists such that $x_{|(-\infty,t)} = 0, x(t) = 1$ makes (5.3) be false in t.

$t = 0$.

$$\overline{x(0-0)} \cdot x(0) = 1, \tag{5.5}$$

$$x(0-0) \cdot \overline{x(0)} = 0, \tag{5.6}$$

thus $x(0) = 1$.

$t \in (0, \delta)$.

$$\overline{x(t-0)} \cdot x(t) = \overline{x(t-\delta_f-0)} \cdot \bigcap_{\xi \in [t-\delta_f,t)} \overline{x(\xi)}, \tag{5.7}$$

$$x(t-0) \cdot \overline{x(t)} = 0 \tag{5.8}$$

implies $x(t) = 1$.

$t \in [\delta, \delta_r]$.

$$\overline{x(t-0)} \cdot x(t) = 0, \tag{5.9}$$

$$x(t-0) \cdot \overline{x(t)} = x(t-\delta_r-0) \cdot \bigcap_{\xi \in [t-\delta_r,t)} x(\xi) \tag{5.10}$$

and, in (5.10), $x(t-\delta_r-0) = 0$, thus $x(t) = 1$.

$t > \delta_r$.

$$\overline{x(t-0)} \cdot x(t) = 0, \tag{5.11}$$

$$x(t-0) \cdot \overline{x(t)} = x(t-\delta_r-0) \cdot \bigcap_{\xi \in [t-\delta_r,t)} x(\xi). \tag{5.12}$$

Equation (5.12) with $\forall t > \delta_r, x(t-\delta_r-0) \cdot \bigcap\limits_{\xi \in [t-\delta_r,t)} x(\xi) = 1$ is contradictory and the assumption that x has a switch from 1 to 0 somewhere at the right of δ_r gives a contradiction too.

Similarly, $\delta_f > 0$ gives a contradiction, thus $\delta_r = \delta_f = 0$ and the system (5.1), (5.2) becomes

$$\overline{x(t-0)} \cdot x(t) = \overline{x(t-0)} \cdot u(t), \tag{5.13}$$

$$x(t-0) \cdot \overline{x(t)} = x(t-0) \cdot \overline{u(t)}. \tag{5.14}$$

Let $t_0 \in \mathbf{R}$ be the number with the property that $x_{|(-\infty,t_0)} = x(-\infty+0), u_{|(-\infty,t_0)} = u(-\infty+0)$. The substitution in (5.13), (5.14), for $t < t_0$, of $x(t-0), x(t)$, with $x(-\infty+0)$, and of $u(t-0), u(t)$, with $u(-\infty+0)$, shows that $x(-\infty+0) = u(-\infty+0)$. If we suppose against all reason that (5.13), (5.14) differs from I, then there is

$t_1 \geq t_0$ such that $x_{|(-\infty,t_1)} = u_{|(-\infty,t_1)}$ and $x(t_1) \neq u(t_1)$. However, this assumption implies that

$$\overline{x(t_1 - 0)} \cdot x(t_1) = \overline{x(t_1 - 0)} \cdot \overline{x(t_1)}, \tag{5.15}$$

$$x(t_1 - 0) \cdot \overline{x(t_1)} = x(t_1 - 0) \cdot x(t_1) \tag{5.16}$$

is an inconsistent system. Thus (5.13), (5.14) coincides with I. □

CHAPTER 11

Bounded delays

The bounded delays f are those for which the switch of the input from 0 to 1 at $t_0 : \overline{u(t_0-0)} \cdot u(t_0) = 1$ produces a switch from 0 to 1 of the corresponding state in a bounded time interval

$$\forall x \in f(u), \exists t_1 \in \mathbf{R}, t_1 - t_0 \in [d_{r,\min}, d_{r,\max}] \text{ and } \overline{x(t_1-0)} \cdot x(t_1) = 1$$

where $0 \le d_{r,\min} \le d_{r,\max}$ are independent of the remainder variables. The bounded delays f are also asked to fulfill the dual of the previous property relative to the parameters $0 \le d_{f,\min} \le d_{f,\max}$[1].

The purpose of this chapter is that of giving several definitions of the bounded delays.

1. The first definition of the bounded delays

THEOREM 266. *Let be the numbers $0 \le m_r \le d_r$, $0 \le m_f \le d_f$. The following statements are equivalent:*

a) the inequality

$$\bigcap_{\xi \in [t-d_r, t-d_r+m_r]} u(\xi) \le x(t) \le \bigcup_{\xi \in [t-d_f, t-d_f+m_f]} u(\xi) \tag{1.1}$$

has a solution $x \in S$, for any $u \in S$;

b) for any $u \in S$, we have

$$\bigcap_{\xi \in [t-d_r, t-d_r+m_r]} u(\xi) \le \bigcup_{\xi \in [t-d_f, t-d_f+m_f]} u(\xi); \tag{1.2}$$

c) the following inequalities

$$d_r - m_r \le d_f \text{ and } d_f - m_f \le d_r \tag{1.3}$$

are fulfilled.

PROOF. $a) \Longrightarrow b)$ is obvious.

$b) \Longrightarrow a)$ For any $u \in S$, the functions $\bigcap_{\xi \in [t-d_r, t-d_r+m_r]} u(\xi)$, $\bigcup_{\xi \in [t-d_f, t-d_f+m_f]} u(\xi)$ belong to S and they are solutions of (1.1).

$b) \Longrightarrow c)$ Show that b) implies

$$\forall t \in \mathbf{R}, [t-d_r, t-d_r+m_r] \cap [t-d_f, t-d_f+m_f] \ne \emptyset. \tag{1.4}$$

Indeed, if (1.4) is false then

$$\exists t \in \mathbf{R}, [t-d_r, t-d_r+m_r] \cap [t-d_f, t-d_f+m_f] = \emptyset.$$

[1] In these few words we only proposed to stand out the boundedness of the delays. This is not a correct definition.

Let t be such a fixed time instant. Then there is some $u \in S$ with the property that *supp* $u \supset [t-d_r, t-d_r+m_r]$, *supp* $u \cap [t-d_f, t-d_f+m_f] = \emptyset$, making the hypothesis b) false, whence a contradiction.

From (1.4) we infer that for any $t \in \mathbf{R}$ we can write

$$not\ (t-d_r+m_r < t-d_f \text{ or } t-d_f+m_f < t-d_r)$$

$$\Longleftrightarrow t-d_r+m_r \geq t-d_f \text{ and } t-d_f+m_f \geq t-d_r,$$

i.e. c) is true.

c) $\Longrightarrow$ *b)* We run backwards the previous reasoning: c) implies (1.4) and let $t \in \mathbf{R}$ be arbitrary and fixed, for which there is some $\xi_0 \in [t-d_r, t-d_r+m_r] \cap [t-d_f, t-d_f+m_f]$. For any $u \in S$, we can write that

$$\bigcap_{\xi \in [t-d_r, t-d_r+m_r]} u(\xi) \leq u(\xi_0) \leq \bigcup_{\xi \in [t-d_f, t-d_f+m_f]} u(\xi).$$

□

Definition 109. *In Theorem 266, any of the equivalent properties a), b), c) is called the* ***consistency condition of the bounded delays*** *(CC_{BD}). If one of them is true, we use to say that the system (1.1)* ***fulfills*** *CC_{BD} or that* ***the numbers*** *m_r, d_r, m_f, d_f* ***fulfill that condition****.*

Remark 110. *We give some special cases of fulfillment of CC_{BD}, written under the form (1.3):*

a) $d_r = d_f = d$; CC_{BD} is fulfilled under the form $m_r \geq 0$, $m_f \geq 0$;

b) $m_r = d_r$ and $m_f = d_f$; CC_{BD} takes the form $d_r \geq 0$, $d_f \geq 0$ and it is true;

c) $m_r = m_f = 0$; CC_{BD} is equivalent to $d_r = d_f$.

Theorem 267. *Let be $0 \leq m_r \leq d_r$, $0 \leq m_f \leq d_f$. The inequality (1.1) defines a delay iff CC_{BD} is fulfilled.*

Proof. *If* The two functions $\bigcap_{\xi \in [t-d_r, t-d_r+m_r]} u(\xi) \leq \bigcup_{\xi \in [t-d_f, t-d_f+m_f]} u(\xi)$ are delays and by Theorem 265, (1.1) defines also a delay.

Only if The assumption that CC_{BD} is false means the existence of some non-admissible $u \in S$, i.e. (1.1) does not define a system. □

Definition 110. *Given the real numbers $0 \leq m_r \leq d_r$, $0 \leq m_f \leq d_f$ such that $d_r - m_r \leq d_f$, $d_f - m_f \leq d_r$, the delay*

$$\bigcap_{\xi \in [t-d_r, t-d_r+m_r]} u(\xi) \leq x(t) \leq \bigcup_{\xi \in [t-d_f, t-d_f+m_f]} u(\xi),$$

denoted $f_{BD}^{m_r, d_r, m_f, d_f}$, is called the ***boundedness property****. m_r, m_f are the* ***(rising, falling) inertial delays (thresholds true for cancellation)****; d_r, d_f are the* ***(rising, falling) upper bounds of the transport delays (of the transmission delays for transitions)*** *and the differences $d_f - m_f$, respectively $d_r - m_r$ are called the* ***(rising, falling) lower bounds of the transport delays (of the transmission delays for transitions)****.*

Definition 111. *A delay f is called* ***bounded*** *if there are $0 \leq m_r \leq d_r$, $0 \leq m_f \leq d_f$ such that CC_{BD} is fulfilled and f is a subdelay of $f_{BD}^{m_r, d_r, m_f, d_f}$. If $f \subset f_{BD}^{m_r, d_r, m_f, d_f}$, we say that f* ***satisfies the boundedness property*** *$f_{BD}^{m_r, d_r, m_f, d_f}$.*

REMARK 111. *We give an interpretation to the bounded delays by supposing that the input is the constant function. Then we have* $\forall\lambda \in \mathbf{B}$, $f_{BD}^{m_r,d_r,m_f,d_f}(\lambda) = \{\lambda\}$, *meaning that any bounded delay* f *has the same property:* $\forall\lambda \in \mathbf{B}, \forall x \in f(\lambda), x(t) = \lambda$.

Let $u = \chi_{[0,\delta)}, \delta > 0$ *be an input for which we analyze the states of the system* $f_{BD}^{m_r,d_r,m_f,d_f}$ *in two situations,* $\delta \leq m_r$ *and* $\delta > m_r$.

a) $\delta \leq m_r$,

$$0 \leq x(t) \leq \chi_{[d_f-m_f,d_f+\delta)}(t).$$

a.1) $t \in (-\infty, d_f - m_f), x(t) = 0$; *the 1-pulse did not propagate from the input to the states,*

a.2) $t \in [d_f - m_f, d_f + \delta), x(t) = 0$ *and* $x(t) = 1$ *are both possible, the 1-pulse may have propagated to the states,*

a.3) $t \in [d_f + \delta, \infty), x(t) = 0$, *the 1-pulse on the input cannot affect any longer the states.*

b) $\delta > m_r$,

$$\chi_{[d_r,d_r-m_r+\delta)}(t) \leq x(t) \leq \chi_{[d_f-m_f,d_f+\delta)}(t).$$

b.1) $t \in (-\infty, d_f - m_f), x(t) = 0$; *the 1-pulse did not propagate to the states,*

b.2) $t \in [d_f-m_f, d_r), x(t) = 0$ *or* $x(t) = 1$; *the 1-pulse may have propagated to the states,*

b.3) $t \in [d_r, d_r - m_r + \delta), x(t) = 1$; *the 1-pulse has surely propagated from the input to the states,*

b.4) $t \in [d_r - m_r + \delta, d_f + \delta), x(t) = 0$ *or* $x(t) = 1$; *the 1-pulse may still produce effects,*

b.5) $t \in [d_f + \delta, \infty), x(t) = 0$; *the 1-pulse on the input cannot influence any longer the states.*

Here we add the dual remarks as followed by taking $u = \chi_{(-\infty,0)\cup[\delta,\infty)}$ *as well as the fact that the states of a subdelay* $f \subset f_{BD}^{m_r,d_r,m_f,d_f}$ *may run only through some of the previous possibilities.*

THEOREM 268. *Let be* $0 \leq m_r \leq d_r$, $0 \leq m_f \leq d_f$ *such that* CC_{BD} *is fulfilled. For any* $u \in S$ *and any* $x \in S$, *the following statements are equivalent:*

a) $x \in f_{BD}^{m_r,d_r,m_f,d_f}(u)$;

b) $\exists y \in S, x(t) = \bigcap_{\xi\in[t-d_r,t-d_r+m_r]} u(\xi) \cup y(t) \cdot \bigcup_{\xi\in[t-d_f,t-d_f+m_f]} u(\xi)$.

PROOF. *a)* $\Longrightarrow$ *b)* It is verified the fact that for any $t \in \mathbf{R}$ and any values $\bigcap_{\xi\in[t-d_r,t-d_r+m_r]} u(\xi) = \bigcup_{\xi\in[t-d_f,t-d_f+m_f]} u(\xi) = 0$; $\bigcap_{\xi\in[t-d_r,t-d_r+m_r]} u(\xi) = 0$, $\bigcup_{\xi\in[t-d_f,t-d_f+m_f]} u(\xi) = 1$ and $\bigcap_{\xi\in[t-d_r,t-d_r+m_r]} u(\xi) = \bigcup_{\xi\in[t-d_f,t-d_f+m_f]} u(\xi) = 1$, from $x \in f_{BD}^{m_r,d_r,m_f,d_f}(u)$ we get that

$$x(t) = \bigcap_{\xi\in[t-d_r,t-d_r+m_r]} u(\xi) \cup x(t) \cdot \bigcup_{\xi\in[t-d_f,t-d_f+m_f]} u(\xi).$$

b) $\Longrightarrow$ *a)* For any t and any $y \in S$ we have

$$\bigcap_{\xi\in[t-d_r,t-d_r+m_r]} u(\xi) \leq \bigcap_{\xi\in[t-d_r,t-d_r+m_r]} u(\xi) \cup y(t) \cdot \bigcup_{\xi\in[t-d_f,t-d_f+m_f]} u(\xi) \leq$$

$$\leq \bigcap_{\xi\in[t-d_r,t-d_r+m_r]} u(\xi) \cup \bigcup_{\xi\in[t-d_f,t-d_f+m_f]} u(\xi) = \bigcup_{\xi\in[t-d_f,t-d_f+m_f]} u(\xi).$$

□

2. The equality between the initial values of the input and of the state

THEOREM 269. *Given the numbers $0 \leq m_r \leq d_r, 0 \leq m_f \leq d_f$ such that CC_{BD} is true, then $\forall u \in S, \forall x \in f_{BD}^{m_r,d_r,m_f,d_f}(u)$, we have $x(-\infty+0) = u(-\infty+0)$.*

PROOF. Let $u \in S$ and $x \in f_{BD}^{m_r,d_r,m_f,d_f}(u)$ be arbitrarily chosen. The case $u(-\infty+0) = 0$ is equivalent to $\exists d \in \mathbf{R}, u(t) \leq \chi_{[d,\infty)}(t)$. In this case we have

$$x(t) \leq \bigcup_{\xi\in[t-d_f,t-d_f+m_f]} u(\xi) \leq \bigcup_{\xi\in[t-d_f,t-d_f+m_f]} \chi_{[d,\infty)}(\xi) = \chi_{[d+d_f-m_f,\infty)}(t),$$

i.e. $x(-\infty+0) = 0$.

The case $u(-\infty+0) = 1$ is equivalent to $\exists d \in \mathbf{R}, u(t) \geq \chi_{(-\infty,d)}(t)$, so we have

$$x(t) \geq \bigcap_{\xi\in[t-d_r,t-d_r+m_r]} u(\xi) \geq \bigcap_{\xi\in[t-d_r,t-d_r+m_r]} \chi_{(-\infty,d)}(\xi) = \chi_{(-\infty,d+d_r-m_r)}(t),$$

i.e. $x(-\infty+0) = 1$. □

COROLLARY 3. *$f_{BD}^{m_r,d_r,m_f,d_f}$ has race-free initial states and a bounded initial time:*

$$\forall u \in S, \exists t_0 \in \mathbf{R}, \forall x \in f_{BD}^{m_r,d_r,m_f,d_f}(u), x_{|(-\infty,t_0)} = u(-\infty+0).$$

PROOF. The first statement is inferred from Theorem 269, while the second statement follows from the proof of the same theorem. □

COROLLARY 4. *The initial state function of $f_{BD}^{m_r,d_r,m_f,d_f}$, $\phi_0 : S \to \mathbf{B}$ is defined by*

$$\forall u \in S, \phi_0(u) = u(-\infty+0).$$

COROLLARY 5. *If f is a bounded delay, then it has race-free initial states, a bounded initial time and its initial state function $\phi_0 : S \to \mathbf{B}$ is defined by*

$$\forall u \in S, \phi_0(u) = u(-\infty+0).$$

PROOF. We take into account Corollary 3 and Theorem 35; Corollary 3 and Theorem 37; Corollary 4 and Theorem 39. □

3. Order

THEOREM 270. *Let be $0 \leq m_r \leq d_r$, $0 \leq m_f \leq d_f$ and $0 \leq m'_r \leq d'_r$, $0 \leq m'_f \leq d'_f$ such that CC_{BD} is fulfilled for each of $f_{BD}^{m_r,d_r,m_f,d_f}$, $f_{BD}^{m'_r,d'_r,m'_f,d'_f}$. The following statements are equivalent:*

a) we have

$$f_{BD}^{m_r,d_r,m_f,d_f} \subset f_{BD}^{m'_r,d'_r,m'_f,d'_f};$$

b) the following inequalities

$$d'_r - m'_r \leq d_r - m_r \leq d_f \leq d'_f,$$
$$d'_f - m'_f \leq d_f - m_f \leq d_r \leq d'_r$$

are true.

PROOF. $a)$ is equivalent to $\forall u \in S, \forall t \in \mathbf{R}$,

$$\bigcap_{\xi\in[t-d'_r,t-d'_r+m'_r]} u(\xi) \leq \bigcap_{\xi\in[t-d_r,t-d_r+m_r]} u(\xi) \leq$$
$$\leq \bigcup_{\xi\in[t-d_f,t-d_f+m_f]} u(\xi) \leq \bigcup_{\xi\in[t-d'_f,t-d'_f+m'_f]} u(\xi)$$

iff the following inclusions

$$[t-d'_r, t-d'_r+m'_r] \supset [t-d_r, t-d_r+m_r],$$
$$[t-d_f, t-d_f+m_f] \subset [t-d'_f, t-d'_f+m'_f]$$

hold for all $t \in \mathbf{R}$ iff $b)$ is true, taking into account CC_{BD}, too. □

REMARK 112. *Due to statement b) of the previous theorem, from the conjunction of the statements* $f \subset f_{BD}^{m_r,d_r,m_f,d_f}$ *and* $f_{BD}^{m_r,d_r,m_f,d_f} \subset f_{BD}^{m'_r,d'_r,m'_f,d'_f}$, *we conclude that:*

- if the bounded delay f has the upper bounds of the transmission delays for transitions d_r, d_f, then it also has the upper bounds d'_r, d'_f;

- if the bounded delay f has the lower bounds of the transmission delays for transitions $d_f - m_f$, $d_r - m_r$, then it has also the lower bounds $d'_f - m'_f$, $d'_r - m'_r$.

On the other hand, given the bounded delay f, it is interesting the study of that delay $f_{BD}^{m_r,d_r,m_f,d_f}$ with

i) $f \subset f_{BD}^{m_r,d_r,m_f,d_f}$

ii) for any $f_{BD}^{m'_r,d'_r,m'_f,d'_f}$ such that $f \subset f_{BD}^{m'_r,d'_r,m'_f,d'_f}$, we have $f_{BD}^{m_r,d_r,m_f,d_f} \subset f_{BD}^{m'_r,d'_r,m'_f,d'_f}$

i.e. $f_{BD}^{m_r,d_r,m_f,d_f}$ is the smallest boundedness property, in the sense of the inclusion, satisfied by f.

THEOREM 271. *Let be $f, g : S \to P^*(S)$. If $f \subset g$ and g is a bounded delay, then f is a bounded delay too.*

PROOF. Suppose that $g \subset f_{BD}^{m_r,d_r,m_f,d_f}$, where m_r, d_r, m_f, d_f fulfill CC_{BD}. By Theorem 258, the system f is a delay. Obviously, it is bounded. □

4. Duality

THEOREM 272. *The dual delay of $f_{BD}^{m_r,d_r,m_f,d_f}$ is $f_{BD}^{m_f,d_f,m_r,d_r}$.*

PROOF. First of all we note that CC_{BD} for the two systems is the same. Thus the bounded delay $f_{BD}^{m_r,d_r,m_f,d_f}$ exists iff the bounded delay $f_{BD}^{m_f,d_f,m_r,d_r}$ exists. We prove the duality between the two delays. We have:

$$\forall u \in S, (f_{BD}^{m_r,d_r,m_f,d_f})^*(u) =$$
$$= \{\overline{x} | \bigcap_{\xi\in[t-d_r,t-d_r+m_r]} \overline{u(\xi)} \leq x(t) \leq \bigcup_{\xi\in[t-d_f,t-d_f+m_f]} \overline{u(\xi)}\} =$$
$$= \{\overline{x} | \overline{\bigcap_{\xi\in[t-d_r,t-d_r+m_r]} \overline{u(\xi)}} \geq \overline{x(t)} \geq \overline{\bigcup_{\xi\in[t-d_f,t-d_f+m_f]} \overline{u(\xi)}}\} =$$
$$= \{x | \bigcap_{\xi\in[t-d_f,t-d_f+m_f]} u(\xi) \leq x(t) \leq \bigcup_{\xi\in[t-d_r,t-d_r+m_r]} u(\xi)\} = f_{BD}^{m_f,d_f,m_r,d_r}(u).$$

□

THEOREM 273. *If f is a bounded delay, then f^* is a bounded delay too.*

PROOF. The inclusion $f \subset f_{BD}^{m_r,d_r,m_f,d_f}$ implies

$$\forall u \in S, f^*(u) = \{\overline{x} | x \in f(\overline{u})\} \subset \{\overline{x} | x \in f_{BD}^{m_r,d_r,m_f,d_f}(\overline{u})\} =$$
$$= (f_{BD}^{m_r,d_r,m_f,d_f})^*(u) = f_{BD}^{m_f,d_f,m_r,d_r}(u).$$

But this fact results directly from Theorems 49 and 272 also because we have $f \subset f_{BD}^{m_r,d_r,m_f,d_f} \Longleftrightarrow f^* \subset f_{BD}^{m_f,d_f,m_r,d_r}$. □

THEOREM 274. *The system $f_{BD}^{m_r,d_r,m_f,d_f}$ is self-dual iff $d_r = d_f$ and $m_r = m_f$.*

PROOF. *If* Obvious.
Only if Because (1.1) and

$$\bigcap_{\xi \in [t-d_f, t-d_f+m_f]} u(\xi) \leq x(t) \leq \bigcup_{\xi \in [t-d_r, t-d_r+m_r]} u(\xi)$$

must have the same solutions, we infer that $d_r = d_f$ and $d_r - m_r = d_f - m_f$, i.e. the statement of the theorem. □

5. Serial connection

THEOREM 275. *Let be the numbers $0 \leq m_r \leq d_r$, $0 \leq m_f \leq d_f$ and $0 \leq m'_r \leq d'_r$, $0 \leq m'_f \leq d'_f$ such that $d_r \geq d_f - m_f$, $d_f \geq d_r - m_r$ and $d'_r \geq d'_f - m'_f$, $d'_f \geq d'_r - m'_r$ are true. Then $f_{BD}^{m_r+m'_r,d_r+d'_r,m_f+m'_f,d_f+d'_f}$ is a bounded delay and we have*

$$f_{BD}^{m'_r,d'_r,m'_f,d'_f} \circ f_{BD}^{m_r,d_r,m_f,d_f} = f_{BD}^{m_r+m'_r,d_r+d'_r,m_f+m'_f,d_f+d'_f}.$$

PROOF. We remark that $d_f+d'_f \geq d_r+d'_r-m_r-m'_r$, $d_r+d'_r \geq d_f+d'_f-m_f-m'_f$, thus CC_{BD} is fulfilled again and $f_{BD}^{m_r+m'_r,d_r+d'_r,m_f+m'_f,d_f+d'_f}$ makes a sense.
We prove

$$f_{BD}^{m'_r,d'_r,m'_f,d'_f} \circ f_{BD}^{m_r,d_r,m_f,d_f} \subset f_{BD}^{m_r+m'_r,d_r+d'_r,m_f+m'_f,d_f+d'_f}.$$

Let $u \in S$ and $y \in (f_{BD}^{m'_r,d'_r,m'_f,d'_f} \circ f_{BD}^{m_r,d_r,m_f,d_f})(u)$ be arbitrary, for which $x \in f_{BD}^{m_r,d_r,m_f,d_f}(u)$ exists, such that $y \in f_{BD}^{m'_r,d'_r,m'_f,d'_f}(x)$. The following statements

$$\bigcap_{\omega \in [\xi-d_r, \xi-d_r+m_r]} u(\omega) \leq x(\xi) \leq \bigcup_{\omega \in [\xi-d_f, \xi-d_f+m_f]} u(\omega), \tag{5.1}$$

$$\bigcap_{\xi \in [t-d'_r, t-d'_r+m'_r]} x(\xi) \leq y(t) \leq \bigcup_{\xi \in [t-d'_f, t-d'_f+m'_f]} x(\xi) \tag{5.2}$$

are true for all $t \in \mathbf{R}$ and $\xi \in \mathbf{R}$, wherefrom we get

$$\bigcap_{\xi \in [t-d_r-d'_r, t-d_r-d'_r+m_r+m'_r]} u(\xi) = \bigcap_{\xi \in [t-d'_r, t-d'_r+m'_r]} \bigcap_{\omega \in [\xi-d_r, \xi-d_r+m_r]} u(\omega) \leq$$
$$\leq \bigcap_{\xi \in [t-d'_r, t-d'_r+m'_r]} x(\xi) \leq y(t) \leq \bigcup_{\xi \in [t-d'_f, t-d'_f+m'_f]} x(\xi) \leq$$

$$\leq \bigcup_{\xi\in[t-d'_f,t-d'_f+m'_f]}\bigcup_{\omega\in[\xi-d_f,\xi-d_f+m_f]} u(\omega) = \bigcup_{\xi\in[t-d_f-d'_f,t-d_f-d'_f+m_f+m'_f]} u(\xi).$$

Thus $y \in f_{BD}^{m_r+m'_r,d_r+d'_r,m_f+m'_f,d_f+d'_f}(u)$.

We prove that

$$f_{BD}^{m_r+m'_r,d_r+d'_r,m_f+m'_f,d_f+d'_f} \subset f_{BD}^{m'_r,d'_r,m'_f,d'_f} \circ f_{BD}^{m_r,d_r,m_f,d_f}.$$

We must show that for $u \in S$ and $y \in S$ with

$$\bigcap_{\xi\in[t-d_r-d'_r,t-d_r-d'_r+m_r+m'_r]} u(\xi) \leq y(t) \leq \bigcup_{\xi\in[t-d_f-d'_f,t-d_f-d'_f+m_f+m'_f]} u(\xi), \tag{5.3}$$

both of them arbitrary and fixed, there is some $x \in S$ such that (5.1), (5.2) be fulfilled for all $t \in \mathbf{R}$ and all $\xi \in \mathbf{R}$. There is $t_0 \in \mathbf{R}$ such that $\forall y \in f_{BD}^{m_r+m'_r,d_r+d'_r,m_f+m'_f,d_f+d'_f}(u)$, $\forall x \in f_{BD}^{m_r,d_r,m_f,d_f}(u)$, $\forall y' \in f_{BD}^{m'_r,d'_r,m'_f,d'_f}(x)$, by Theorem 269 we have

$$y_{|(-\infty,t_0)} = x_{|(-\infty,t_0)} = y'_{|(-\infty,t_0)} = u_{|(-\infty,t_0)} = u(-\infty+0).$$

Against all reason, we can assume that there is $t_1 \geq t_0$ such that the previous property be true for $t < t_1$ and false for $t = t_1$. Suppose that $y(t_1) = 0$. From (5.3) we have

$$\bigcap_{\xi\in[t_1-d'_r,t_1-d'_r+m'_r]}\bigcap_{\omega\in[\xi-d_r,\xi-d_r+m_r]} u(\omega) = 0,$$

i.e.

$$\exists \xi_1 \in [t_1-d'_r, t_1-d'_r+m'_r], \bigcap_{\omega\in[\xi_1-d_r,\xi_1-d_r+m_r]} u(\omega) = 0.$$

The relation (5.1), written for $\xi = \xi_1$, shows that we can choose $x(\xi_1) = 0$ and (5.2), written for $t = t_1$, is true because

$$\bigcap_{\xi\in[t_1-d'_r,t_1-d'_r+m'_r]} x(\xi) = x(\xi_1) = 0,$$

in contradiction with the assumption made on the existence of t_1. The same result is obtained if we suppose that $y(t_1) = 1$. □

THEOREM 276. *Relative to the serial connection, the set of the bounded delays is a monoid, where the unit is I.*

PROOF. From $f \subset f_{BD}^{m_r,d_r,m_f,d_f}$ and $g \subset f_{BD}^{m'_r,d'_r,m'_f,d'_f}$, taking into account Theorems 74 and 275 we obtain that

$$g \circ f \subset g \circ f_{BD}^{m_r,d_r,m_f,d_f} \subset f_{BD}^{m'_r,d'_r,m'_f,d'_f} \circ f_{BD}^{m_r,d_r,m_f,d_f} = f_{BD}^{m_r+m'_r,d_r+d'_r,m_f+m'_f,d_f+d'_f}.$$

Remark that I belongs to the monoid, because $I = f_{BD}^{0,0,0,0}$. □

REMARK 113. *If f is an arbitrary delay, then by its serial connection with a bounded delay, in general, we obtain an arbitrary delay (not a bounded delay).*

Now combine the order of the boundedness properties from Theorem 270 with Theorem 74, giving the compatibility between the serial connection and the order, and with Theorem 275, related to the serial connection $f_{BD}^{m'_r,d'_r,m'_f,d'_f} \circ f_{BD}^{m_r,d_r,m_f,d_f}$ in the following way. Let be the bounded delays $f_{BD}^{m_r,d_r,m_f,d_f}$, $f_{BD}^{m'_r,d'_r,m'_f,d'_f}$ and

$f_{BD}^{m_r'',d_r'',m_f'',d_f''}$, *where* $0 \leq m_r \leq d_r$, $0 \leq m_f \leq d_f$, $0 \leq m_r' \leq d_r'$, $0 \leq m_f' \leq d_f'$, $0 \leq m_r'' \leq d_r''$, $0 \leq m_f'' \leq d_f''$, *such that* CC_{BD} *is fulfilled three times. The implication*

$$f_{BD}^{m_r',d_r',m_f',d_f'} \subset f_{BD}^{m_r'',d_r'',m_f'',d_f''} \Longrightarrow$$
$$\Longrightarrow f_{BD}^{m_r',d_r',m_f',d_f'} \circ f_{BD}^{m_r,d_r,m_f,d_f} \subset f_{BD}^{m_r'',d_r'',m_f'',d_f''} \circ f_{BD}^{m_r,d_r,m_f,d_f}$$

means that

$$\begin{aligned} d_r'' - m_r'' &\leq d_r' - m_r' \leq d_f' \leq d_f'', \\ d_f'' - m_f'' &\leq d_f' - m_f' \leq d_r' \leq d_r'' \end{aligned}$$

implies

$$\begin{aligned} d_r + d_r'' - m_r - m_r'' &\leq d_r + d_r' - m_r - m_r' \leq d_f + d_f' \leq d_f + d_f'', \\ d_f + d_f'' - m_f - m_f'' &\leq d_f + d_f' - m_f - m_f' \leq d_r + d_r' \leq d_r + d_r''. \end{aligned}$$

The other situation

$$f_{BD}^{m_r',d_r',m_f',d_f'} \subset f_{BD}^{m_r'',d_r'',m_f'',d_f''} \Longrightarrow$$
$$\Longrightarrow f_{BD}^{m_r,d_r,m_f,d_f} \circ f_{BD}^{m_r',d_r',m_f',d_f'} \subset f_{BD}^{m_r,d_r,m_f,d_f} \circ f_{BD}^{m_r'',d_r'',m_f'',d_f''}$$

is similar.

6. Intersection

Theorem 277. *If the numbers* $0 \leq m_r \leq d_r$, $0 \leq m_f \leq d_f$ *respectively* $0 \leq m_r' \leq d_r'$, $0 \leq m_f' \leq d_f'$ *satisfy* CC_{BD} *twice:* $d_r \geq d_f - m_f$, $d_f \geq d_r - m_r$, *respectively* $d_r' \geq d_f' - m_f'$, $d_f' \geq d_r' - m_r'$, *then the following statements are equivalent:*

$$\forall u \in S, f_{BD}^{m_r,d_r,m_f,d_f}(u) \cap f_{BD}^{m_r',d_r',m_f',d_f'}(u) \neq \emptyset;$$
$$d_r \geq d_f' - m_f',\ d_f' \geq d_r - m_r,\ d_r' \geq d_f - m_f,\ d_f \geq d_r' - m_r'.$$

If one of them is true, then there is the delay $f_{BD}^{m_r,d_r,m_f,d_f} \cap f_{BD}^{m_r',d_r',m_f',d_f'}$ *and it is given by*

$$\bigcap_{\xi \in [t-d_r, t-d_r+m_r]} u(\xi) \cup \bigcap_{\xi \in [t-d_r', t-d_r'+m_r']} u(\xi) \leq x(t) \leq$$
$$\leq \bigcup_{\xi \in [t-d_f, t-d_f+m_f]} u(\xi) \cdot \bigcup_{\xi \in [t-d_f', t-d_f'+m_f']} u(\xi). \tag{6.1}$$

Proof. We show the first statement. For an arbitrary $u \in S$ we can write:

$$\exists x \in f_{BD}^{m_r,d_r,m_f,d_f}(u) \cap f_{BD}^{m_r',d_r',m_f',d_f'}(u) \Longleftrightarrow$$
$$\Longleftrightarrow \exists x \in S, \bigcap_{\xi \in [t-d_r, t-d_r+m_r]} u(\xi) \leq x(t) \leq \bigcup_{\xi \in [t-d_f, t-d_f+m_f]} u(\xi) \text{ and}$$
$$\text{and} \bigcap_{\xi \in [t-d_r', t-d_r'+m_r']} u(\xi) \leq x(t) \leq \bigcup_{\xi \in [t-d_f', t-d_f'+m_f']} u(\xi)$$
$$\Longleftrightarrow \bigcap_{\xi \in [t-d_r, t-d_r+m_r]} u(\xi) \leq \bigcup_{\xi \in [t-d_f, t-d_f+m_f]} u(\xi) \text{ and}$$

$$\text{and} \bigcap_{\xi\in[t-d'_r,t-d'_r+m'_r]} u(\xi) \leq \bigcup_{\xi\in[t-d'_f,t-d'_f+m'_f]} u(\xi) \text{ and}$$

$$\text{and} \bigcap_{\xi\in[t-d_r,t-d_r+m_r]} u(\xi) \leq \bigcup_{\xi\in[t-d'_f,t-d'_f+m'_f]} u(\xi) \text{ and}$$

$$\text{and} \bigcap_{\xi\in[t-d'_r,t-d'_r+m'_r]} u(\xi) \leq \bigcup_{\xi\in[t-d_f,t-d_f+m_f]} u(\xi)$$

$$\iff d_r \geq d_f - m_f, d_f \geq d_r - m_r \text{ and } d'_r \geq d'_f - m'_f, d'_f \geq d'_r - m'_r \text{ and}$$

$$\text{and } d_r \geq d'_f - m'_f,\ d'_f \geq d_r - m_r \text{ and } d'_r \geq d_f - m_f,\ d_f \geq d'_r - m'_r$$

$$\iff d_r \geq d'_f - m'_f,\ d'_f \geq d_r - m_r \text{ and } d'_r \geq d_f - m_f,\ d_f \geq d'_r - m'_r,$$

where we have taken into account Theorem 266 and the hypothesis.

The second statement of the theorem is obvious, but we can make the following reasoning too. We fix arbitrarily $t \in \mathbf{R}, u \in S$ and assign to $\bigcap_{\xi\in[t-d_r,t-d_r+m_r]} u(\xi)$, $\bigcup_{\xi\in[t-d_f,t-d_f+m_f]} u(\xi)$, $\bigcap_{\xi\in[t-d'_r,t-d'_r+m'_r]} u(\xi)$, $\bigcup_{\xi\in[t-d'_f,t-d'_f+m'_f]} u(\xi)$ all possible values from 0,0,0,0 to 1,1,1,1 following seven situations. In each of them, the values $x(t), x \in f_{BD}^{m_r,d_r,m_f,d_f}(u) \cap f_{BD}^{m'_r,d'_r,m'_f,d'_f}(u)$ and the solutions $x(t)$ of (6.1) coincide. $\square$

Remark 114. *The delay* $f_{BD}^{m_r,d_r,m_f,d_f} \cap f_{BD}^{m'_r,d'_r,m'_f,d'_f}$ *is bounded, for example* $f_{BD}^{m_r,d_r,m_f,d_f} \cap f_{BD}^{m'_r,d'_r,m'_f,d'_f} \subset f_{BD}^{m_r,d_r,m_f,d_f}$, *but it is not of the form* $f_{BD}^{m''_r,d''_r,m''_f,d''_f}$.

Theorem 278. *If f is a bounded delay and $g : S \to P^*(S)$ is an arbitrary system, such that $\forall u \in S, f(u) \cap g(u) \neq \emptyset$, then $f \cap g$ is a bounded delay.*

Proof. Let be $0 \leq m_r \leq d_r$, $0 \leq m_f \leq d_f$ such that CC_{BD} be fulfilled and suppose that $f \subset f_{BD}^{m_r,d_r,m_f,d_f}$. By hypothesis, $f \cap g$ is a delay and, because $f \cap g \subset f$, we have obtained that $f \cap g$ is bounded. $\square$

Corollary 6. *The intersection of the bounded delays is a bounded delay.*

7. Union

Theorem 279. *Give the numbers $0 \leq m_r \leq d_r$, $0 \leq m_f \leq d_f, 0 \leq m'_r \leq d'_r$, $0 \leq m'_f \leq d'_f$ such that CC_{BD} is fulfilled twice. Then there is the delay* $f_{BD}^{m_r,d_r,m_f,d_f} \cup f_{BD}^{m'_r,d'_r,m'_f,d'_f}$ *and it is given by the system*

$$\bigcap_{\xi\in[t-d_r,t-d_r+m_r]} u(\xi) \cdot \bigcap_{\xi\in[t-d'_r,t-d'_r+m'_r]} u(\xi) \leq x(t) \leq \tag{7.1}$$

$$\leq \bigcup_{\xi\in[t-d_f,t-d_f+m_f]} u(\xi) \cup \bigcup_{\xi\in[t-d'_f,t-d'_f+m'_f]} u(\xi).$$

Proof. Let $t \in \mathbf{R}, u \in S$ be arbitrary and fixed. We assign to the numbers $\bigcap_{\xi\in[t-d_r,t-d_r+m_r]} u(\xi)$, $\bigcup_{\xi\in[t-d_f,t-d_f+m_f]} u(\xi)$, $\bigcap_{\xi\in[t-d'_r,t-d'_r+m'_r]} u(\xi)$, $\bigcup_{\xi\in[t-d'_f,t-d'_f+m'_f]} u(\xi)$ all possible values from 0,0,0,0 to 1,1,1,1. As a result we obtain nine situations. In each situation, the values $x(t)$, $x \in f_{BD}^{m_r,d_r,m_f,d_f}(u) \cup f_{BD}^{m'_r,d'_r,m'_f,d'_f}(u)$ and the solutions $x(t)$ of (7.1) coincide. $\square$

REMARK 115. *The delay* $f_{BD}^{m_r,d_r,m_f,d_f} \cup f_{BD}^{m'_r,d'_r,m'_f,d'_f}$ *is not a boundedness property* $f_{BD}^{m''_r,d''_r,m''_f,d''_f}$, *but it is bounded. A possibility of choosing* $f_{BD}^{m''_r,d''_r,m''_f,d''_f}$ *such that* $f_{BD}^{m_r,d_r,m_f,d_f} \cup f_{BD}^{m'_r,d'_r,m'_f,d'_f} \subset f_{BD}^{m''_r,d''_r,m''_f,d''_f}$ *is the following one:*

$$d''_r = d''_f = m''_r = m''_f = \max\{d_r, d_f\} \overset{not}{=} d.$$

Indeed, $\forall t \in \mathbf{R}$, *we have then*

$$[t-d,t] \supset [t-d_r, t-d_r+m_r], [t-d,t] \supset [t-d_f, t-d_f+m_f],$$

$$[t-d,t] \supset [t-d'_r, t-d'_r+m'_r], [t-d,t] \supset [t-d'_f, t-d'_f+m'_f],$$

thus $\forall t \in \mathbf{R}, \forall u \in S$ *we infer that*

$$\bigcap_{\xi\in[t-d,t]} u(\xi) \leq \bigcap_{\xi\in[t-d_r,t-d_r+m_r]} u(\xi) \cdot \bigcap_{\xi\in[t-d'_r,t-d'_r+m'_r]} u(\xi) \leq x(t) \leq$$

$$\leq \bigcup_{\xi\in[t-d_f,t-d_f+m_f]} u(\xi) \cup \bigcup_{\xi\in[t-d'_f,t-d'_f+m'_f]} u(\xi) \leq \bigcup_{\xi\in[t-d,t]} u(\xi).$$

THEOREM 280. *The union of the bounded delays is a bounded delay too.*

PROOF. From $f \subset f_{BD}^{m_r,d_r,m_f,d_f}$, $g \subset f_{BD}^{m'_r,d'_r,m'_f,d'_f}$ we have that $f \cup g \subset f_{BD}^{m_r,d_r,m_f,d_f} \cup f_{BD}^{m'_r,d'_r,m'_f,d'_f} \subset f_{BD}^{m''_r,d''_r,m''_f,d''_f}$. On the other hand we have supposed that the consistency conditions are fulfilled for both delays $f_{BD}^{m_r,d_r,m_f,d_f}$ and $f_{BD}^{m'_r,d'_r,m'_f,d'_f}$. The parameters $m''_r, d''_r, m''_f, d''_f$ may be chosen like in the previous remark. □

8. Determinism

THEOREM 281. *Let be* $0 \leq m_r \leq d_r$, $0 \leq m_f \leq d_f$ *such that* CC_{BD} *is true. The following statements are equivalent:*

a) $f_{BD}^{m_r,d_r,m_f,d_f}$ *is deterministic;*

b) the upper bounds and the lower bounds of the transport delays coincide

$$d_r = d_f - m_f,\ d_f = d_r - m_r;$$

c) the inertial delays are null

$$m_r = m_f = 0;$$

d) the boundedness property degenerates in a time translation

$$\exists d \geq 0, f_{BD}^{m_r,d_r,m_f,d_f} = I_d.$$

PROOF. a) $\Longrightarrow$ b) The hypothesis states that $\forall u \in S$, $f_{BD}^{m_r,d_r,m_f,d_f}(u)$ has exactly one element:

$$\forall u \in S, \bigcap_{\xi\in[t-d_r,t-d_r+m_r]} u(\xi) = \bigcup_{\xi\in[t-d_f,t-d_f+m_f]} u(\xi). \tag{8.1}$$

We assign to u the values $\chi_{[0,\infty)}, \chi_{(-\infty,0)}$ and we obtain

$$\bigcap_{\xi\in[t-d_r,t-d_r+m_r]} \chi_{[0,\infty)}(\xi) = \chi_{[d_r,\infty)}(t), \tag{8.2}$$

$$\bigcup_{\xi\in[t-d_f,t-d_f+m_f]}\chi_{[0,\infty)}(\xi)=\chi_{[d_f-m_f,\infty)}(t), \tag{8.3}$$

$$\bigcap_{\xi\in[t-d_r,t-d_r+m_r]}\chi_{(-\infty,0)}(\xi)=\chi_{(-\infty,d_r-m_r)}(t), \tag{8.4}$$

$$\bigcup_{\xi\in[t-d_f,t-d_f+m_f]}\chi_{(-\infty,0)}(\xi)=\chi_{(-\infty,d_f)}(t). \tag{8.5}$$

The relation (8.1) implies the equality of the functions from (8.2) and (8.3) and of the functions from (8.4) and (8.5). This indicates the fulfillment of b).

b) $\Longrightarrow$ c) We add the two relations term by term and we get $m_r+m_f=0$, whence c) is true.

c) $\Longrightarrow$ d) By hypothesis c) and from CC_{BD}, we infer $d_r\geq d_f$, $d_f\geq d_r$, i.e. $d_r=d_f=d$ and (1.1) becomes

$$u(t-d)\leq x(t)\leq u(t-d).$$

In other words, $f_{BD}^{0,d,0,d}(u)=u\circ\tau^d=I_d(u)$.

d) $\Longrightarrow$ a) Obvious, since I_d is deterministic. □

Corollary 7. *Any bounded delay f having the property that $f\subset f_{BD}^{m_r,d_r,m_f,d_f}$, while $f_{BD}^{m_r,d_r,m_f,d_f}$ satisfies one of the conditions a),...,d) of Theorem 281 is deterministic and coincides with I_d, where we have put $d=d_r=d_f$.*

9. Time invariance

Theorem 282. *The system $f_{BD}^{m_r,d_r,m_f,d_f}$ is time invariant.*

Proof. Because $\forall d\in\mathbf{R},\forall u\in S,\forall x\in f_{BD}^{m_r,d_r,m_f,d_f}(u)$ we have

$$\bigcap_{\xi\in[t-d_r,t-d_r+m_r]}(u\circ\tau^d)(\xi)=\bigcap_{\xi\in[t-d_r,t-d_r+m_r]}u(\xi-d)=\bigcap_{\xi+d\in[t-d_r,t-d_r+m_r]}u(\xi)=$$

$$=\bigcap_{\xi\in[t-d-d_r,t-d-d_r+m_r]}u(\xi)\leq x(t-d)\leq\bigcup_{\xi\in[t-d-d_f,t-d-d_f+m_f]}u(\xi)=$$

$$=\bigcup_{\xi+d\in[t-d_f,t-d_f+m_f]}u(\xi)=\bigcup_{\xi\in[t-d_f,t-d_f+m_f]}u(\xi-d)=\bigcup_{\xi\in[t-d_f,t-d_f+m_f]}(u\circ\tau^d)(\xi),$$

we infer that

$$\{x\circ\tau^d|x\in f_{BD}^{m_r,d_r,m_f,d_f}(u)\}\subset f_{BD}^{m_r,d_r,m_f,d_f}(u\circ\tau^d).$$

□

Remark 116. *In general, the bounded delays are not time invariant. A counterexample is represented by the delay $f\subset f_{BD}^{1,2,1,2}$ defined by*

$$\forall u\in S,f(u)=\begin{cases}f_{BD}^{1,2,1,2}(u),u\neq\chi_{[0,\infty)}\\ \chi_{[1,\infty)},u=\chi_{[0,\infty)}\end{cases},$$

for which we note that $\chi_{[1,\infty)}\in f_{BD}^{1,2,1,2}(\chi_{[0,\infty)})$, thus $f\subset f_{BD}^{1,2,1,2}$ indeed. We have

$$f(\chi_{[0,\infty)}\circ\tau^1)=f(\chi_{[1,\infty)})=\{x|\chi_{[3,\infty)}(t)\leq x(t)\leq\chi_{[2,\infty)}(t)\},$$

$$\{x\circ\tau^1|x\in f(\chi_{[0,\infty)})\}=\chi_{[2,\infty)}.$$

10. Non-anticipation

THEOREM 283. *For any numbers $0 \leq m_r \leq d_r, 0 \leq m_f \leq d_f$ such that CC_{BD} be fulfilled, $f_{BD}^{m_r,d_r,m_f,d_f}$ is non-anticipatory in the sense of Definition 63 and of Definition 65, items v),...,ix).*

PROOF. a) We prove the non-anticipation in the sense of Definition 63. We must show that $\forall u \in S, \forall x \in f_{BD}^{m_r,d_r,m_f,d_f}(u)$, we have:

a.1) x is constant

or

a.2) u, x are variable and

$$\min\{t|u(t-0) \neq u(t)\} \leq \min\{t|x(t-0) \neq x(t)\}.$$

If $u(-\infty+0) = 0$, there is some $d \in \mathbf{R}$ with $u \leq \chi_{[d,\infty)}$ such that we get

$$x(t) \leq \bigcup_{\xi \in [t-d_f, t-d_f+m_f]} u(\xi) \leq \bigcup_{\xi \in [t-d_f, t-d_f+m_f]} \chi_{[d,\infty)}(\xi) = \chi_{[d+d_f-m_f,\infty)}(t).$$

If $u = 0$, we have that $x = 0$ and a.1) is fulfilled. If $u \neq 0$, either $x = 0$ and a.1) is fulfilled, or $x \neq 0$ and we can choose d such that

$$\min\{t|u(t-0) \neq u(t)\} = d \leq d + d_f - m_f \leq \min\{t|x(t-0) \neq x(t)\}$$

be true, showing the fulfillment of a.2). The situation when $u(-\infty+0) = 1$ is similar.

b) We show the non-anticipation in the sense of Definition 65 item ix): $\exists d$, $\exists d'$, $0 \leq d \leq d'$ and

$$\forall t \in \mathbf{R}, \forall u \in U, \forall v \in U, u_{|[t-d',t-d]} = v_{|[t-d',t-d]} \Longrightarrow$$

$$\Longrightarrow \{x(t)|x \in f_{BD}^{m_r,d_r,m_f,d_f}(u)\} = \{y(t)|y \in f_{BD}^{m_r,d_r,m_f,d_f}(v)\}.$$

Indeed, let us take $d = 0, d' = \max\{d_r, d_f\}$ and $t \in \mathbf{R}$ arbitrary. Because

$$[t-d', t] \supset [t-d_r, t-d_r+m_r], [t-d', t] \supset [t-d_f, t-d_f+m_f],$$

we infer that

$$\forall u \in U, \forall v \in U, u_{|[t-d',t-d]} = v_{|[t-d',t-d]} \Longrightarrow$$

$$\Longrightarrow \bigcap_{\xi \in [t-d_r, t-d_r+m_r]} u(\xi) = \bigcap_{\xi \in [t-d_r, t-d_r+m_r]} v(\xi)$$

and

$$\bigcup_{\xi \in [t-d_f, t-d_f+m_f]} u(\xi) = \bigcup_{\xi \in [t-d_f, t-d_f+m_f]} v(\xi),$$

thus

$$\{x(t)|x \in f_{BD}^{m_r,d_r,m_f,d_f}(u)\} = \{y(t)|y \in f_{BD}^{m_r,d_r,m_f,d_f}(v)\}.$$

□

THEOREM 284. *If $f \subset f_{BD}^{m_r,d_r,m_f,d_f}$ is a bounded delay, then f is also non-anticipatory in the sense of Definition 63.*

PROOF. This fact follows from Theorem 172. □

REMARK 117. *Sometimes the bounded delays are anticipatory in the senses of this concept other than Definition 63 and the delay from Remark 116 gives a counterexample for the non-anticipation in the sense of Definition 65, v). For the inputs* $\chi_{[0,\infty)}$ *and* $\chi_{[0,3)}$ *we have* $\chi_{[0,\infty)|(-\infty,2]} = \chi_{[0,3)|(-\infty,2]}$, *but*

$$\{x_{|(-\infty,2]}|x \in f(\chi_{[0,\infty)})\} = \chi_{[1,\infty)|(-\infty,2]} \neq \{y_{|(-\infty,2]}|\chi_{[2,4)}(t) \leq y(t) \leq \chi_{[1,5)}(t)\} =$$
$$= \{y_{|(-\infty,2]}|y \in f(\chi_{[0,3)})\}.$$

11. Fixed delays and inertial delays

COROLLARY 8. *(of Theorem 281) The deterministic bounded delays are given by the equation*

$$x(t) = u(t-d), \tag{11.1}$$

where $d \geq 0$; *the non-deterministic bounded delays consist in the system (1.1), where* $m_r + m_f > 0$.

DEFINITION 112. *For* $u, x \in S$ *and* $d \geq 0$, *the delay (11.1) is called the* ***fixed delay*** *(**condition**, **property**, **model**). Other acceptable term is that of **pure**, **ideal** or **non-inertial delay**.*

*A delay different from the fixed delay is called **inertial**.*

REMARK 118. *For the fixed and the pure delays, the informal Definitions 101, 102 coincide.*

In Definition 112 the inertia was defined to be the property of the delays of being not ideal. In particular, the non-deterministic delays, for example the bounded delays where $m_r + m_f > 0$, *are inertial.*

A special case of inclusion in Theorem 270, a) consists in the situation when the left bounded delay is deterministic. Let $d \in [d'_r - m'_r, d'_r] \cap [d'_f - m'_f, d'_f]$ *(item b) of that theorem states* $d \in [d'_r - m'_r, d'_f] \cap [d'_f - m'_f, d'_r]$ *and we can prove that* $[d'_r - m'_r, d'_f] \cap [d'_f - m'_f, d'_r] = [d'_r - m'_r, d'_r] \cap [d'_f - m'_f, d'_f]$*); then from the statements*

$$\begin{aligned} t - d'_r &\leq t - d \leq t - d'_r + m'_r, \\ t - d'_f &\leq t - d \leq t - d'_f + m'_f, \end{aligned}$$

$$\bigcap_{\xi \in [t-d'_r, t-d'_r+m'_r]} u(\xi) \leq u(t-d) \leq \bigcup_{\xi \in [t-d'_f, t-d'_f+m'_f]} u(\xi)$$

we conclude that $I_d \subset f_{BD}^{m'_r, d'_r, m'_f, d'_f}$, *indeed.*

We give now some properties of the fixed delays.

The delay I_d *has race-free initial states, a bounded initial time and its initial state function* $\phi_0 : S \to \mathbf{B}$ *is defined as:* $\forall u \in S, \phi_0(u) = u(-\infty + 0)$, *Corollary 5. The delay* I_d *is self-dual, Theorem 274. The inverse of* I_d *is* I_{-d} *that, in general, is a delay but not a bounded delay (it is bounded only if* $d = 0$*). The serial connection of the fixed delays is a fixed delay and it coincides with the composition of the translations: for* $d \geq 0, d' \geq 0$, *we have* $d + d' \geq 0$ *and*

$$I_d \circ I_{d'} = I_{d'} \circ I_d = I_{d+d'},$$

special case of Theorem 275. The delay I_d *is time invariant (Theorem 282) and non-anticipatory in the sense of Definition 63 and of Definition 65, v),...,ix) (Theorem 283).*

The delay I_d is also injective

$$\forall u \in S, \forall v \in S, u \neq v \Longrightarrow I_d(u) \neq I_d(v),$$

$$\forall u \in S, \forall v \in S, u \neq v \Longrightarrow I_d(u) \cap I_d(v) = \emptyset$$

and surjective

$$\forall x \in S, \exists u \in S, I_d(u) = x,$$

$$\exists t \in \mathbf{R}, \forall \lambda \in \mathbf{B}, \exists u \in S, I_d(u)(t) = \lambda.$$

12. Other definitions of the bounded delays

THEOREM 285. *Let be the numbers $d_r > 0, d_f > 0$. For any $u \in S$, we have*

$$\lim_{m_r \nearrow d_r} \bigcap_{\xi \in [t-d_r, t-d_r+m_r]} u(\xi) = \bigcap_{\xi \in [t-d_r, t)} u(\xi),$$

$$\lim_{m_f \nearrow d_f} \bigcup_{\xi \in [t-d_f, t-d_f+m_f]} u(\xi) = \bigcup_{\xi \in [t-d_f, t)} u(\xi).$$

PROOF. Prove the first relation. We fix arbitrarily $t \in R, u \in S$. Then there is an $\varepsilon > 0$ with

$$\forall \xi \in [t-\varepsilon, t), u(\xi) = u(t-0).$$

For any $m_r \in (d_r - \varepsilon, d_r)$, we obtain $u(t - d_r + m_r) = u(t-0)$, thus

$$\bigcap_{\xi \in [t-d_r, t-d_r+m_r]} u(\xi) = \bigcap_{\xi \in [t-d_r, t-d_r+m_r]} u(\xi) \cdot u(t-0) =$$

$$= \bigcap_{\xi \in [t-d_r, t-d_r+m_r]} u(\xi) \cdot \bigcap_{\xi \in [t-d_r+m_r, t)} u(\xi) = \bigcap_{\xi \in [t-d_r, t)} u(\xi).$$

The second relation is proved similarly. □

REMARK 119. *By using the previous result and the Convention from Remark 96, we take the limits in (1.1) as $m_r \nearrow d_r$ and $m_f \nearrow d_f$. Then the double inequality becomes*[2]

$$\bigcap_{\xi \in [t-d_r, t)} u(\xi) \leq x(t) \leq \bigcup_{\xi \in [t-d_f, t)} u(\xi). \tag{12.1}$$

The system needs no consistency condition since

$$\bigcap_{\xi \in [t-d_r, t)} u(\xi) \leq u(t-d) \leq \bigcup_{\xi \in [t-d_f, t)} u(\xi)$$

is true for any $d \in (0, \min\{d_r, d_f\})$.

THEOREM 286. *For any $d_r > 0, d_f > 0$, the inequality (12.1) defines a delay.*

PROOF. The inequalities (12.1), by the previously discussed facts, define a system. Suppose that $\exists \lambda \in \mathbf{B}$, such that $u \in S_c(\lambda)$ i.e. $\exists t_f \in \mathbf{R}, u_{|[t_f, \infty)} = \lambda$. Because $\bigcap_{\xi \in [\cdot - d_r, \cdot)} u(\xi)_{|[t_f + d_r, \infty)} = \bigcup_{\xi \in [\cdot - d_f, \cdot)} u(\xi)_{|[t_f + d_f, \infty)} = \lambda$, we have that any $x \in S$ satisfying (12.1) belongs to $S_c(\lambda)$. □

[2]This is just a possible interpretation of that Convention, when m_r, m_f and d_r, d_f are not equal, but infinitely close.

Definition 113. *Give the numbers* $d_r > 0, d_f > 0$. *The delay*

$$\bigcap_{\xi\in[t-d_r,t)} u(\xi) \leq x(t) \leq \bigcup_{\xi\in[t-d_f,t)} u(\xi)$$

is called the ***boundedness property*** *and is denoted by* $f_{BD'}^{d_r,d_f}$. *The parameters* d_r, d_f *are the* (***rising, falling***) ***upper bounds of the transport delays*** (***transmission delays for transitions***).

Definition 114. *A delay* f *is called* ***bounded*** *if there are* $d_r > 0, d_f > 0$ *such that* $f \subset f_{BD'}^{d_r,d_f}$.

Remark 120. *Another term for* $f \subset f_{BD'}^{d_r,d_f}$ *might be that of* ***upper bounded, lower unbounded delay***.

In general, the properties of $f_{BD'}^{d_r,d_f}$ *repeat the properties of* $f_{BD}^{m_r,d_r,m_f,d_f}$. *For example* $f_{BD'}^{d_r,d_f} \subset f_{BD'}^{d'_r,d'_f}$ *is equivalent to* $d_f \leq d'_f, d_r \leq d'_r$; *the dual of* $f_{BD'}^{d_r,d_f}$ *is* $f_{BD'}^{d_f,d_r}$ *and self-duality means* $d_r = d_f$; $f_{BD'}^{d'_r,d'_f} \circ f_{BD'}^{d_r,d_f} = f_{BD'}^{d_r+d'_r,d_f+d'_f}$ *and so on.*

Of course, other variants of the concept of bounded delay may be defined, starting with the properties

$$\bigcap_{\xi\in[t-d_r,t)} u(\xi) \leq x(t) \leq \bigcup_{\xi\in[t-d_f,t-d_f+m_f]} u(\xi), \tag{12.2}$$

$$u(t-0) \leq x(t) \leq \bigcup_{\xi\in[t-d_f,t-d_f+m_f]} u(\xi), \tag{12.3}$$

$$\bigcap_{\xi\in[t-d_r,t-d_r+m_r]} u(\xi) \leq x(t) \leq \bigcup_{\xi\in[t-d_f,t)} u(\xi), \tag{12.4}$$

$$\bigcap_{\xi\in[t-d_r,t-d_r+m_r]} u(\xi) \leq x(t) \leq u(t-0) \tag{12.5}$$

where, in principle, $0 \leq m_r \leq d_r, 0 \leq m_f \leq d_f$ *are still true. The relation (12.3) has followed by passing to the limit in (12.2) as* $d_r \searrow 0$. *Similarly, (12.5) has followed by passing to the limit in (12.4) as* $d_f \searrow 0$. *For example, (12.3) has solutions iff it is of the form*

$$u(t-0) \leq x(t) \leq \bigcup_{\xi\in[t-d_f,t]} u(\xi),$$

where $d_f > 0$.

CHAPTER 12

Absolutely inertial delays

Let $\delta_r \geq 0, \delta_f \geq 0$ be two parameters. The delay f is absolutely inertial if $\forall u \in U, \forall x \in f(u)$ when x switches from 0 to 1 it remains there more than δ_r time units and dually when x switches from 1 to 0 it remains there more than δ_f time units. This type of inertia was called 'absolute' because the property is independent on the choice of u. In the chapter, several definitions for absolute inertial delays are provided together with their related properties. A version of the absolute inertia is indicated at the end of the chapter together with zenoness, representing, in some sense, the absence of the absolute inertia.

1. The first definition of the absolutely inertial delays

Theorem 287. *Let be the numbers $\delta_r \geq 0, \delta_f \geq 0$. When $x \in S$, the following statements are equivalent:*

$$(1.1)\qquad \begin{aligned} \overline{x(t-0)} \cdot x(t) &\leq \bigcap_{\xi \in [t, t+\delta_r]} x(\xi), \\ x(t-0) \cdot \overline{x(t)} &\leq \bigcap_{\xi \in [t, t+\delta_f]} \overline{x(\xi)}; \end{aligned}$$

$$(1.2)\qquad \begin{aligned} \overline{x(t-0)} \cdot x(t) &\leq \overline{x(t-0)} \cdot \bigcap_{\xi \in [t, t+\delta_r]} x(\xi), \\ x(t-0) \cdot \overline{x(t)} &\leq x(t-0) \cdot \bigcap_{\xi \in [t, t+\delta_f]} \overline{x(\xi)}; \end{aligned}$$

$$(1.3)\qquad \begin{aligned} \overline{x(t-0)} \cdot x(t) &= \overline{x(t-0)} \cdot \bigcap_{\xi \in [t, t+\delta_r]} x(\xi), \\ x(t-0) \cdot \overline{x(t)} &= x(t-0) \cdot \bigcap_{\xi \in [t, t+\delta_f]} \overline{x(\xi)}; \end{aligned}$$

$\forall t \in \mathbf{R}, \forall t' \in \mathbf{R},$

$$(1.4)\qquad \begin{aligned} (t &< t' \text{ and } \overline{x(t-0)} \cdot x(t) = 1 \text{ and } x(t'-0) \cdot \overline{x(t')} = 1) \Longrightarrow t' - t > \delta_r, \\ (t &< t' \text{ and } x(t-0) \cdot \overline{x(t)} = 1 \text{ and } \overline{x(t'-0)} \cdot x(t') = 1) \Longrightarrow t' - t > \delta_f; \end{aligned}$$

$$(1.5)\qquad \begin{aligned} \overline{x(t-0)} \cdot x(t) &\leq \overline{x(t-\delta_f-0)} \cdot \bigcap_{\xi \in [t-\delta_f, t)} \overline{x(\xi)}, \\ x(t-0) \cdot \overline{x(t)} &\leq x(t-\delta_r-0) \cdot \bigcap_{\xi \in [t-\delta_r, t)} x(\xi). \end{aligned}$$

PROOF. (1.1) $\Longrightarrow$ (1.2) Both sides of the first inequality from (1.1) are multiplied by $\overline{x(t-0)}$ to follow the first inequality from (1.2).

(1.2) $\Longrightarrow$ (1.1) From the first inequality of (1.2) we get the first inequality of (1.1):

$$\overline{x(t-0)} \cdot x(t) \leq \overline{x(t-0)} \cdot \bigcap_{\xi \in [t, t+\delta_r]} x(\xi) \leq \bigcap_{\xi \in [t, t+\delta_r]} x(\xi).$$

(1.1) $\Longrightarrow$ (1.3) In the following inequalities

$$\overline{x(t-0)} \cdot x(t) \leq \bigcap_{\xi \in [t, t+\delta_r]} x(\xi) \leq x(t),$$

obtained from the first inequality (1.1), we multiply all the terms by $\overline{x(t-0)}$ to get the first inequality (1.3).

(1.3) $\Longrightarrow$ (1.1) The first inequality (1.3) implies the first inequality (1.1):

$$\overline{x(t-0)} \cdot x(t) = \overline{x(t-0)} \cdot \bigcap_{\xi \in [t, t+\delta_r]} x(\xi) \leq \bigcap_{\xi \in [t, t+\delta_r]} x(\xi).$$

(1.1) $\Longrightarrow$ (1.4) Let t, t' be arbitrary, such that

$$t < t' \ and \ \overline{x(t-0)} \cdot x(t) = 1 \ and \ x(t'-0) \cdot \overline{x(t')} = 1.$$

(1.1) states that $\bigcap_{\xi \in [t, t+\delta_r]} x(\xi) = 1$ and $\bigcap_{\xi \in [t', t'+\delta_f]} \overline{x(\xi)} = 1$, thus $[t, t+\delta_r] \cap [t', t' + \delta_f] = \emptyset$ wherefrom $t + \delta_r < t'$. The conclusion is $t' - t > \delta_r$. The first implication (1.4) was proved.

(1.4) $\Longrightarrow$ (1.1) Suppose that the first inequality of (1.1) is not true, i.e. there is $t \in \mathbf{R}$ with

$$\overline{x(t-0)} \cdot x(t) = 1 \ and \bigcap_{\xi \in [t, t+\delta_r]} x(\xi) = 0,$$

meaning the existence of $t' \in (t, t+\delta_r]$, where x switches from 1 to 0

$$x(t'-0) \cdot \overline{x(t')} = 1.$$

We have $t' - t \leq \delta_r$, contradiction with the first implication from (1.4).

(1.4) $\Longrightarrow$ (1.5) If the first inequality of (1.5) is not true, then there is $t' \in \mathbf{R}$ such that

$$\overline{x(t'-0)} \cdot x(t') = 1 \ and \ \overline{x(t'-\delta_f-0)} \cdot \bigcap_{\xi \in [t'-\delta_f, t')} \overline{x(\xi)} = 0.$$

This means that for any $\varepsilon > 0$ there is some $t \in [t' - \delta_f - \varepsilon, t')$ with

$$x(t-0) \cdot \overline{x(t)} = 1.$$

The inequality $t' - \delta_f - \varepsilon \leq t$, true for all $\varepsilon > 0$, gives $t' - t \leq \delta_f$. Thus the second implication (1.4) is not true.

(1.5) $\Longrightarrow$ (1.4) Let us take two numbers t, t', such that

$$t < t' \ and \ \overline{x(t-0)} \cdot x(t) = 1 \ and \ x(t'-0) \cdot \overline{x(t')} = 1,$$

implying

$$\overline{x(t-\delta_f-0)} \cdot \bigcap_{\xi \in [t-\delta_f, t)} \overline{x(\xi)} = 1 \ and \ x(t'-\delta_r-0) \cdot \bigcap_{\xi \in [t'-\delta_r, t')} x(\xi) = 1.$$

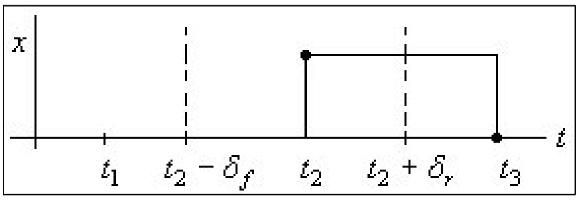

FIGURE 1. The interpretation of absolute inertia

In other words, there are $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ with the property

$$\forall \xi \in [t-\delta_f-\varepsilon_1, t), x(\xi) = 0,$$
$$\forall \xi \in [t'-\delta_r-\varepsilon_2, t'), x(\xi) = 1,$$
$$[t-\delta_f-\varepsilon_1, t) \cap [t'-\delta_r-\varepsilon_2, t') = \emptyset.$$

The last empty intersection leads to the conclusion that

$$t \leq t'-\delta_r-\varepsilon_2$$

is true, i.e. $t'-t \geq \delta_r+\varepsilon_2 > \delta_r$ (the other possibility $t' \leq t-\delta_f-\varepsilon_1$ is false since in $t'-t \leq -\delta_f-\varepsilon_1$ the left-hand side is positive and the right-hand side is negative). We have proved the first implication of (1.4). □

DEFINITION 115. *The autonomous system $f_{AI}^{\delta_r,\delta_f} \subset S$ defined by any of the equivalent properties (1.1),...,(1.5) is called the* ***absolute inertia property****. The numbers δ_r, δ_f are called the* ***(rising, falling) inertial parameters****. When $\delta_r = \delta_f = 0$, $f_{AI}^{\delta_r,\delta_f}$ is called* ***trivial*** *otherwise we say that it is* ***non-trivial****.*

REMARK 121. *The absolute inertia is the property of the signals to keep their value constant more than a given time interval after each switch. Thus they are characterized by a certain slowness. This interpretation follows from Figure 1. We note how the switch $\overline{x(t_2-0)} \cdot x(t_2) = 1$ in the (1.1) version assures that x will remain 1 during a time interval of length $t_3-t_2 > \delta_r$ while in the (1.5) version it assures that x has remained 0 for a time interval of length $t_2-t_1 > \delta_f$. When t runs over $\mathbf{R}$, the two properties are equivalent.*

Remark the way that any of (1.1),...,(1.5) degenerates in the trivial situation $\delta_r = \delta_f = 0 : f_{AI}^{0,0} = S$. Remark also the intermediary situations when one of $\delta_r > 0, \delta_f = 0$ and $\delta_r = 0, \delta_f > 0$ is true and the inclusions $f_{AI}^{\delta_r,\delta_f} \subset S$ are strict.

The set $f_{AI}^{\delta_r,\delta_f}$ is not closed under the Boolean laws if $\delta_r > 0$ or $\delta_f > 0$. For example, $\chi_{[0,2)}, \chi_{[1,3)} \in f_{AI}^{1,0}$, but $\chi_{[0,2)} \cdot \chi_{[1,3)} = \chi_{[1,2)} \notin f_{AI}^{1,0}$.

For any $\delta_r \geq 0$, $\delta_f \geq 0$ and $x \in S$, there is some $t_0 \in \mathbf{R}$ such that for $t < t_0$ any of (1.1),...,(1.5) is fulfilled since $\overline{x(t-0)} \cdot x(t) = x(t-0) \cdot \overline{x(t)} = 0$. This property represents the consistency between the absolute inertia and the way the signals were defined. It is somehow similar to the fact that at the bounded delays we had $\forall u \in S, \forall x \in f_{BD}^{m_r,d_r,m_f,d_f}(u)$, $x(-\infty+0) = u(-\infty+0)$.

DEFINITION 116. *The delay f is called* ***absolutely inertial*** *if there are $\delta_r \geq 0, \delta_f \geq 0$ such that*

$$\forall u \in S, f(u) \subset f_{AI}^{\delta_r,\delta_f}.$$

Sometimes we say that f ***satisfies the absolute inertia property*** *$f_{AI}^{\delta_r,\delta_f}$.*

THEOREM 288. *Let f be some delay and $\delta_r \geq 0, \delta_f \geq 0$, such that $\forall u \in S, f(u) \cap f_{AI}^{\delta_r,\delta_f} \neq \emptyset$. Then f defines the absolutely inertial delay $f \cap f_{AI}^{\delta_r,\delta_f}$.*

PROOF. The relationship $f \cap f_{AI}^{\delta_r,\delta_f} \subset f$ shows that $f \cap f_{AI}^{\delta_r,\delta_f}$ is a delay, while $f \cap f_{AI}^{\delta_r,\delta_f} \subset f_{AI}^{\delta_r,\delta_f}$ shows that the delay is absolutely inertial. □

DEFINITION 117. *The delay $f \cap f_{AI}^{\delta_r,\delta_f}$ defined in the conditions of the previous theorem is called the* ***absolutely inertial delay induced by*** f.

REMARK 122. *Intuitively, we say that the absolutely inertial delay f expresses a cause-effect relationship between an input and a family of inertial states, such that for any $u \in S$, the variations of the delayed signal $x \in f(u)$ cannot be faster than the fulfillment of $f_{AI}^{\delta_r,\delta_f}$ allows.*

Because $f_{AI}^{0,0} = S$, each delay f is trivially absolutely inertial: $f \subset f_{AI}^{0,0}$.

2. Order

THEOREM 289. *Given the non-negative numbers $\delta_r, \delta_f, \delta'_r, \delta'_f$, the following statements are equivalent:*

a) $f_{AI}^{\delta_r,\delta_f} \subset f_{AI}^{\delta'_r,\delta'_f}$;

b) $\delta_r \geq \delta'_r$ *and* $\delta_f \geq \delta'_f$.

PROOF. In (1.1), $\bigcap\limits_{\xi \in [t,t+\delta_r]} x(\xi) \leq \bigcap\limits_{\xi \in [t,t+\delta'_r]} x(\xi)$ is true for all $x \in S$ iff $[t, t+\delta_r] \supset [t, t+\delta'_r]$, meaning that $\delta_r \geq \delta'_r$. Similarly for $\delta_f \geq \delta'_f$. □

REMARK 123. *From the previous theorem and from the conjunction of the statements $f \subset f_{AI}^{\delta_r,\delta_f}$ and $f_{AI}^{\delta_r,\delta_f} \subset f_{AI}^{\delta'_r,\delta'_f}$ we conclude that if the delay f has the property $f_{AI}^{\delta_r,\delta_f}$, then it has also the property $f_{AI}^{\delta'_r,\delta'_f}$, when $\delta_r \geq \delta'_r$ and $\delta_f \geq \delta'_f$.*

On the other hand, let be the absolutely inertial delay f: $\exists \delta_r \geq 0, \exists \delta_f \geq 0, f \subset f_{AI}^{\delta_r,\delta_f}$. It is interesting the study of that property $f_{AI}^{\delta_r,\delta_f}$ with

i) $f \subset f_{AI}^{\delta_r,\delta_f}$

ii) for any $f_{AI}^{\delta'_r,\delta'_f}$ with $f \subset f_{AI}^{\delta'_r,\delta'_f}$ we have $f_{AI}^{\delta_r,\delta_f} \subset f_{AI}^{\delta'_r,\delta'_f}$

i.e. $f_{AI}^{\delta_r,\delta_f}$ is the smallest absolute inertia property in the sense of the inclusion satisfied by f.

COROLLARY 9. *If $g \subset f_{AI}^{\delta'_r,\delta'_f}$ is an absolutely inertial delay, then for any subdelay $f \subset g$, there are $\delta_r \geq \delta'_r$ and $\delta_f \geq \delta'_f$ such that $f \subset f_{AI}^{\delta_r,\delta_f}$.*

3. Duality

THEOREM 290. *Let be $\delta_r \geq 0, \delta_f \geq 0$. The dual of the system $f_{AI}^{\delta_r,\delta_f}$ is $f_{AI}^{\delta_f,\delta_r}$.*

PROOF. For any $u \in S$, we have

$$(f_{AI}^{\delta_r,\delta_f})^*(u) = \{\overline{x} | \overline{x(t-0)} \cdot x(t) \leq \bigcap_{\xi \in [t,t+\delta_r]} x(\xi), x(t-0) \cdot \overline{x(t)} \leq \bigcap_{\xi \in [t,t+\delta_f]} \overline{x(\xi)}\} =$$

$$= \{x | \overline{\overline{x(t-0)}} \cdot \overline{x(t)} \leq \bigcap_{\xi \in [t,t+\delta_r]} \overline{x(\xi)}, \overline{x(t-0)} \cdot \overline{\overline{x(t)}} \leq \bigcap_{\xi \in [t,t+\delta_f]} \overline{\overline{x(\xi)}}\} =$$

$$= \{x|x(t-0)\cdot\overline{x(t)} \leq \bigcap_{\xi\in[t,t+\delta_r]} \overline{x(\xi)}, \overline{x(t-0)}\cdot x(t) \leq \bigcap_{\xi\in[t,t+\delta_f]} x(\xi)\} = f_{AI}^{\delta_f,\delta_r}(u).$$

□

THEOREM 291. *Suppose that $f \subset f_{AI}^{\delta_r,\delta_f}$ is an absolutely inertial delay. Then $f^* \subset f_{AI}^{\delta_f,\delta_r}$ is an absolutely inertial delay.*

PROOF. From Theorem 261, f^* is a delay. The inclusion $f^* \subset f_{AI}^{\delta_f,\delta_r}$ follows from Theorem 49 and also from the previous theorem. However, we can show this fact directly:

$$\forall u \in S, f^*(u) = \{\overline{x}|x \in f(\overline{u})\} \subset \{\overline{x}|x \in f_{AI}^{\delta_r,\delta_f}\} = \{x|\overline{x} \in f_{AI}^{\delta_r,\delta_f}\} = f_{AI}^{\delta_f,\delta_r}.$$

□

THEOREM 292. *The property $f_{AI}^{\delta_r,\delta_f}$ is self-dual iff $\delta_r = \delta_f$.*

PROOF. *If* is obvious.

Only if From $f_{AI}^{\delta_r,\delta_f} \subset f_{AI}^{\delta_f,\delta_r}$ we infer (see Theorem 289) that $\delta_r \geq \delta_f, \delta_f \geq \delta_r$. □

4. Serial connection

THEOREM 293. *Let be the delays f, g. The inclusion $f \subset f_{AI}^{\delta_r,\delta_f}$ implies $f \circ g \subset f_{AI}^{\delta_r,\delta_f}$. In particular, the serial connection of the absolutely inertial delays with the parameters δ_r, δ_f is an absolutely inertial delay with the parameters δ_r, δ_f.*

PROOF. Obvious. □

5. Intersection

THEOREM 294. *Let be the non-negative numbers $\delta_r, \delta_f, \delta'_r, \delta'_f$. We have:*

$$f_{AI}^{\delta_r,\delta_f} \cap f_{AI}^{\delta'_r,\delta'_f} = f_{AI}^{\max\{\delta_r,\delta'_r\},\max\{\delta_f,\delta'_f\}}. \tag{5.1}$$

PROOF. Let be t_0 such that $\overline{x(t_0-0)}\cdot x(t_0) = 1$. If the inequalities

$$\overline{x(t-0)}\cdot x(t) \leq \bigcap_{\xi\in[t,t+\delta_r]} x(\xi),$$

$$\overline{x(t-0)}\cdot x(t) \leq \bigcap_{\xi\in[t,t+\delta'_r]} x(\xi)$$

are both true, we infer

$$1 = \bigcap_{\xi\in[t_0,t_0+\delta_r]} x(\xi)\cdot \bigcap_{\xi\in[t_0,t_0+\delta'_r]} x(\xi) = \bigcap_{\xi\in[t_0,t_0+\max\{\delta_r,\delta'_r\}]} x(\xi)$$

and, similarly, if t_1 is chosen such that $x(t_1-0)\cdot\overline{x(t_1)} = 1$ be true, the inequalities

$$x(t-0)\cdot\overline{x(t)} \leq \bigcap_{\xi\in[t,t+\delta_f]} \overline{x(\xi)},$$

$$x(t-0)\cdot\overline{x(t)} \leq \bigcap_{\xi\in[t,t+\delta'_f]} \overline{x(\xi)}$$

being both fulfilled, then

$$1 = \bigcap_{\xi\in[t_1,t_1+\delta_f]} \overline{x(\xi)} \cdot \bigcap_{\xi\in[t_1,t_1+\delta'_f]} \overline{x(\xi)} = \bigcap_{\xi\in[t_1,t_1+\max\{\delta_f,\delta'_f\}]} \overline{x(\xi)}$$

is true. We have proved the inclusion

$$f_{AI}^{\delta_r,\delta_f} \cap f_{AI}^{\delta'_r,\delta'_f} \subset f_{AI}^{\max\{\delta_r,\delta'_r\},\max\{\delta_f,\delta'_f\}}.$$

On the other hand, from Theorem 289 we infer $f_{AI}^{\max\{\delta_r,\delta'_r\},\max\{\delta_f,\delta'_f\}} \subset f_{AI}^{\delta_r,\delta_f}$, $f_{AI}^{\max\{\delta_r,\delta'_r\},\max\{\delta_f,\delta'_f\}} \subset f_{AI}^{\delta'_r,\delta'_f}$. Thus

$$f_{AI}^{\max\{\delta_r,\delta'_r\},\max\{\delta_f,\delta'_f\}} \subset f_{AI}^{\delta_r,\delta_f} \cap f_{AI}^{\delta'_r,\delta'_f}.$$

□

THEOREM 295. *Let $f \subset f_{AI}^{\delta_r,\delta_f}$ be an absolutely inertial delay and $g : S \to P^*(S)$ some system. If $\forall u \in S, f(u) \cap g(u) \neq \emptyset$, then $f \cap g \subset f_{AI}^{\delta_r,\delta_f}$ is an absolutely inertial delay.*

PROOF. From Theorem 258 it follows that $f \cap g$ is a delay and we have $f \cap g \subset f \subset f_{AI}^{\delta_r,\delta_f}$. □

THEOREM 296. *Let be the absolutely inertial delays $f \subset f_{AI}^{\delta_r,\delta_f}, g \subset f_{AI}^{\delta'_r,\delta'_f}$ and suppose that $\forall u \in S, f(u) \cap g(u) \neq \emptyset$. The delay $f \cap g$ is absolutely inertial with $f \cap g \subset f_{AI}^{\max\{\delta_r,\delta'_r\},\max\{\delta_f,\delta'_f\}}$.*

PROOF. The intersection $f \cap g$ is a delay satisfying $f \cap g \subset f \subset f_{AI}^{\delta_r,\delta_f}$ and $f \cap g \subset g \subset f_{AIC}^{\delta'_r,\delta'_f}$. From this we get that $f \cap g \subset f_{AI}^{\delta_r,\delta_f} \cap f_{AI}^{\delta'_r,\delta'_f} = f_{AI}^{\max\{\delta_r,\delta'_r\},\max\{\delta_f,\delta'_f\}}$. We have taken into account (5.1). □

THEOREM 297. *For all $\delta_r \geq 0, \delta_f \geq 0$ the intersection $f_{UD} \cap f_{AI}^{\delta_r,\delta_f}$ is a delay and the following property of serial connection*

$$(f_{UD} \cap f_{AI}^{\delta_r,\delta_f}) \circ (f_{UD} \cap f_{AI}^{\delta'_r,\delta'_f}) = f_{UD} \cap f_{AI}^{\delta_r,\delta_f}$$

is true, where $\delta'_r \geq 0, \delta'_f \geq 0$.

PROOF. Let $u \in S$ be arbitrary. If $\exists \lambda \in \mathbf{B}, u \in S_c(\lambda)$, then $\lambda \in f_{UD}(u) \cap f_{AI}^{\delta_r,\delta_f}$ and if $u \in S \setminus S_c$, then $f_{UD}(u) \cap f_{AI}^{\delta_r,\delta_f} = S \cap f_{AI}^{\delta_r,\delta_f} = f_{AI}^{\delta_r,\delta_f} \neq \emptyset$. We have proved that $\forall \delta_r \geq 0, \forall \delta_f \geq 0, \forall u \in S, f_{UD}(u) \cap f_{AI}^{\delta_r,\delta_f} \neq \emptyset$. Thus, for any $\delta_r \geq 0, \delta_f \geq 0$, $f_{UD} \cap f_{AI}^{\delta_r,\delta_f}$ is a function $S \to P^*(S)$ that is a delay because it is included in f_{UD}.

We have

$$((f_{UD} \cap f_{AI}^{\delta_r,\delta_f}) \circ (f_{UD} \cap f_{AI}^{\delta'_r,\delta'_f}))(u) = \bigcup_{x\in(f_{UD}\cap f_{AI}^{\delta'_r,\delta'_f})(u)} (f_{UD} \cap f_{AI}^{\delta_r,\delta_f})(x) =$$

$$= f_{AI}^{\delta_r,\delta_f} \cap \bigcup_{x \in f_{UD}(u) \cap f_{AI}^{\delta'_r,\delta'_f}} f_{UD}(x) = f_{AI}^{\delta_r,\delta_f} \cap \begin{cases} \bigcup\limits_{x \in S_c(0) \cap f_{AI}^{\delta'_r,\delta'_f}} f_{UD}(x), u \in S_c(0) \\ \bigcup\limits_{x \in S_c(1) \cap f_{AI}^{\delta'_r,\delta'_f}} f_{UD}(x), u \in S_c(1) \\ \bigcup\limits_{x \in S \cap f_{AI}^{\delta'_r,\delta'_f}} f_{UD}(x), u \in S \setminus S_c \end{cases} =$$

$$= f_{AI}^{\delta_r,\delta_f} \cap \begin{cases} \bigcup\limits_{x \in S_c(0) \cap f_{AI}^{\delta'_r,\delta'_f}} S_c(0), u \in S_c(0) \\ \bigcup\limits_{x \in S_c(1) \cap f_{AI}^{\delta'_r,\delta'_f}} S_c(1), u \in S_c(1) \\ \bigcup\limits_{x \in f_{AI}^{\delta'_r,\delta'_f}} S, u \in S \setminus S_c \end{cases} = f_{AI}^{\delta_r,\delta_f} \cap \begin{cases} S_c(0), u \in S_c(0) \\ S_c(1), u \in S_c(1) \\ S, u \in S \setminus S_c \end{cases} =$$

$$= (f_{UD} \cap f_{AI}^{\delta_r,\delta_f})(u).$$

□

THEOREM 298. *Let be the delays f, g and the numbers $\delta_r \geq 0, \delta_f \geq 0$, such that $\forall u \in S, f(u) \cap f_{AI}^{\delta_r,\delta_f} \neq \emptyset$. The formula*

$$(f \cap f_{AI}^{\delta_r,\delta_f}) \circ g = (f \circ g) \cap f_{AI}^{\delta_r,\delta_f}$$

is true.

PROOF. For any $u \in S$, we get

$$((f \cap f_{AI}^{\delta_r,\delta_f}) \circ g)(u) = \{y | \exists x, y \in f(x) \text{ and } y \in f_{AI}^{\delta_r,\delta_f} \text{ and } x \in g(u)\} =$$

$$= ((f \circ g) \cap f_{AI}^{\delta_r,\delta_f})(u).$$

□

6. Union

THEOREM 299. *For any numbers $\delta_r \geq 0, \delta_f \geq 0, \delta'_r \geq 0, \delta'_f \geq 0$, we have*

$$f_{AI}^{\delta_r,\delta_f} \cup f_{AI}^{\delta'_r,\delta'_f} = f_{AI}^{\min\{\delta_r,\delta'_r\},\min\{\delta_f,\delta'_f\}}. \tag{6.1}$$

PROOF. Similar with the proof of Theorem 294. □

THEOREM 300. *Given the absolutely inertial delays $f \subset f_{AI}^{\delta_r,\delta_f}$ and $g \subset f_{AI}^{\delta'_r,\delta'_f}$, the delay $f \cup g$ is absolutely inertial with $f \cup g \subset f_{AI}^{\min\{\delta_r,\delta'_r\},\min\{\delta_f,\delta'_f\}}$.*

PROOF. The delay $f \cup g$ is included in $f_{AI}^{\delta_r,\delta_f} \cup f_{AI}^{\delta'_r,\delta'_f}$ and we take into account (6.1). □

7. Time invariance

THEOREM 301. *For any $\delta_r \geq 0, \delta_f \geq 0$ the system $f_{AI}^{\delta_r,\delta_f}$ is time invariant.*

PROOF. Let $u \in S$ and $d \in \mathbf{R}$ be arbitrary. We have that (1.1) and

$$\overline{x(t+d-0)} \cdot x(t+d) \leq \bigcap_{\xi \in [t,t+\delta_r]} x(\xi+d),$$

$$x(t+d-0) \cdot \overline{x(t+d)} \leq \bigcap_{\xi \in [t,t+\delta_f]} \overline{x(\xi+d)}$$

are equivalent (in the sense that they have the same solutions), wherefrom

$$f_{AI}^{\delta_r,\delta_f}(u \circ \tau^d) = f_{AI}^{\delta_r,\delta_f} = \{x | x \circ \tau^{-d} \in f_{AI}^{\delta_r,\delta_f}\} = \{x \circ \tau^d | x \in f_{AI}^{\delta_r,\delta_f}\}.$$

□

THEOREM 302. *Let f be some time invariant delay and suppose that the property $\forall u \in S, f(u) \cap f_{AI}^{\delta_r,\delta_f} \neq \emptyset$ is fulfilled. Then the induced delay $f \cap f_{AI}^{\delta_r,\delta_f}$ is time invariant.*

PROOF. Let $u \in S$ and $d \in \mathbf{R}$ be arbitrary. From the previous theorem we have

$$(f \cap f_{AI}^{\delta_r,\delta_f})(u \circ \tau^d) = f(u \circ \tau^d) \cap f_{AI}^{\delta_r,\delta_f} = \{x \circ \tau^d | x \in f(u)\} \cap \{x \circ \tau^d | x \in f_{AI}^{\delta_r,\delta_f}\} =$$

$$= \{x \circ \tau^d | x \in f(u) \cap f_{AI}^{\delta_r,\delta_f}\} = \{x \circ \tau^d | x \in (f \cap f_{AI}^{\delta_r,\delta_f})(u)\}.$$

The same statement follows from the fact that $f \cap f_{AI}^{\delta_r,\delta_f}$ is a subdelay of f, that is time invariant as intersection of time invariant systems (Theorem 168). □

8. Examples of absolutely inertial delays

THEOREM 303. *$I_d \cap f_{AI}^{\delta_r,\delta_f}, d \in \mathbf{R}$ is an absolutely inertial delay for $\delta_r = \delta_f = 0$ only.*

PROOF. Suppose against all reason that there is $\delta_r > 0$ with $\forall u \in S, I_d(u) \cap f_{AI}^{\delta_r,\delta_f} \neq \emptyset$. The input $u = \chi_{[0,\delta)}$, where $0 < \delta \leq \delta_r$, satisfies the property

$$I_d(u) = \chi_{[0,\delta)} \circ \tau^d \notin f_{AI}^{\delta_r,\delta_f},$$

contradiction. The situation is similar if we suppose that there is $\delta_f > 0$ such that $I_d \cap f_{AI}^{\delta_r,\delta_f}$ is an absolutely inertial delay. □

EXAMPLE 104. *The two deterministic delays from Example 98 as well as the following non-deterministic delays*

$$f(u) = \begin{cases} \{1\} \cup \{\chi_{[d,\infty)} | d \in \mathbf{R}\}, \ if \ u \in S_c(1) \\ \{0\} \cup \{\chi_{(-\infty,d)} | d \in \mathbf{R}\}, \ otherwise \end{cases},$$

$$f(u) = \begin{cases} \{0\} \cup \{\chi_{(-\infty,d)} | d \in \mathbf{R}\}, \ if \ u \in S_c(0) \\ \{1\} \cup \{\chi_{[d,\infty)} | d \in \mathbf{R}\}, \ otherwise \end{cases}$$

satisfy $f_{AI}^{\delta_r,\delta_f}$ for all $\delta_r \geq 0, \delta_f \geq 0$.

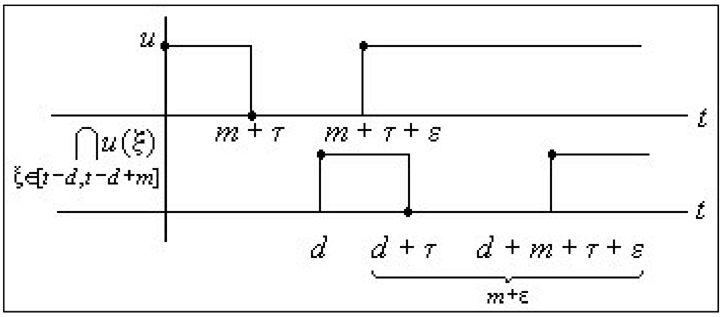

FIGURE 2. Deterministic delay, Theorem 304 a)

THEOREM 304. *Let be* $0 \leq m \leq d$.
a) The deterministic delay (from Example 101)

$$x(t) = \bigcap_{\xi\in[t-d,t-d+m]} u(\xi)$$

satisfies $x \in f_{AI}^{0,m}$ *and for any* δ_r, δ_f *such that* $\delta_r > 0$ *or* $\delta_f > m$*, there is an* $u \in S$ *with* $x \notin f_{AI}^{\delta_r,\delta_f}$. *In other words,* $0, m$ *are the largest values that* δ_r, δ_f *can take making* $\forall u \in S, x \in f_{AI}^{\delta_r,\delta_f}$ *true.*
b) The delay (from Example 101)

$$x(t) = \bigcup_{\xi\in[t-d,t-d+m]} u(\xi)$$

satisfies $x \in f_{AI}^{m,0}$*; for any* δ_r, δ_f *with one of* $\delta_r > m$*,* $\delta_f > 0$ *true, there is an input* $u \in S$ *such that* $x \notin f_{AI}^{\delta_r,\delta_f}$*, meaning that* $m, 0$ *are the largest values of* δ_r, δ_f *making* $\forall u \in S, x \in f_{AI}^{\delta_r,\delta_f}$ *true.*

PROOF. a) In the second property (1.4) the hypothesis states

$$t' > t \text{ and } x(t-0) \cdot \overline{x(t)} = 1 \text{ and } \overline{x(t'-0)} \cdot x(t') = 1,$$

wherefrom we infer (see Theorem 22)

$$u(t-d-0) \cdot \bigcap_{\xi\in[t-d,t-d+m)} u(\xi) \cdot \overline{u(t-d+m)} = 1,$$

$$\overline{u(t'-d-0)} \cdot \bigcap_{\xi\in[t'-d,t'-d+m]} u(\xi) = 1,$$

i.e. $t-d+m = t'-d-\varepsilon$ for some $\varepsilon > 0$, and, eventually $t'-t > m$.

Suppose that there are $\delta_r > 0$ and $\delta_f \geq 0$ such that $x \in f_{AI}^{\delta_r,\delta_f}$. We take some $\varepsilon \in (0, \delta_r)$ and $u = \chi_{[0,m+\varepsilon)}$ for which we have

$$x(t) = \bigcap_{\xi\in[t-d,t-d+m]} \chi_{[0,m+\varepsilon)}(\xi) = \chi_{[d,d+\varepsilon)}(t).$$

Thus $x \notin f_{AI}^{\delta_r,\delta_f}$, contradiction. Similarly, the assumption that there are $\delta_r \geq 0$ and $\delta_f > m$ with $x \in f_{AI}^{\delta_r,\delta_f}$ brings us to the following counterexample. Let be

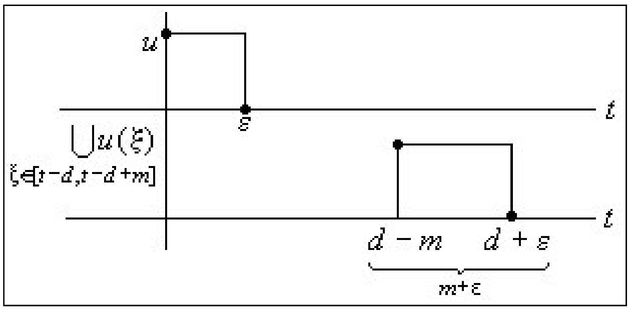

FIGURE 3. Deterministic delay, Theorem 304 b)

$u = \chi_{(-\infty,0)\cup[\varepsilon,\infty)}$, where $\varepsilon \in (0, \delta_f - m)$. We have

$$x(t) = \bigcap_{\xi\in[t-d,t-d+m]} \chi_{(-\infty,0)\cup[\varepsilon,\infty)}(\xi) = \chi_{(-\infty,d-m)\cup[d+\varepsilon,\infty)}(t).$$

Because $d + \varepsilon - (d - m) = \varepsilon + m < \delta_f$, we infer the contradiction $x \notin f_{AI}^{\delta_r,\delta_f}$. □

THEOREM 305. *The delay* $f_{UD} \cap f_{AI}^{\delta_r,\delta_f}$ *is a time invariant absolutely inertial delay included in* $f_{AI}^{\delta_r,\delta_f}$ *for any* $\delta_r \geq 0, \delta_f \geq 0$*;* $f_{UD} \cap f_{AI}^{\delta_r,\delta_f}$ *is self-dual iff* $\delta_r = \delta_f$*.*

PROOF. The intersection $f_{UD} \cap f_{AI}^{\delta_r,\delta_f}$ is a delay (Theorem 297) and the fact that it is included in $f_{AI}^{\delta_r,\delta_f}$ is clear.

In order to prove the time invariance, we note that f_{UD} is time invariant (Theorem 252) and $f_{AI}^{\delta_r,\delta_f}$ is itself time invariant (Theorem 301). Thus their intersection is time invariant (Theorem 168).

Prove now the self-duality statement. In order to show the necessity, we can write

$$(f_{UD} \cap f_{AI}^{\delta_r,\delta_f})^* = f_{UD}^* \cap f_{AI}^{\delta_r,\delta_f *} = f_{UD} \cap f_{AI}^{\delta_f,\delta_r} = f_{UD} \cap f_{AI}^{\delta_r,\delta_f}$$

and suppose against all reason that $\delta_r = \delta_f$ is false, for example, $\delta_r < \delta_f$. Then, for $u = 0$, $\chi_{[0,\delta_f)} \in (f_{UD} \cap f_{AI}^{\delta_r,\delta_f})(u)$ and $\chi_{[0,\delta_f)} \notin (f_{UD} \cap f_{AI}^{\delta_f,\delta_r})(u)$, contradiction showing that $\delta_r \geq \delta_f$ is necessary. Similarly, we have $\delta_r \leq \delta_f$. Thus the equality $\delta_r = \delta_f$ is necessary. Obviously this equality is sufficient in order that $f_{UD} \cap f_{AI}^{\delta_r,\delta_f}$ be self-dual. □

9. Other definitions of absolute inertia

REMARK 124. *Another definition of absolute inertia is given by the replacement of inequalities (1.1) with the inequalities*

$$\overline{x(t-0)} \cdot x(t) \leq \bigcap_{\xi\in[t,t+\delta_r)} x(\xi) \tag{9.1}$$
$$x(t-0) \cdot \overline{x(t)} \leq \bigcap_{\xi\in[t,t+\delta_f)} \overline{x(\xi)}$$

where $\delta_r \geq 0, \delta_f \geq 0$ *again, autonomous system that is denoted by* $f_{AI'}^{\delta_r,\delta_f}$*. When* $\delta_r = 0, \delta_f = 0$ *the previous inequalities take the trivial form, like at* $f_{AI}^{\delta_r,\delta_f}$*. The* *study of $f_{AI'}^{\delta_r,\delta_f}$ is similar to that of $f_{AI}^{\delta_r,\delta_f}$. The difference is that in (1.4) $t'-t>\delta_r, t'-t>\delta_f$ are replaced by $t'-t\geq\delta_r, t'-t\geq\delta_f$.*

There is still the possibility of combining the first inequality from (1.1) with the second inequality from (9.1), or the first inequality from (9.1) with the second inequality from (1.1).

EXAMPLE 105. *The system defined by the inequalities (Theorem 322 will analyze a similar system)*

$$\overline{x(t-0)}\cdot x(t)=\bigcap_{\xi\in[t-\delta_f,t)}\overline{x(\xi)}\cdot u(t), \tag{9.2}$$

$$x(t-0)\cdot\overline{x(t)}=\bigcap_{\xi\in[t-\delta_r,t)}x(\xi)\cdot\overline{u(t)}, \tag{9.3}$$

$\delta_r>0,\delta_f>0$, is a deterministic time invariant delay. Its solution satisfies $x\in f_{AI'}^{\delta_r,\delta_f}$; it is self-dual iff $\delta_r=\delta_f$.

PROOF. We can suppose that a solution of (9.2), (9.3) always exists. We show that the system is a delay and let $u\in S_c(1)$. Then there is $t_f\in\mathbf{R}$ such that $u_{|[t_f,\infty)}=1$ and $\forall t\geq t_f$ (9.2), (9.3) take the form

$$\overline{x(t-0)}\cdot x(t)=\bigcap_{\xi\in[t-\delta_f,t)}\overline{x(\xi)}, \tag{9.4}$$

$$x(t-0)\cdot\overline{x(t)}=0. \tag{9.5}$$

If $\bigcap\limits_{\xi\in[t_f-\delta_f,t_f)}\overline{x(\xi)}=1$, then $x(t_f)=1$. In addition we have $x_{|[t_f,\infty)}=1$.

For $\bigcap\limits_{\xi\in[t_f-\delta_f,t_f)}\overline{x(\xi)}=0$, there are two possibilities:

a) $x(t_f-0)=0$. Because $\exists\xi\in[t_f-\delta_f,t_f),x(\xi)=1$, we put $\xi_f=\sup\{\xi|\xi\in[t_f-\delta_f,t_f),x(\xi)=1\}$ and we have that $x_{|[\xi_f+\delta_f,\infty)}=1$;

b) $x(t_f-0)=1$. Then $x_{|[t_f,\infty)}=1$.

Similarly, we can show that if $u\in S_c(0)$, then any solution x of (9.2), (9.3) belongs to $S_c(0)$.

The determinism of the delay is proved like this. There is $t_0\in\mathbf{R}$ such that $x_{|(-\infty,t_0)}=x(-\infty+0),u_{|(-\infty,t_0)}=u(-\infty+0)$. Hence we have $x(-\infty+0)=u(-\infty+0)$. The assumption of existence of $t_1\geq t_0$ and of two distinct solutions x,y of (9.2), (9.3) satisfying $x_{|(-\infty,t_1)}=y_{|(-\infty,t_1)},x(t_1)\neq y(t_1)$ gives a contradiction.

The time invariance follows from the fact that if x is the solution of the system, then for any $d\in\mathbf{R}$ we get that $x\circ\tau^d$ is the solution of the system

$$\overline{y(t-0)}\cdot y(t)=\bigcap_{\xi\in[t-\delta_f,t)}\overline{y(\xi)}\cdot u(t-d),$$

$$y(t-0)\cdot\overline{y(t)}=\bigcap_{\xi\in[t-\delta_r,t)}y(\xi)\cdot\overline{u(t-d)}.$$

The absolute inertia is a consequence of the remark that

$$\overline{x(t-0)}\cdot x(t)=\bigcap_{\xi\in[t-\delta_f,t)}\overline{x(\xi)}\cdot u(t)\leq\bigcap_{\xi\in[t-\delta_f,t)}\overline{x(\xi)},$$

$$x(t-0)\cdot\overline{x(t)}=\bigcap_{\xi\in[t-\delta_r,t)} x(\xi)\cdot\overline{u(t)}\leq\bigcap_{\xi\in[t-\delta_r,t)} x(\xi)$$

and these two inequalities are a version of (1.5).

The self-duality implies the equivalence between (9.2), (9.3) and the following system

$$\overline{x(t-0)}\cdot x(t)=\bigcap_{\xi\in[t-\delta_r,t)}\overline{x(\xi)}\cdot u(t),$$

$$x(t-0)\cdot\overline{x(t)}=\bigcap_{\xi\in[t-\delta_f,t)} x(\xi)\cdot\overline{u(t)}.$$

We can prove that this occurs iff $\delta_r=\delta_f$. □

10. Zeno delays

DEFINITION 118. *If the system $f:S\to P^*(S)$ satisfies one of the properties*
i) $\forall\varepsilon>0,\exists t\in\mathbf{R},\exists t'\in\mathbf{R},\exists u\in S,\exists x\in f(u)$,

$$\overline{x(t-0)}\cdot x(t)=1 \text{ and } x(t'-0)\cdot\overline{x(t')}=1 \text{ and } 0<t'-t<\varepsilon,$$

ii) $\forall\varepsilon>0,\exists t\in\mathbf{R},\exists t'\in\mathbf{R},\exists u\in S,\exists x\in f(u)$,

$$x(t-0)\cdot\overline{x(t)}=1 \text{ and } \overline{x(t'-0)}\cdot x(t')=1 \text{ and } 0<t'-t<\varepsilon,$$

then it is called ***Zeno****.*

THEOREM 306. *If the delay f is Zeno, then any delay $g\supset f$ is Zeno. If f is not Zeno, then all its subdelays $g\subset f$ are not Zeno.*

REMARK 125. *Zenoness (named so after the name of Zeno of Elea 495?-435? B.C.) is considered a non-natural property for the delay f, in the sense that the existence of the inputs that produce pulses of the states with an arbitrarily short length does not correspond to the behavior of the devices from the digital electrical engineering, that are characterized by a certain slowness. For example the delay I_d is Zeno.*

In general, in literature the concept of Zenoness is slightly differently presented, mainly due to the fact that the authors do not use the same notion of signal like us. Let us quote Karl Henrik Johansson[1] *saying 'Zeno hybrid automata accept executions*[2] *with infinitely many discrete transitions within a finite time interval. Real physical systems are, of course, not Zeno, but hybrid automata modeling real systems may be Zeno. The Zeno phenomena is often due to a too high level of abstraction'. Other authors deal with the Zeno signals. We reproduce here the opinion on the Zeno conditions expressed by Edward A. Lee*[3]*: the model 'illustrates a Zeno condition, where an infinite number of events occur before t_0 and hence the clock actor is unable to ever produce its output at time t_0'. 'Eventually, execution stops advancing time'.*

In modeling, we prefer to use the delays that are not Zeno.

THEOREM 307. *For the delay f the following statements are equivalent:*
a) f is not Zeno;
b) there are $\delta_r>0$ and $\delta_f>0$ such that $f\subset f_{AI'}^{\delta_r,\delta_f}$.

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[1]Hybrid systems, EECS291E, UC Berkeley, Spring 2000, Lecture #4, Zeno Hybrid Automata
[2]I.e. have states $x\in f(u)$.
[3]Advanced Topics in Systems Theory, EECS290N, UC Berkeley, Fall 2004.

Proof. The negation of the zenoness property from Definition 118 may be written under the form: the following statements are true:

i) $\exists\delta_r > 0, \forall t \in \mathbf{R}, \forall t' \in \mathbf{R}, \forall u \in S, \forall x \in f(u)$,

$$(t < t' \textit{ and } \overline{x(t-0)} \cdot x(t) = 1 \textit{ and } x(t'-0) \cdot \overline{x(t')} = 1) \Longrightarrow t' - t \geq \delta_r;$$

ii) $\exists\delta_f > 0, \forall t \in \mathbf{R}, \forall t' \in \mathbf{R}, \forall u \in S, \forall x \in f(u)$,

$$(t < t' \textit{ and } x(t-0) \cdot \overline{x(t)} = 1 \textit{ and } \overline{x(t'-0)} \cdot x(t') = 1) \Longrightarrow t' - t \geq \delta_f$$

and this is equivalent to $f \subset f_{AI'}^{\delta_r,\delta_f}$ (see also (1.4)). □

CHAPTER 13

Relatively inertial delays

Relative inertia is the property of the states of having their speed of variation limited by a function depending on the input, while the relatively inertial delays are these delays the states of which are relatively inertial. Even if the concept has close connections with the published literature, it has a severe shortcoming, that we have called the paradox of inertia: the serial connection of two relatively inertial delays is not a relatively inertial delay. A counterexample is given.

Some major properties of these delays are presented with examples and the relationship with the absolute inertia and zenoness is analyzed.

1. Relative inertia

THEOREM 308. *Let be the numbers* $0 \leq \mu_r \leq \delta_r$, $0 \leq \mu_f \leq \delta_f$. *The following statements are equivalent:*

$$\begin{aligned} \overline{x(t-0)} \cdot x(t) &\leq \bigcap_{\xi \in [t-\delta_r, t-\delta_r+\mu_r]} u(\xi), \\ x(t-0) \cdot \overline{x(t)} &\leq \bigcap_{\xi \in [t-\delta_f, t-\delta_f+\mu_f]} \overline{u(\xi)}, \end{aligned} \tag{1.1}$$

$$\begin{aligned} \overline{x(t-0)} \cdot x(t) &\leq \overline{x(t-0)} \cdot \bigcap_{\xi \in [t-\delta_r, t-\delta_r+\mu_r]} u(\xi), \\ x(t-0) \cdot \overline{x(t)} &\leq x(t-0) \cdot \bigcap_{\xi \in [t-\delta_f, t-\delta_f+\mu_f]} \overline{u(\xi)}, \end{aligned} \tag{1.2}$$

where $u, x \in S$.

PROOF. Similar to (1.1) $\Longleftrightarrow$ (1.2) part of the proof from Theorem 287. $\square$

DEFINITION 119. *The system* $f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f} : S \to P^*(S)$ *defined by any of the properties (1.1), (1.2) is called the* ***relative inertia property***. *The numbers* $\mu_r, \delta_r, \mu_f, \delta_f$ *are the* ***(rising, falling) inertial parameters***.

REMARK 126. *The attribute 'relative' given to inertia refers to the fact that the property* $f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$ *depends on* u, *as opposed to the former case of 'absolute' inertia* $f_{AI}^{\delta_r,\delta_f}$, *when the system was autonomous.*

$f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$ *is a system, indeed. For example* $\forall u \in S$, *the constant functions* $0, 1 \in S$ *belong to* $f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}(u)$.

The consistency between the relative inertia property and the initial values $u(-\infty+0)$, $x(-\infty+0)$ *is given by the fact that there is* t_0 *such that for any* $t < t_0$, *we have* $\overline{x(t-0)} \cdot x(t) = x(t-0) \cdot \overline{x(t)} = 0$. *Thus the relative inertia property is trivially fulfilled on some set* $t \in (-\infty, t_0)$.

On the other hand, for any $\lambda \in \mathbf{B}$, *we have that* $f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}(\lambda) = \{0,1\} \cup \{\overline{\lambda} \oplus \chi_{[d,\infty)} | d \in \mathbf{R}\}$, *where we have identified again the Boolean constant with the constant signal. This shows that when the input is constant the relatively inertial states are monotonous.*

Remark what the relative inertia property becomes in the 'trivial' case when $\delta_r = \delta_f = \mu_r = \mu_f = 0$

$$\overline{x(t-0)} \cdot x(t) \leq u(t), \tag{1.3}$$

$$x(t-0) \cdot \overline{x(t)} \leq \overline{u(t)}. \tag{1.4}$$

This is not trivial at all. The inequalities (1.3), (1.4) describe the situation when x *may switch only if it becomes equal to* u. *A situation more general than (1.3), (1.4), when* $\delta_r \geq 0$, $\delta_f \geq 0$, $\mu_r = \mu_f = 0$, *was called by us in previous works the* ***property of constancy*** *and the delays that fulfill it were called* ***constant***.

There is a variant of $f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$, *similar to the* $f_{BD'}^{d_r,d_f}$ *and* $f_{AI'}^{\delta_r,\delta_f}$ *variants of* $f_{BD}^{m_r,d_r,m_f,d_f}$ *and* $f_{AI}^{\delta_r,\delta_f}$, *i.e. we can replace (1.1) by*

$$\begin{aligned} \overline{x(t-0)} \cdot x(t) &\leq \bigcap_{\xi \in [t-\delta_r, t)} u(\xi), \\ x(t-0) \cdot \overline{x(t)} &\leq \bigcap_{\xi \in [t-\delta_f, t)} \overline{u(\xi)}, \end{aligned}$$

where $\delta_r > 0, \delta_f > 0$. *This property is denoted by* $f_{RI'}^{\delta_r,\delta_f}$ *and is analyzed similarly to* $f_{RI}^{\delta_r,\delta_f}$.

2. What other authors say

REMARK 127. *Again we interpret the relative inertia condition by quoting some informal definitions.*

[**15**], [**16**] *(see Definition 103, a)) state: the inertial delays 'model the fact that the practical circuits will not respond' at the output 'to two transitions' on the input 'which are very close together'. In case that two transitions of* u *are 'very close together', i.e. if* u *has a* 1*-pulse of length* $\leq \mu_r$, *then the function* $\bigcap_{\xi \in [t-\delta_r, t-\delta_r+\mu_r]} u(\xi)$ *is null on some set, where the function* $\overline{x(t-0)} \cdot x(t)$ *is null also: there are* $t_1 < t_2 < t_3 < t_4$ *such that* $t_3 - t_2 \leq \mu_r$ *and*

$$u(t) = u(t) \cdot \chi_{(-\infty,t_1)}(t) \oplus \chi_{[t_2,t_3)}(t) \oplus u(t) \cdot \chi_{[t_4,\infty)}(t),$$

$$\bigcap_{\xi \in [t-\delta_r, t-\delta_r+\mu_r]} u(\xi) = \bigcap_{\xi \in [t-\delta_r, t-\delta_r+\mu_r]} u(\xi) \cdot \chi_{(-\infty, t_1+\delta_r-\mu_r) \cup [t_4+\delta_r, \infty)}(t)$$

imply

$$\forall \xi \in [t_1 + \delta_r - \mu_r, t_4 + \delta_r), \overline{x(\xi-0)} \cdot x(\xi) = 0.$$

The situation is similar if u *has a* 0*-pulse of length* $\leq \mu_f$.

In this context we rewrite a quotation from [**14**] *(see Definition 103 b)): 'pulses shorter than or equal to the delay magnitude are not transmitted' in the following manner: 'pulses shorter than or equal to* μ_r *(respectively to* μ_f*) are not transmitted and pulses strictly longer than* μ_r *(respectively than* μ_f*) may be transmitted'.*

On the other hand, Definition 104, ii) and Definition 105, i) look, keeping the notations from the latter, like $f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$ *with* $\delta_r = \delta_f = d_{\min}$, $\mu_r = \mu_f = d_{\min} - 0$, *i.e. like* $f_{RI'}^{\delta_r,\delta_f}$. *This point of view agrees with the one from* [**15**], [**16**] *see Convention from Remark 96 stating that: 'the transmission delay for transitions is the same as the threshold for cancellation'. Here* δ_r, δ_f *act as 'transmission delays for transitions', even if they are rather 'minimum transmission delays for transitions' and* μ_r, μ_f *act as 'thresholds true for cancellation'. 'The same as' means that the two quantities differ by an infinitesimal.*

Furthermore, let us recall [**1**] *(see Remark 101) the quotation: 'changes should persist for at least* l_1 *time units but propagated after* $l_2, l_2 > l_1$ *time'. In our formalism represented by* $f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$ *if we accept the self-duality, we have:* $l_1 = \mu_r = \mu_f, l_2 = \delta_r = \delta_f$ *and 'changes should persist for strictly more than* l_1 *time units but propagated after more than or equal to* $l_2, l_2 \geq l_1$ *time'.*

3. The relationship between relative inertia and absolute inertia

THEOREM 309. *Let* $0 \leq \mu_r \leq \delta_r$, $0 \leq \mu_f \leq \delta_f$ *be arbitrary. If* $\delta_f \geq \delta_r - \mu_r$, $\delta_r \geq \delta_f - \mu_f$, *then* $\forall u \in S, f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}(u) \subset f_{AI}^{\delta_f-\delta_r+\mu_r,\delta_r-\delta_f+\mu_f}(u)$.

PROOF. Let $t, t' \in \mathbf{R}$, $u \in S$ and $x \in f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}(u)$ be arbitrary, such that

$$t < t' \textit{ and } \overline{x(t-0)} \cdot x(t) = 1 \textit{ and } x(t'-0) \cdot \overline{x(t')} = 1.$$

We get

$$\bigcap_{\xi \in [t-\delta_r, t-\delta_r+\mu_r]} u(\xi) = \bigcap_{\xi \in [t'-\delta_f, t'-\delta_f+\mu_f]} \overline{u(\xi)} = 1$$
$$\Longrightarrow [t-\delta_r, t-\delta_r+\mu_r] \cap [t'-\delta_f, t'-\delta_f+\mu_f] = \emptyset$$
$$\Longrightarrow t-\delta_r+\mu_r < t'-\delta_f \textit{ or } t'-\delta_f+\mu_f < t-\delta_r$$
$$\Longrightarrow t'-t > \delta_f-\delta_r+\mu_r \textit{ or } t'-t < \delta_f-\mu_f-\delta_r$$
$$\Longrightarrow t'-t > \delta_f-\delta_r+\mu_r$$

(The inequality $t'-t < \delta_f-\mu_f-\delta_r$ is false, because the left-hand side is strictly positive and the right-hand side is non-positive).

The proof is similar for the second inequality. □

REMARK 128. *The inequalities* $\delta_f \geq \delta_r - \mu_r, \delta_r \geq \delta_f - \mu_f$ *from the hypothesis of Theorem 309 are similar to* CC_{BD}.

4. Relatively inertial delays

DEFINITION 120. *Let be the numbers* $0 \leq \mu_r \leq \delta_r$, $0 \leq \mu_f \leq \delta_f$ *and the delay* f. *If* f *satisfies the condition*

$$\forall u \in S, f(u) \subset f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}(u),$$

then it is called the ***relatively inertial delay****. We use to say that* f ***satisfies the relative inertia property*** $f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$.

THEOREM 310. *Let* f *be a delay and* $0 \leq \mu_r \leq \delta_r, 0 \leq \mu_f \leq \delta_f$ *with the property that* $\forall u \in S, f(u) \cap f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}(u) \neq \emptyset$. *Then* f *defines the relatively inertial delay* $f \cap f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$.

Proof. $\forall u \in S, f(u) \cap f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}(u) \neq \emptyset$ and $f \cap f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f} \subset f$ show that $f \cap f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$ is a delay and from $f \cap f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f} \subset f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$ we infer that the delay is relatively inertial. □

Definition 121. *The delay $f \cap f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$ defined in the conditions of the previous theorem is called the* ***relatively inertial delay induced by*** f.

Remark 129. *The relatively inertial delays are the widest accepted inertial models and, at the same time, the most controversial. The controversies are generated by the fact that, in the non-formalized theories where they are used, it is not provable that the serial connection of two relatively inertial delays is a relatively inertial delay, a normal closure property. We shall refer to this important aspect later.*

Let be $0 \leq \mu_r \leq \delta_r$ and $0 \leq \mu_f \leq \delta_f$. The fact that the delay f is relatively inertial $f \subset f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$ and that $\delta_f \geq \delta_r - \mu_r, \delta_r \geq \delta_f - \mu_f$, implies by Theorem 309, that f is absolutely inertial $f \subset f_{AI}^{\delta_f-\delta_r+\mu_r,\delta_r-\delta_f+\mu_f}$. Moreover, if f' is an arbitrary delay with $\forall u \in S, f'(u) \cap f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}(u) \neq \emptyset$, then the induced delay $f' \cap f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$ is a subdelay of the induced delay $f' \cap f_{AI}^{\delta_f-\delta_r+\mu_r,\delta_r-\delta_f+\mu_f}$.

Example 106. *The extreme situations of relatively inertial delays $f \cap f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$ induced by the delay f are created by $f = f_{UD}$ and by $f = I_d$ respectively, that define relatively inertial delays for all $\mu_r, \delta_r, \mu_f, \delta_f$ such that $0 \leq \mu_r \leq \delta_r$, $0 \leq \mu_f \leq \delta_f$, and for $\mu_r = \mu_f = 0$, $\delta_r = \delta_f = d$ respectively.*

Example 107. *The two delays from Example 98 satisfy the relative inertia property $f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$ for all $0 \leq \mu_r \leq \delta_r$, $0 \leq \mu_f \leq \delta_f$.*

Example 108. *Let be $0 \leq m \leq d$. The delay $x(t) = \bigcap\limits_{\xi \in [t-d,t-d+m]} u(\xi)$ satisfies the relative inertia property under the form*

$$\begin{aligned} \overline{x(t-0)} \cdot x(t) &\leq \bigcap_{\xi \in [t-d,t-d+m]} u(\xi), \\ x(t-0) \cdot \overline{x(t)} &\leq \overline{u(t-d+m)}. \end{aligned}$$

Dually, the delay $x(t) = \bigcup\limits_{\xi \in [t-d,t-d+m]} u(\xi)$ is relatively inertial also:

$$\begin{aligned} \overline{x(t-0)} \cdot x(t) &\leq u(t-d+m), \\ x(t-0) \cdot \overline{x(t)} &\leq \bigcap_{\xi \in [t-d,t-d+m]} \overline{u(\xi)}. \end{aligned}$$

These facts follow from Theorem 22.

5. Order

Theorem 311. *Given the numbers $0 \leq \mu_r \leq \delta_r$, $0 \leq \mu_f \leq \delta_f$, $0 \leq \mu'_r \leq \delta'_r$, $0 \leq \mu'_f \leq \delta'_f$, the following statements are true:*

a) $f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f} \subset f_{RI}^{\mu'_r,\delta'_r,\mu'_f,\delta'_f}$;

b) $\delta_r \geq \delta'_r$, $\delta_f \geq \delta'_f$, $\delta_r - \mu_r \leq \delta'_r - \mu'_r$, $\delta_f - \mu_f \leq \delta'_f - \mu'_f$.

PROOF. $a) \Longrightarrow b)$ The inclusion from a) means that for any $u \in S$ we have:

$$\bigcap_{\xi\in[t-\delta_r,t-\delta_r+\mu_r]} u(\xi) \le \bigcap_{\xi\in[t-\delta'_r,t-\delta'_r+\mu'_r]} u(\xi);$$

$$\bigcap_{\xi\in[t-\delta_f,t-\delta_f+\mu_f]} \overline{u(\xi)} \le \bigcap_{\xi\in[t-\delta'_f,t-\delta'_f+\mu'_f]} \overline{u(\xi)},$$

i.e. $[t-\delta_r, t-\delta_r+\mu_r] \supset [t-\delta'_r, t-\delta'_r+\mu'_r]$, $[t-\delta_f, t-\delta_f+\mu_f] \supset [t-\delta'_f, t-\delta'_f+\mu'_f]$ and eventually b). At this moment, the implication $b) \Longrightarrow a)$ is obvious. $\square$

REMARK 130. *Suppose that* $f \subset f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$ *and* $f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f} \subset f_{RI}^{\mu'_r,\delta'_r,\mu'_f,\delta'_f}$ *are true. From Theorem 311 we have that if* $\mu_r, \delta_r, \mu_f, \delta_f$ *are the inertial parameters of* f*, then* $\mu'_r, \delta'_r, \mu'_f, \delta'_f$ *are also the inertial parameters of* f *whenever* $\delta_r \ge \delta'_r$, $\delta_f \ge \delta'_f$, $\delta_r - \mu_r \le \delta'_r - \mu'_r$, $\delta_f - \mu_f \le \delta'_f - \mu'_f$ *are true.*

Consider now the relatively inertial delay f*. We state the problem of finding that system* $f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$ *satisfying*

i) $f \subset f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$

ii) for any $f_{RI}^{\mu'_r,\delta'_r,\mu'_f,\delta'_f}$ *with* $f \subset f_{RI}^{\mu'_r,\delta'_r,\mu'_f,\delta'_f}$*, we have* $f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f} \subset f_{RI}^{\mu'_r,\delta'_r,\mu'_f,\delta'_f}$

i.e. $f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$ *is the smallest relative inertia property in the sense of the inclusion satisfied by* f*.*

6. Duality

THEOREM 312. *The dual of* $f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$ *is* $f_{RI}^{\mu_f,\delta_f,\mu_r,\delta_r}$.

PROOF. For any $u \in S$ we have:

$$(f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f})^*(u) =$$

$$= \{\overline{x} | \overline{x(t-0)} \cdot x(t) \le \bigcap_{\xi\in[t-\delta_r,t-\delta_r+\mu_r]} \overline{u(\xi)},\ x(t-0) \cdot \overline{x(t)} \le \bigcap_{\xi\in[t-\delta_f,t-\delta_f+\mu_f]} u(\xi)\} =$$

$$= \{x | x(t-0) \cdot \overline{x(t)} \le \bigcap_{\xi\in[t-\delta_r,t-\delta_r+\mu_r]} \overline{u(\xi)},\ \overline{x(t-0)} \cdot x(t) \le \bigcap_{\xi\in[t-\delta_f,t-\delta_f+\mu_f]} u(\xi)\} =$$

$$= f_{RI}^{\mu_f,\delta_f,\mu_r,\delta_r}(u).$$

$\square$

THEOREM 313. *If* $f \subset f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$ *is a relatively inertial delay, then* f^* *is a relatively inertial delay with* $f^* \subset f_{RI}^{\mu_f,\delta_f,\mu_r,\delta_r}$.

PROOF. For any $u \in S$ we can write

$$f^*(u) = \{\overline{x} | x \in f(\overline{u})\} \subset \{\overline{x} | x \in f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}(\overline{u})\} =$$

$$= \{x | \overline{x} \in f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}(\overline{u})\} = (f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f})^*(u) = f_{RI}^{\mu_f,\delta_f,\mu_r,\delta_r}(u)$$

and we have made use of Theorem 312. $\square$

7. Serial connection. The paradox of inertia

EXAMPLE 109. *The following deterministic delays (see Example 105)*

$$\overline{x(t-0)} \cdot x(t) = \bigcap_{\xi\in[t-2,t)} \overline{x(\xi)} \cdot u(t), \tag{7.1}$$

$$x(t-0) \cdot \overline{x(t)} = \bigcap_{\xi\in[t-2,t)} x(\xi) \cdot \overline{u(t)} \tag{7.2}$$

and

$$\overline{y(t-0)} \cdot y(t) = \bigcap_{\xi\in[t-4,t)} \overline{y(\xi)} \cdot x(t), \tag{7.3}$$

$$y(t-0) \cdot \overline{y(t)} = \bigcap_{\xi\in[t-4,t)} y(\xi) \cdot \overline{x(t)} \tag{7.4}$$

are relatively inertial:

$$\overline{x(t-0)} \cdot x(t) \leq u(t),$$
$$x(t-0) \cdot \overline{x(t)} \leq \overline{u(t)}$$

and respectively

$$\overline{y(t-0)} \cdot y(t) \leq x(t),$$
$$y(t-0) \cdot \overline{y(t)} \leq \overline{x(t)}.$$

For symmetry reasons, if their serial connection is relatively inertial, then it satisfies

$$\overline{y(t-0)} \cdot y(t) \leq \bigcap_{\xi\in[t-\delta,t-\delta+\mu]} u(\xi), \tag{7.5}$$

$$y(t-0) \cdot \overline{y(t)} \leq \bigcap_{\xi\in[t-\delta,t-\delta+\mu]} \overline{u(\xi)}, \tag{7.6}$$

where $0 \leq \mu \leq \delta$. *We choose* $u(t) = \chi_{[0,1)\cup[2,3)\cup[4,\infty)}(t)$, *for which (7.1), (7.2) give* $x(t) = \chi_{[0,3)\cup[5,\infty)}(t)$ *and, from (7.3), (7.4) we infer* $y(t) = \chi_{[0,4)\cup[8,\infty)}(t)$. *We have*

$$\bigcap_{\xi\in[t-\delta,t-\delta+\mu]} u(\xi) = \chi_{[\delta,1+\delta-\mu)\cup[2+\delta,3+\delta-\mu)\cup[4+\delta,\infty)}(t),$$

$$\bigcap_{\xi\in[t-\delta,t-\delta+\mu]} \overline{u(\xi)} = \chi_{(-\infty,\delta-\mu)\cup[1+\delta,2+\delta-\mu)\cup[3+\delta,4+\delta-\mu)}(t),$$

where, in principle, the intervals $[\delta, 1+\delta-\mu), [2+\delta, 3+\delta-\mu), [1+\delta, 2+\delta-\mu), [3+\delta, 4+\delta-\mu)$ *may be empty or non-empty. By taking into account (7.5), (7.6) we get:*

$$\chi_{\{0,8\}}(t) \leq \chi_{[\delta,1+\delta-\mu)\cup[2+\delta,3+\delta-\mu)\cup[4+\delta,\infty)}(t), \tag{7.7}$$

$$\chi_{\{4\}}(t) \leq \chi_{(-\infty,\delta-\mu)\cup[1+\delta,2+\delta-\mu)\cup[3+\delta,4+\delta-\mu)}(t). \tag{7.8}$$

(7.7) implies $\delta \leq 0$, *thus* $\delta = \mu = 0$. *This represents a contradiction with (7.8), that becomes*

$$\chi_{\{4\}}(t) \leq \chi_{(-\infty,0)\cup[1,2)\cup[3,4)}(t).$$

The conclusion is that the serial connection of the relatively inertial delays, in this case (7.1), (7.2) and (7.3), (7.4), is not always a relatively inertial delay. This failure was previously called by the author the 'paradox of inertia'.

8. Intersection

THEOREM 314. *For any numbers $0 \leq \mu_r \leq \delta_r, 0 \leq \mu_f \leq \delta_f$ and $0 \leq \mu'_r \leq \delta'_r, 0 \leq \mu'_f \leq \delta'_f$, the intersection $f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f} \cap f_{RI}^{\mu'_r,\delta'_r,\mu'_f,\delta'_f}$ is a $S \to P^*(S)$ system defined by the inequalities*

$$\overline{x(t-0)} \cdot x(t) \leq \bigcap_{\xi\in[t-\delta_r,t-\delta_r+\mu_r]} u(\xi) \cdot \bigcap_{\xi\in[t-\delta'_r,t-\delta'_r+\mu'_r]} u(\xi),$$

$$x(t-0) \cdot \overline{x(t)} \leq \bigcap_{\xi\in[t-\delta_f,t-\delta_f+\mu_f]} \overline{u(\xi)} \cdot \bigcap_{\xi\in[t-\delta'_f,t-\delta'_f+\mu'_f]} \overline{u(\xi)}.$$

PROOF. For any $u \in S$, the constant functions $0, 1 \in S$ belong to the intersection $f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}(u) \cap f_{RI}^{\mu'_r,\delta'_r,\mu'_f,\delta'_f}(u)$. The statement is obvious. □

REMARK 131. *If in the intersection $f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f} \cap f_{RI}^{\mu'_r,\delta'_r,\mu'_f,\delta'_f}$ we have $\forall t \in \mathbf{R}, [t-\delta_r, t-\delta_r+\mu_r] \cap [t-\delta'_r, t-\delta'_r+\mu'_r] \neq \emptyset$ and $[t-\delta_f, t-\delta_f+\mu_f] \cap [t-\delta'_f, t-\delta'_f+\mu'_f] \neq \emptyset$, then this intersection is a relative inertia property but, in general, the previous property is not true.*

If $f \subset f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$ is a delay and $g : S \to P^*(S)$ is a system with the property $\forall u \in S, f(u) \cap g(u) \neq \emptyset$, then we note that $f \cap g$ is a relatively inertial delay $\subset f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$.

9. Union

THEOREM 315. *Given the numbers $0 \leq \mu_r \leq \delta_r, 0 \leq \mu_f \leq \delta_f$, $0 \leq \mu'_r \leq \delta'_r, 0 \leq \mu'_f \leq \delta'_f$, the union $f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f} \cup f_{RI}^{\mu'_r,\delta'_r,\mu'_f,\delta'_f} : S \to P^*(S)$ is expressed by the inequalities*

$$\overline{x(t-0)} \cdot x(t) \leq \bigcap_{\xi\in[t-\delta_r,t-\delta_r+\mu_r]} u(\xi) \cup \bigcap_{\xi\in[t-\delta'_r,t-\delta'_r+\mu'_r]} u(\xi),$$

$$x(t-0) \cdot \overline{x(t)} \leq \bigcap_{\xi\in[t-\delta_f,t-\delta_f+\mu_f]} \overline{u(\xi)} \cup \bigcap_{\xi\in[t-\delta'_f,t-\delta'_f+\mu'_f]} \overline{u(\xi)}.$$

10. Non-anticipation

THEOREM 316. *For $0 \leq \mu_r \leq \delta_r, 0 \leq \mu_f \leq \delta_f$ the relative inertia property $f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$ is non-anticipatory in the sense of Definition 65, items v),...,ix).*

PROOF. The property of non-anticipation from Definition 65, v)

$$\forall t \in \mathbf{R}, \forall u \in S, \forall v \in S, u_{|(-\infty,t]} = v_{|(-\infty,t]} \Longrightarrow$$

$$\Longrightarrow \{x_{|(-\infty,t]} | x \in f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}(u)\} = \{y_{|(-\infty,t]} | y \in f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}(v)\}$$

takes place because for any $t \in \mathbf{R}$ and $u \in S$, the functions $\bigcap\limits_{\xi\in[t-\delta_r,t-\delta_r+\mu_r]} u(\xi)$, $\bigcap\limits_{\xi\in[t-\delta_f,t-\delta_f+\mu_f]} \overline{u(\xi)}$ depend on $u_{|(-\infty,t]}$ only etc. □

THEOREM 317. *Let f be a non-anticipatory delay in the sense of Definition 65, either of items v),...,ix) and suppose that* $\forall u \in S, f(u) \cap f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}(u) \neq \emptyset$. *Then the relatively inertial induced delay* $f \cap f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$ *satisfies the same non-anticipation property.*

PROOF. We can make use of Theorem 316 and of the version of Theorem 185 that is true for this definition. □

11. Time invariance

THEOREM 318. *The relative inertia property* $f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$ *is time invariant.*

PROOF. For any $d \in \mathbf{R}$ and any $u \in S$ we infer that

$$f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}(u \circ \tau^d) =$$

$$= \{x | \overline{x(t-0)} \cdot x(t) \leq \bigcap_{\xi \in [t-\delta_r, t-\delta_r+\mu_r]} (u \circ \tau^d)(\xi),$$

$$x(t-0) \cdot \overline{x(t)} \leq \bigcap_{\xi \in [t-\delta_f, t-\delta_f+\mu_f]} \overline{(u \circ \tau^d)(\xi)}\} =$$

$$= \{x | \overline{x(t-0)} \cdot x(t) \leq \bigcap_{\xi \in [t-d-\delta_r, t-d-\delta_r+\mu_r]} u(\xi),$$

$$x(t-0) \cdot \overline{x(t)} \leq \bigcap_{\xi \in [t-d-\delta_f, t-d-\delta_f+\mu_f]} \overline{u(\xi)}\} =$$

$$= \{x | \overline{x(t+d-0)} \cdot x(t+d) \leq \bigcap_{\xi \in [t-\delta_r, t-\delta_r+\mu_r]} u(\xi),$$

$$x(t+d-0) \cdot \overline{x(t+d)} \leq \bigcap_{\xi \in [t-\delta_f, t-\delta_f+\mu_f]} \overline{u(\xi)}\} =$$

$$= \{x \circ \tau^d | \overline{x(t-0)} \cdot x(t) \leq \bigcap_{\xi \in [t-\delta_r, t-\delta_r+\mu_r]} u(\xi), x(t-0) \cdot \overline{x(t)} \leq \bigcap_{\xi \in [t-\delta_f, t-\delta_f+\mu_f]} \overline{u(\xi)}\} =$$

$$= \{x \circ \tau^d | x \in f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}(u)\}.$$

□

THEOREM 319. *Let f be a time invariant delay. If* $\forall u \in S, f(u) \cap f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}(u) \neq \emptyset$, *then the relatively inertial delay induced by f,* $f \cap f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$ *is time invariant.*

PROOF. The function $f \cap f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$ is a delay, because it is included in f and it is non-empty. It is also time invariant as intersection of two time invariant systems. □

THEOREM 320. *Let be the numbers* $0 \leq \mu_r \leq \delta_r$, $0 \leq \mu_f \leq \delta_f$.
a) $f_{UD} \cap f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$ *is time invariant.*
b) $f_{UD} \cap f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$ *is self-dual iff* $\mu_r = \mu_f, \delta_r = \delta_f$.

PROOF. a) By Theorem 252, f_{UD} is time invariant and $\forall u \in S, f_{UD}(u) \cap f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}(u) \neq \emptyset$ is true because a constant function belongs to the intersection. $f_{UD} \cap f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$ is the intersection of two time invariant systems.

b) *Only if*

$$f_{UD} \cap f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f} = (f_{UD} \cap f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f})^* = f_{UD}^* \cap (f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f})^* = f_{UD} \cap f_{RI}^{\mu_f,\delta_f,\mu_r,\delta_r}$$

and we prove that this is true only if $\mu_r = \mu_f, \delta_r = \delta_f$. □

12. Zeno delays

THEOREM 321. *The condition necessary and sufficient in order that $f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$ be not Zeno is that $\delta_f > \delta_r - \mu_r$, $\delta_r > \delta_f - \mu_f$.*

PROOF. *The necessity* Suppose against all reason that $f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$ is not Zeno and $\delta_f \leq \delta_r - \mu_r$ and let be $u = \chi_{(-\infty,0)}$ for which

$$\begin{aligned}
\bigcap_{\xi \in [t-\delta_r, t-\delta_r+\mu_r]} u(\xi) &= \bigcap_{\xi \in [t-\delta_r, t-\delta_r+\mu_r]} \chi_{(-\infty,0)}(\xi) = \chi_{(-\infty,\delta_r-\mu_r)}(t), \\
\bigcap_{\xi \in [t-\delta_f, t-\delta_f+\mu_f]} \overline{u(\xi)} &= \bigcap_{\xi \in [t-\delta_f, t-\delta_f+\mu_f]} \chi_{[0,\infty)}(\xi) = \chi_{[\delta_f,\infty)}(t).
\end{aligned}$$

For any $\varepsilon > 0$ we have that there are $\varepsilon' \in (0, \varepsilon)$ and $x = \chi_{[\delta_f - \varepsilon', \delta_f)} \in f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}(u)$ contradiction with the hypothesis stating that $f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$ is not Zeno. The other assumption that $f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$ is not Zeno, but $\delta_r \leq \delta_f - \mu_f$, gives a similar contradiction.

The sufficiency We have

$$f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f} \subset f_{AI}^{\delta_f-\delta_r+\mu_r,\delta_r-\delta_f+\mu_f} \subset f_{AI'}^{\delta_f-\delta_r+\mu_r,\delta_r-\delta_f+\mu_f},$$

where the first inclusion follows from Theorem 309, while the second, from the fact that $[t, t+\delta_f-\delta_r+\mu_r] \supset [t, t+\delta_f-\delta_r+\mu_r), [t, t+\delta_r-\delta_f+\mu_f] \supset [t, t+\delta_r-\delta_f+\mu_f)$. The statement is a consequence of Theorem 307. □

COROLLARY 10. *No Zeno delay $f \subset f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$ exists iff $\delta_f > \delta_r - \mu_r$, $\delta_r > \delta_f - \mu_f$.*

13. The study of a deterministic delay

THEOREM 322. *Let $0 \leq m_r \leq d_r$, $0 \leq m_f \leq d_f$ be such that $d_f \geq d_r - m_r$, $d_r \geq d_f - m_f$ is true. The equations*

$$\overline{x(t-0)} \cdot x(t) = \overline{x(t-0)} \cdot \bigcap_{\xi \in [t-d_r, t-d_r+m_r]} u(\xi), \tag{13.1}$$

$$x(t-0) \cdot \overline{x(t)} = x(t-0) \cdot \bigcap_{\xi \in [t-d_f, t-d_f+m_f]} \overline{u(\xi)} \tag{13.2}$$

satisfy the following properties:

a) the consistency with the initial conditions: $\forall u \in S, \forall x \in S$, *if* x *satisfies (13.1), (13.2), then* $x(-\infty+0) = u(-\infty+0)$ *and there is* $t_0 \in \mathbf{R}$ *such that, for any* $t < t_0$, *we have*

$$\overline{x(t-0)} \cdot x(t) = \overline{x(t-0)} \cdot \bigcap_{\xi \in [t-d_r, t-d_r+m_r]} u(\xi) = 0, \tag{13.3}$$

$$x(t-0) \cdot \overline{x(t)} = x(t-0) \cdot \bigcap_{\xi \in [t-d_f, t-d_f+m_f]} \overline{u(\xi)} = 0; \tag{13.4}$$

b) (13.1), (13.2) define a deterministic system f*;*

c) the consistency of the system with the final conditions: f *is a delay and* $\forall u \in S_c$, *there is* $t_1 \in \mathbf{R}$ *such that* $\forall t \geq t_1$ *we have (13.3), (13.4) fulfilled;*

d) $f \subset f_{RI}^{m_r, d_r, m_f, d_f}$*;*

e) $f \subset f_{AI}^{d_f - d_r + m_r, d_r - d_f + m_f}$*;*

f) $f \subset f_{BD}^{m_r, d_r, m_f, d_f}$*;*

g) f *is time invariant;*

h) f *is self-dual iff* $d_r = d_f$ *and* $m_r = m_f$*;*

i) f *is non-anticipatory in the sense of Definitions 63 and 65, items v),...,ix).*

Proof. a) Let $u \in S, x \in S$ be arbitrary for which there are $t_0', t_0'' \in \mathbf{R}$ and $\lambda, \mu \in \mathbf{B}$ such that $x_{|(-\infty, t_0')} = \mu, u_{|(-\infty, t_0'')} = \lambda$. Thus, there is $t_0 < \min\{t_0', t_0'' + d_r - m_r, t_0'' + d_f - m_f\}$ such that (13.1), (13.2) become for $t < t_0$

$$\overline{\mu} \cdot \mu = \overline{\mu} \cdot \lambda,$$
$$\mu \cdot \overline{\mu} = \mu \cdot \overline{\lambda}.$$

We get $\mu = \lambda$.

b) For any $u \in S$, from a) we know that there is $t_0 \in \mathbf{R}$ such that for $t < t_0$ the solution is unique and is given by $x(t) = u(-\infty+0)$. The supposition that (13.1), (13.2) do not define a system means the existence of $t_1 \geq t_0$ such that at least a solution x exists for $t < t_1$ and at t_1 there is no solution. Let be $x(t_1 - 0) = 0$, for which (13.1) gives

$$x(t_1) = \bigcap_{\xi \in [t_1-d_r, t_1-d_r+m_r]} u(\xi),$$

contradiction; if $x(t_1 - 0) = 1$, (13.2) gives

$$x(t_1) = \overline{\bigcap_{\xi \in [t_1-d_f, t_1-d_f+m_f]} \overline{u(\xi)}},$$

contradiction again with the supposition that at t_1 there is no solution. As t_1 was arbitrary, the solution $x \in f(u)$ exists and because u was arbitrary, f is a system.

The proof of the uniqueness is similar to that of the existence, the difference being that the supposition 'at t_1 there is no solution' is replaced by 'at t_1 there are two solutions $x(t_1) = 0, x(t_1) = 1$'.

c) We fix some arbitrary $u \in S_c$ and we suppose that $\exists t_1 \in \mathbf{R}, \forall t \geq t_1, u(t) = 1$. Because $\forall t \geq t_1 + d_r$, $\bigcap\limits_{\xi \in [t-d_r, t-d_r+m_r]} u(\xi) = 1$ and $\bigcap\limits_{\xi \in [t-d_f, t-d_f+m_f]} \overline{u(\xi)} = 0$, we get that the unique solution of (13.1), (13.2) fulfills $\forall t \geq t_1 + d_r, x(t) = 1$. The situation is similar for $\exists t_1 \in \mathbf{R}, \forall t \geq t_1, u(t) = 0$, when $\forall t \geq t_1 + d_f, x(t) = 0$. We conclude that $\forall t \geq t_1 + \max\{d_r, d_f\}$, (13.3), (13.4) are true.

d) From (13.1), (13.2) we infer

$$\overline{x(t-0)} \cdot x(t) \leq \bigcap_{\xi \in [t-d_r, t-d_r+m_r]} u(\xi),$$

$$x(t-0) \cdot \overline{x(t)} \leq \bigcap_{\xi \in [t-d_f, t-d_f+m_f]} \overline{u(\xi)}.$$

e) We take into account item d), CC_{BD} and we apply Theorem 309.

f) We suppose against all reason that there are $t \in \mathbf{R}, u \in S$ such that $\bigcap_{\xi \in [t-d_r, t-d_r+m_r]} u(\xi) = 1$ and $x(t) = 0$; from (13.1) we get $\overline{x(t-0)} = 0$. Thus $x(t-0) = 1$ and, on the other hand, because $\bigcap_{\xi \in [t-d_f, t-d_f+m_f]} \overline{u(\xi)} = 0$ (from CC_{BD}), equation (13.2) gives $\overline{x(t)} = 0$, thus $x(t) = 1$, contradiction. In other words $\bigcap_{\xi \in [t-d_r, t-d_r+m_r]} u(\xi) \leq x(t)$. The inequality $x(t) \leq \bigcup_{\xi \in [t-d_f, t-d_f+m_f]} u(\xi)$ is similarly proved.

g) Let $u \in S, d \in \mathbf{R}$ be arbitrary and we replace x, u in (13.1), (13.2) by $y, u \circ \tau^d$. From the first equation we have the following equivalent properties:

$$\overline{y(t-0)} \cdot y(t) = \overline{y(t-0)} \cdot \bigcap_{\xi \in [t-d_r, t-d_r+m_r]} (u \circ \tau^d)(\xi);$$

$$\overline{y(t-0)} \cdot y(t) = \overline{y(t-0)} \cdot \bigcap_{\xi \in [t-d_r, t-d_r+m_r]} u(\xi - d);$$

$$\overline{y(t-0)} \cdot y(t) = \overline{y(t-0)} \cdot \bigcap_{\xi + d \in [t-d_r, t-d_r+m_r]} u(\xi);$$

$$\overline{y(t-0)} \cdot y(t) = \overline{y(t-0)} \cdot \bigcap_{\xi \in [t-d-d_r, t-d-d_r+m_r]} u(\xi);$$

$$\overline{y(t+d-0)} \cdot y(t+d) = \overline{y(t+d-0)} \cdot \bigcap_{\xi \in [t-d_r, t-d_r+m_r]} u(\xi), \tag{13.5}$$

while from the second equation respectively

$$y(t+d-0) \cdot \overline{y(t+d)} = y(t+d-0) \cdot \bigcap_{\xi \in [t-d_f, t-d_f+m_f]} \overline{u(\xi)}. \tag{13.6}$$

At this moment we compare (13.1), (13.2) with (13.5), (13.6), deterministic systems. We infer the fact that $y(t+d) = x(t)$, i.e.

$$y(t) = x(t-d) = (x \circ \tau^d)(t).$$

h) $\forall u \in S, f^*(u) = \overline{f(\overline{u})} = f(u)$, where $x = f(u)$, implies the fact that (13.1), (13.2) and

$$\overline{x(t-0)} \cdot x(t) = \overline{x(t-0)} \cdot \bigcap_{\xi \in [t-d_f, t-d_f+m_f]} u(\xi),$$

$$x(t-0) \cdot \overline{x(t)} = x(t-0) \cdot \bigcap_{\xi \in [t-d_r, t-d_r+m_r]} \overline{u(\xi)}$$

have the same solution. We choose $u = \chi_{[0,\delta)}$, where $\delta > \max\{m_r, m_f\}$ for which the unique solutions of the two systems are $\chi_{[d_r, \delta+d_f)}$ and $\chi_{[d_f, \delta+d_r)}$. Thus $d_r = d_f$. By supposing against all reason that $m_r \neq m_f$, we choose for example $m_r < m_f$, then $\delta \in (m_r, m_f)$; in this situation the unique solutions of the two systems become $\chi_{[d_r,\delta+d_f)}$ and 0, contradiction. The assumption that $m_r > m_f$ gives a contradiction too. We proved that $m_r = m_f$.

i) We show the non-anticipation in the sense of Definition 63. If x is constant, then the system is non-anticipatory. Thus we suppose that x is variable and this implies that u is variable too. Let $d \in \mathbf{R}$ be such that $d = \min\{t|u(t-0) \neq u(t)\}$ and take, for example, $u(-\infty+0) = 0$. Then, because $u(t) \leq \chi_{[d,\infty)}(t)$, we can write

$$\bigcap_{\xi\in[t-d_r,t-d_r+m_r]} u(\xi) \leq \bigcap_{\xi\in[t-d_r,t-d_r+m_r]} \chi_{[d,\infty)}(\xi) = \chi_{[d+d_r,\infty)}(t)$$

and, from (13.1), we infer that

$$\overline{x(t-0)}\cdot x(t) = \overline{x(t-0)}\cdot \bigcap_{\xi\in[t-d_r,t-d_r+m_r]} u(\xi) \leq \bigcap_{\xi\in[t-d_r,t-d_r+m_r]} u(\xi) \leq \chi_{[d+d_r,\infty)}(t),$$

where $x(-\infty+0) = 0$, from a). These remarks show that

$$\min\{t|u(t-0) \neq u(t)\} = d \leq d + d_r \leq \min\{t|x(t-0) \neq x(t)\}.$$

The situation $u(-\infty+0) = 1$ is treated similarly. Therefore f is non-anticipatory. $\square$

REMARK 132. *If in (13.1), (13.2) we put $m_r = m_f = 0$, then CC_{BD} implies $d_r = d_f = d$ and the system*

$$\begin{aligned} \overline{x(t-0)}\cdot x(t) &= \overline{x(t-0)}\cdot u(t-d), \\ x(t-0)\cdot \overline{x(t)} &= x(t-0)\cdot \overline{u(t-d)} \end{aligned}$$

coincides with I_d, because $x(t) = u(t-d)$ is a solution and the solution is unique.

THEOREM 323. *Let the real numbers $0 \leq m_r \leq d_r$, $0 \leq m_f \leq d_f$ be arbitrary with $d_r - m_r \leq d_f$, $d_f - m_f \leq d_r$. The following systems a),...,g) are equivalent, in the sense that for any $u \in S$, if $x \in S$ satisfies one of them, then it also satisfies any other:*

a)

$$\overline{x(t-0)}\cdot x(t) = \overline{x(t-0)}\cdot \bigcap_{\xi\in[t-d_r,t-d_r+m_r]} u(\xi),$$

$$x(t-0)\cdot \overline{x(t)} = x(t-0)\cdot \bigcap_{\xi\in[t-d_f,t-d_f+m_f]} \overline{u(\xi)};$$

b)

$$\bigcap_{\xi\in[t-d_r,t-d_r+m_r]} u(\xi) \leq x(t) \leq \bigcup_{\xi\in[t-d_f,t-d_f+m_f]} u(\xi),$$

$$\overline{x(t-0)}\cdot x(t) \leq \bigcap_{\xi\in[t-d_r,t-d_r+m_r]} u(\xi),$$

$$x(t-0)\cdot \overline{x(t)} \leq \bigcap_{\xi\in[t-d_f,t-d_f+m_f]} \overline{u(\xi)};$$

c)

$$\bigcap_{\xi\in[t-d_r,t-d_r+m_r]} u(\xi) \leq x(t),$$

$$\bigcap_{\xi\in[t-d_f,t-d_f+m_f]} \overline{u(\xi)} \leq \overline{x(t)},$$

$$\overline{\bigcap_{\xi\in[t-d_r,t-d_r+m_r]} u(\xi)} \cdot \overline{\bigcap_{\xi\in[t-d_f,t-d_f+m_f]} \overline{u(\xi)}} \leq \overline{x(t-0)} \cdot \overline{x(t)} \cup x(t-0) \cdot x(t);$$

d)

$$x(t) = \begin{cases} 1, if \bigcap_{\xi\in[t-d_r,t-d_r+m_r]} u(\xi) = 1 \\ 0, if \bigcap_{\xi\in[t-d_f,t-d_f+m_f]} \overline{u(\xi)} = 1 \\ x(t-0), otherwise \end{cases};$$

e)

$$x(t) = \bigcap_{\xi\in[t-d_r,t-d_r+m_r]} u(\xi) \cup x(t-0) \cdot \bigcup_{\xi\in[t-d_f,t-d_f+m_f]} u(\xi);$$

f)

$$Dx(t) = \overline{x(t-0)} \cdot \bigcap_{\xi\in[t-d_r,t-d_r+m_r]} u(\xi) \cup x(t-0) \cdot \bigcap_{\xi\in[t-d_f,t-d_f+m_f]} \overline{u(\xi)};$$

g)

$$\overline{x(t-0)} \cdot x(t) \cdot \bigcap_{\xi\in[t-d_r,t-d_r+m_r]} u(\xi) \cup x(t-0) \cdot \overline{x(t)} \cdot \bigcap_{\xi\in[t-d_f,t-d_f+m_f]} \overline{u(\xi)} \cup$$

$$\cup \overline{x(t-0)} \cdot \overline{x(t)} \cdot \overline{\bigcap_{\xi\in[t-d_r,t-d_r+m_r]} u(\xi)} \cup x(t-0) \cdot x(t) \cdot \overline{\bigcap_{\xi\in[t-d_f,t-d_f+m_f]} \overline{u(\xi)}} = 1.$$

PROOF. Let $t \in \mathbf{R}$ and $u \in S$ be arbitrary and fixed. In a),...,g), due to the satisfaction of CC_{BD}, there are three possibilities:

i) $\bigcap_{\xi\in[t-d_r,t-d_r+m_r]} u(\xi) = \bigcup_{\xi\in[t-d_f,t-d_f+m_f]} u(\xi) = 0;$

ii) $\bigcap_{\xi\in[t-d_r,t-d_r+m_r]} u(\xi) = 0, \bigcup_{\xi\in[t-d_f,t-d_f+m_f]} u(\xi) = 1;$

iii) $\bigcap_{\xi\in[t-d_r,t-d_r+m_r]} u(\xi) = \bigcup_{\xi\in[t-d_f,t-d_f+m_f]} u(\xi) = 1.$

Case i) By taking into account the fact that

$$\overline{\bigcup_{\xi\in[t-d_f,t-d_f+m_f]} u(\xi)} = \bigcap_{\xi\in[t-d_f,t-d_f+m_f]} \overline{u(\xi)} = 1,$$

a) is

$$\begin{aligned} \overline{x(t-0)} \cdot x(t) &= 0, \\ x(t-0) \cdot \overline{x(t)} &= x(t-0), \end{aligned}$$

whose unique solution is $x(t) = 0$. The first requirement from b) gives $x(t) \leq 0$, thus $x(t) = 0$. The second inequality c) shows that $1 \leq \overline{x(t)}$, that is $x(t) = 0$. d) and e) give $x(t) = 0$ too. f) becomes $x(t-0) \oplus x(t) = x(t-0)$, in other words $x(t) = 0$. Because $\overline{\bigcap_{\xi\in[t-d_r,t-d_r+m_r]} u(\xi)} = 1$, in this case g) is

$$x(t-0) \cdot \overline{x(t)} \cup \overline{x(t-0)} \cdot \overline{x(t)} = (x(t-0) \cup \overline{x(t-0)}) \cdot \overline{x(t)} = \overline{x(t)} = 1,$$

i.e. $x(t) = 0$.

The other two cases are similar, with $x(t) = x(t-0)$ for Case ii) and $x(t) = 1$ for Case iii). $\square$

14. The study of a deterministic delay, variant

LEMMA 3. *Let be $d_r > 0, d_f > 0$ and $u \in S$. The following formulae are true:*

$$\bigcap_{\xi\in[t-d_r,t)} u(\xi) = u(t-0)\cdot\overline{\bigcup_{\xi\in(t-d_r,t)} Du(\xi)};$$

$$\bigcap_{\xi\in[t-d_f,t)} \overline{u(\xi)} = \overline{u(t-0)}\cdot\overline{\bigcup_{\xi\in(t-d_f,t)} Du(\xi)}.$$

PROOF. Prove the first of these two relations and let t be arbitrary and fixed. We have:

$$\bigcap_{\xi\in[t-d_r,t)} u(\xi) = 1 \Longleftrightarrow u(t-0) = 1 \textit{ and } u_{|[t-d_r,t)} \text{ is constant} \Longleftrightarrow$$

(we apply the right continuity of u in $t - d_r$)

$$\Longleftrightarrow u(t-0) = 1 \textit{ and } u_{|(t-d_r,t)} \text{ is constant} \Longleftrightarrow$$

$$\Longleftrightarrow u(t-0) = 1 \textit{ and } \forall\xi\in(t-d_r,t), Du(\xi) = 0 \Longleftrightarrow$$

$$\Longleftrightarrow u(t-0)\cdot\overline{\bigcup_{\xi\in(t-d_r,t)} Du(\xi)} = 1.$$

Because t was arbitrary, the equation is proved. $\square$

REMARK 133. *The idea of replacing $f_{BD}^{m_r,d_r,m_f,d_f}$, $f_{AI}^{\delta_r,\delta_f}$, $f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$ by $f_{BD'}^{d_r,d_f}$, $f_{AI'}^{\delta_r,\delta_f}$, $f_{RI'}^{\delta_r,\delta_f}$ has implications in the way of understanding Theorem 322. For example, the delay from Theorem 323, f) takes the form*

$$Dx(t) = \overline{x(t-0)}\cdot\bigcap_{\xi\in[t-d_r,t)} u(\xi) \cup x(t-0)\cdot\bigcap_{\xi\in[t-d_f,t)} \overline{u(\xi)}, \tag{14.1}$$

where $d_r > 0, d_f > 0$. In the self-dual version, when $d_r = d_f = d > 0$ we have:

$$\overline{x(t-0)}\cdot\bigcap_{\xi\in[t-d,t)} u(\xi) \cup x(t-0)\cdot\bigcap_{\xi\in[t-d,t)} \overline{u(\xi)} =$$

(we apply now Lemma 3)

$$= \overline{x(t-0)}\cdot u(t-0)\cdot\overline{\bigcup_{\xi\in(t-d,t)} Du(\xi)} \cup x(t-0)\cdot\overline{u(t-0)}\cdot\overline{\bigcup_{\xi\in(t-d,t)} Du(\xi)} =$$

$$= (\overline{x(t-0)}\cdot u(t-0) \cup x(t-0)\cdot\overline{u(t-0)})\cdot\overline{\bigcup_{\xi\in(t-d,t)} Du(\xi)} =$$

$$= (x(t-0)\oplus u(t-0))\cdot\overline{\bigcup_{\xi\in(t-d,t)} Du(\xi)},$$

such that equation (14.1) becomes

$$Dx(t) = (x(t-0)\oplus u(t-0))\cdot\overline{\bigcup_{\xi\in(t-d,t)} Du(\xi)}. \tag{14.2}$$

As we have already stated in Example 61, the system (14.2) is non-anticipatory in the sense of Definitions 63, 64 and the second property is not necessarily true for the system (13.1), (13.2).

If $x_{|(-\infty,t_0)} = x(-\infty+0) = u(-\infty+0) = u_{|(-\infty,t_0)}$, *then (14.2) is equivalent to*

$$Dx(t) = (x(t-0) \oplus u(t-0)) \cdot \overline{\bigcup_{\xi \in (t-d,t)} Du(\xi)} \cdot \chi_{[t_0+d,\infty)}(t). \tag{14.3}$$

Part 3

Applications

CHAPTER 14

The equations of the ideal latches

1. Ideal latches, the general equation

Theorem 324. *The following two systems*

$$\left\{\begin{array}{c}\overline{x(t-0)}\cdot x(t)=\overline{x(t-0)}\cdot u(t)\\ x(t-0)\cdot\overline{x(t)}=x(t-0)\cdot v(t)\\ u(t)\cdot v(t)=0\end{array}\right.\tag{1.1}$$

and

$$x(t)\cdot u(t)\cdot\overline{v(t)}\cup\overline{x(t)}\cdot\overline{u(t)}\cdot v(t)\cup\tag{1.2}$$

$$\cup(\overline{x(t-0)}\cdot\overline{x(t)}\cup x(t-0)\cdot x(t))\cdot\overline{u(t)}\cdot\overline{v(t)}=1,$$

where $u,v,x\in S$ and x is the indeterminate, are equivalent, i.e. they have the same solutions.

Proof. Let $t\in\mathbf{R}$ be arbitrary and fixed. We have the following possibilities.

Case a) $u(t)=0,v(t)=0$. We must show the equivalence between

$$\left\{\begin{array}{l}\overline{x(t-0)}\cdot x(t)=0\\ x(t-0)\cdot\overline{x(t)}=0\end{array}\right.$$

and

$$\overline{x(t-0)}\cdot\overline{x(t)}\cup x(t-0)\cdot x(t)=1,$$

i.e. $x(t-0)=x(t)$. The first system is equivalent to any of

$$\overline{x(t-0)}\cdot x(t)\cup x(t-0)\cdot\overline{x(t)}=0,$$

$$\overline{\overline{x(t-0)}\cdot x(t)\cup x(t-0)\cdot\overline{x(t)}}=1,$$

$$(x(t-0)\cup\overline{x(t)})\cdot(\overline{x(t-0)}\cup x(t))=1,$$

$$\overline{x(t-0)}\cdot\overline{x(t)}\cup x(t-0)\cdot x(t)=1.$$

Case b) $u(t)=0,v(t)=1$. We must show the equivalence between

$$\left\{\begin{array}{c}\overline{x(t-0)}\cdot x(t)=0\\ x(t-0)\cdot\overline{x(t)}=x(t-0)\end{array}\right.$$

and $x(t)=0$. We infer

$$\begin{aligned}0&=\overline{x(t-0)}\cdot x(t)\cup(x(t-0)\cdot\overline{x(t)}\oplus x(t-0))\\ &=\overline{x(t-0)}\cdot x(t)\cup\overline{x(t-0)\cdot\overline{x(t)}}\cdot x(t-0)\cup x(t-0)\cdot\overline{x(t)}\cdot\overline{x(t-0)}\\ &=\overline{x(t-0)}\cdot x(t)\cup(\overline{x(t-0)}\cup x(t))\cdot x(t-0)\\ &=\overline{x(t-0)}\cdot x(t)\cup x(t-0)\cdot x(t)=(\overline{x(t-0)}\cup x(t-0))\cdot x(t)=x(t).\end{aligned}$$

Case c) $u(t) = 1, v(t) = 0$ Similarly to b), it is shown that

$$\begin{cases} \overline{x(t-0)} \cdot x(t) = \overline{x(t-0)} \\ x(t-0) \cdot \overline{x(t)} = 0 \end{cases}$$

is equivalent to $x(t) = 1$.

Another line of proof consists in:

- giving to $u(t), v(t)$ all possible values and remarking that each time (1.1), (1.2) have the same solutions $x(t-0) = x(t)$

$$\begin{cases} 0 = \overline{x(t)} \cdot u(t) \\ 0 = x(t) \cdot v(t) \\ u(t) \cdot v(t) = 0 \end{cases},$$

$$x(t) \cdot u(t) \cdot \overline{v(t)} \cup \overline{x(t)} \cdot \overline{u(t)} \cdot v(t) \cup \overline{u(t)} \cdot \overline{v(t)} = 1,$$

for some $t_0 \in \mathbf{R}$ and $t < t_0$ (see Table 1);

$u(t)$	$v(t)$	$x(t)$
0	0	0, 1
0	1	0
1	0	1

Table 1

- giving to $u(t), v(t), x(t-0)$ all possible values and remarking again that each time for $t \geq t_0$, (1.1), (1.2) have the same solutions $x(t)$ (see Table 2).

$u(t)$	$v(t)$	$x(t-0)$	$x(t)$
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1

Table 2

□

DEFINITION 122. *The equations (1.1), (1.2) are called the* ***equations of the ideal latch****. The system $f_{IL} : U \to P^*(S)$,*

$$U = \{(u(t), v(t)) | u, v \in S, u(t) \cdot v(t) = 0\},$$

defined by any of them is called the ***ideal latch*** *while the equation*

$$u(t) \cdot v(t) = 0$$

is called the ***admissibility condition of the inputs****.*

REMARK 134. *It is interesting to compare the ideal latch f_{IL} equation (1.1) and the deterministic system f described by (13.1), (13.2), Ch. 13. The admissibility condition of the inputs for f*

$$\bigcap_{\xi \in [t-d_r, t-d_r+m_r]} u(\xi) \cdot \bigcap_{\xi \in [t-d_f, t-d_f+m_f]} \overline{u(\xi)} = 0$$

is a property equivalent to $[t-d_r, t-d_r+m_r] \cap [t-d_f, t-d_f+m_f] \neq \emptyset$ *and to* CC_{BD}. *The fact that* $\forall u \in S, \exists t_0 \in \mathbf{R}, \forall t < t_0$,

$$\bigcap_{\xi \in [t-d_r, t-d_r+m_r]} u(\xi) = u(-\infty+0), \quad \bigcap_{\xi \in [t-d_f, t-d_f+m_f]} \overline{u(\xi)} = \overline{u(-\infty+0)}$$

makes f *be deterministic, unlike* f_{IL}.

THEOREM 325. *The initial state function of* f_{IL} *is defined by*

$$\phi_0 : U \to P^*(\mathbf{B}), \forall (u,v) \in U, \phi_0(u,v) = \begin{cases} \mathbf{B}, if\ u(-\infty+0) = v(-\infty+0) = 0 \\ 0, if\ u(-\infty+0) = 0, v(-\infty+0) = 1 \\ 1, if\ u(-\infty+0) = 1, v(-\infty+0) = 0 \end{cases}.$$

PROOF. For any $(u,v) \in U$, there is a $t_0 \in \mathbf{R}$ such that $u_{|(-\infty,t_0)} = u(-\infty+0), v_{|(-\infty,t_0)} = v(-\infty+0)$ and $\forall t < t_0$ we are in one of the cases a), b), c) of Theorem 324. In case a) we obtain $x(t) \in \mathbf{B}$ constant, because $x(t-0) = x(t)$; in case b), $x(t) = 0$ while in case c), $x(t) = 1$ (see Table 1). □

THEOREM 326. *The system* f_{IL} *is finite and has the following properties.*

a) If $u = v = 0$, *then* $f_{IL}(u,v) = \{0,1\}$.

b) If $u(-\infty+0) = v(-\infty+0) = 0$, *but* $\exists t \in \mathbf{R}, u(t) \cup v(t) = 1$, *then* $f_{IL}(u,v) = \{x', x''\}$ *and the two states* x', x'' *satisfy*

$$x'_{|(-\infty,t_0)} = 0,\ x''_{|(-\infty,t_0)} = 1,\ x'_{|[t_0,\infty)} = x''_{|[t_0,\infty)},$$

where we have denoted $t_0 = \min\ supp(u \cup v)$.

c) For $u(-\infty+0) \cup v(-\infty+0) = 1$, $f_{IL}(u,v)$ *has exactly one element.*

PROOF. a) As we have seen from the proof of Theorem 324, in case a), the equation of the ideal latch is $x(t-0) = x(t)$ and it has the constant solutions $0, 1$.

b) The equation $x(t-0) = x(t), t < t_0$ has the solutions x', x'' satisfying $x'_{|(-\infty,t_0)} = 0,\ x''_{|(-\infty,t_0)} = 1$. At this moment suppose that $u(t_0) = 0, v(t_0) = 1$, making (1.2) to become

$$x'(t_0) = x''(t_0) = 0,$$

with the implication that $\forall t > t_0, x'(t) = x''(t)$. The situation is similar for $u(t_0) = 1, v(t_0) = 0$ and $x'(t_0) = x''(t_0) = 1$.

c) For $u(-\infty+0) \cup v(-\infty+0) = 1$, Theorem 325 implies that the value of $x(-\infty+0)$ is unique for all $x \in f_{IL}(u,v)$, whence the uniqueness of x. □

THEOREM 327. *The system* f_{IL} *is time invariant.*

PROOF. Let $d \in \mathbf{R}$ and $u, v \in S$ be arbitrary and fixed and the system

$$(1.3) \quad \begin{cases} \overline{x(t-d-0)} \cdot x(t-d) = \overline{x(t-d-0)} \cdot u(t-d) \\ x(t-d-0) \cdot \overline{x(t-d)} = x(t-d-0) \cdot v(t-d) \\ u(t-d) \cdot v(t-d) = 0 \end{cases}.$$

By comparing (1.1) with (1.3) we can see that $\forall x \in f_{IL}(u,v)$, we have $x \circ \tau^d \in f_{IL}(u \circ \tau^d, v \circ \tau^d)$. In other words the inclusion

$$\{x \circ \tau^d | x \in f_{IL}(u,v)\} \subset f_{IL}(u \circ \tau^d, v \circ \tau^d)$$

is true. The inverse inclusion is also true. □

THEOREM 328. *The ideal latch is non-anticipatory in the sense of Definitions 63 and 65, items v),...,ix).*

PROOF. Let us show the non-anticipation in the sense of Definition 63.

The first possibility is that $u = v = 0$. In this case x is constant and f_{IL} is non-anticipatory.

The second possibility is that $\exists t_0 \in \mathbf{R}, t_0 = \min\ supp(u \cup v)$, implying that $x_{|(-\infty,t_0)}$ is constant. Thus $\min\ suppDu \cup suppDv = t_0$ and one of the two solutions switches at t_0, the other does not, the property of non-anticipation being fulfilled again.

The third possibility is that $u(-\infty + 0) \cup v(-\infty + 0) = 1$ and the solution $x \in f_{IL}(u, v)$ is unique. If u, v are constant, then x is constant and f_{IL} is non-anticipatory. Otherwise, when there is $t_0 = \min\ suppDu \cup suppDv$, we have two possibilities: that x switches or not in t_0, the non-anticipation property being satisfied each time.

Let us show the non-anticipation in the sense of Definition 65, v). Let $t_0 \in \mathbf{R}$, (u, v), $(u', v') \in U$ be arbitrary and fixed, such that $u_{|(-\infty,t_0]} = u'_{|(-\infty,t_0]}, v_{|(-\infty,t_0]} = v'_{|(-\infty,t_0]}$. Due to the finiteness of f_{IL}, Theorem 325 shows the existence of a $t_1 \in \mathbf{R}$ having the property that $\{x_{|(-\infty,t_1]} | x \in f_{IL}(u, v)\} = \{y_{|(-\infty,t_1]} | y \in f_{IL}(u', v')\}$, i.e. that t_1 with $\forall x \in f_{IL}(u, v), x_{|(-\infty,t_1]} = x(-\infty + 0), \forall y \in f_{IL}(u', v'), y_{|(-\infty,t_1]} = y(-\infty + 0)$. If $t_1 \geq t_0$ the non-anticipation takes place, while if $t_1 < t_0$, then it is a consequence of the function $(u(t), v(t), x(t - 0)) \to x(t)$ from Table 2, applied a finite number of times at the points of the set $t \in (t_1, t_0] \cap (suppDu \cup suppDv)$. □

THEOREM 329. *The system f_{IL} fulfills the surjectivity property*

$$\forall x \in S, \exists (u, v) \in U, x \in f_{IL}(u, v).$$

PROOF. For any $x \in S$, it is sufficient to choose $u = x$ and $v = \overline{x}$, since in this case (1.2) becomes $x \cup \overline{x} = 1$. □

THEOREM 330. *The system f_{IL} is relatively stable with a bounded final time:*

$$\forall (u, v) \in U \cap S_c^{(2)}, \exists t_f \in \mathbf{R}, \forall x \in f_{IL}(u, v), \exists \mu \in \mathbf{B}, x_{|[t_f,\infty)} = \mu.$$

PROOF. Suppose that for arbitrary $(u, v) \in U$, there are $t_f \in \mathbf{R}$ and $\lambda_1, \lambda_2 \in \mathbf{B}$ such that $u_{|[t_f,\infty)} = \lambda_1, v_{|[t_f,\infty)} = \lambda_2$ i.e. for $t \geq t_f$, (1.1) becomes

$$\begin{cases} \overline{x(t-0)} \cdot x(t) = \overline{x(t-0)} \cdot \lambda_1 \\ x(t-0) \cdot \overline{x(t)} = x(t-0) \cdot \lambda_2 \\ \lambda_1 \cdot \lambda_2 = 0 \end{cases}.$$

For any $x \in f_{IL}(u, v)$, we have: if $\lambda_1 = \lambda_2 = 0$, then $x_{|[t_f,\infty)} = x(t_f - 0)$; if $\lambda_1 = 0, \lambda_2 = 1$, then $x_{|[t_f,\infty)} = 0$, and if $\lambda_1 = 1, \lambda_2 = 0$, then $x_{|[t_f,\infty)} = 1$. □

2. C element

Any of the following equivalent statements:

$$\begin{cases} \overline{x(t-0)} \cdot x(t) = \overline{x(t-0)} \cdot u(t) \cdot v(t) \\ x(t-0) \cdot \overline{x(t)} = x(t-0) \cdot \overline{u(t)} \cdot \overline{v(t)} \end{cases} \tag{2.1}$$

and

$$x(t) \cdot u(t) \cdot v(t) \cup \overline{x(t)} \cdot \overline{u(t)} \cdot \overline{v(t)} \cup \tag{2.2}$$

$$\cup (\overline{x(t-0)} \cdot \overline{x(t)} \cup x(t-0) \cdot x(t)) \cdot (\overline{u(t)} \cdot v(t) \cup u(t) \cdot \overline{v(t)}) = 1$$

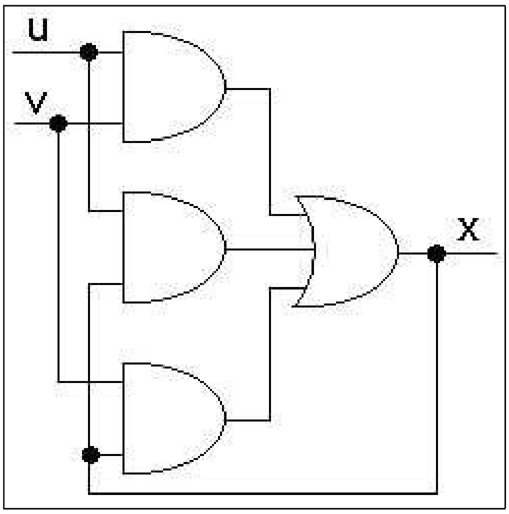

FIGURE 1. The C element

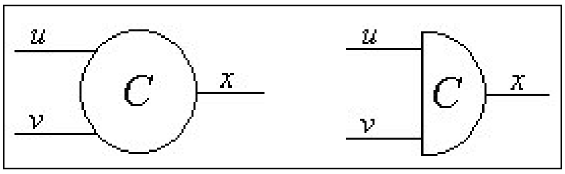

FIGURE 2. The symbols of the C element

are called the **equations of the C element** (**of Muller**), where u, v, x are signals, the first two called **inputs** and the last – **state**. Equations (2.1), (2.2) are the equations of a latch (1.1), (1.2), where $u(t)$ is replaced by $u(t) \cdot v(t)$ and $v(t)$ is replaced by $\overline{u(t)} \cdot \overline{v(t)}$. It is noted the fulfillment of the admissibility condition of the inputs. The study of (2.2) provides: $x(t)$ is 1 if $u(t) = v(t) = 1$, $x(t)$ is 0 if $u(t) = v(t) = 0$, and $x(t) = x(t-0)$, $x(t)$ keeps its previous value, otherwise. The general form of equations (2.1), (2.2) for m inputs $u_1, ..., u_m$ is

$$\begin{cases} \overline{x(t-0)} \cdot x(t) = \overline{x(t-0)} \cdot u_1(t) \cdot ... \cdot u_m(t) \\ x(t-0) \cdot \overline{x(t)} = x(t-0) \cdot \overline{u_1(t)} \cdot ... \cdot \overline{u_m(t)} \end{cases},$$

$$x(t) \cdot u_1(t) \cdot ... \cdot u_m(t) \cup \overline{x(t)} \cdot \overline{u_1(t)} \cdot ... \cdot \overline{u_m(t)}$$
$$\cup(\overline{x(t-0)} \cdot \overline{x(t)} \cup x(t-0) \cdot x(t)) \cdot \overline{u_1(t) \cdot ... \cdot u_m(t)} \cdot (u_1(t) \cup ... \cup u_m(t)) = 1.$$

3. Asymmetric C elements

In [**24**], the circuit from Figure 1 is called the **symmetric C element** and the **asymmetric C elements** from Figures 3, 5 are presented, together with their symbols from Figures 4 and 6. In the first case the equivalent equations are

$$\begin{cases} \overline{x(t-0)} \cdot x(t) = \overline{x(t-0)} \cdot u(t) \cdot v(t) \\ x(t-0) \cdot \overline{x(t)} = x(t-0) \cdot \overline{v(t)} \end{cases}, \tag{3.1}$$

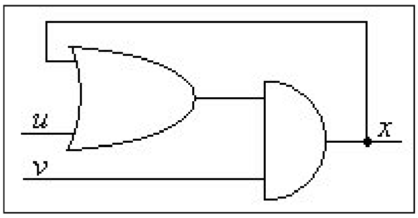

FIGURE 3. Asymmetric C element

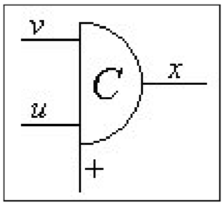

FIGURE 4. The symbol of the circuit from Figure 3

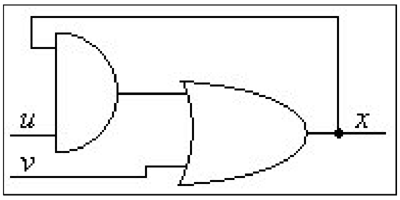

FIGURE 5. Asymmetric C element

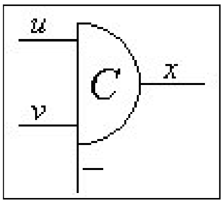

FIGURE 6. The symbol of the circuit from Figure 5

$$
\begin{aligned}
& x(t)\cdot u(t)\cdot v(t)\cup\overline{x(t)}\cdot\overline{v(t)} \\
& \cup(\overline{x(t-0)}\cdot\overline{x(t)}\cup x(t-0)\cdot x(t))\cdot\overline{u(t)}\cdot v(t)=1
\end{aligned}
\tag{3.2}
$$

while in the second case they are

$$
\begin{cases}
\overline{x(t-0)}\cdot x(t)=\overline{x(t-0)}\cdot v(t) \\
x(t-0)\cdot\overline{x(t)}=x(t-0)\cdot\overline{u(t)}\cdot\overline{v(t)}
\end{cases},
\tag{3.4}
$$

$$
\begin{aligned}
& x(t)\cdot v(t)\cup\overline{x(t)}\cdot\overline{u(t)}\cdot\overline{v(t)} \\
& \cup(\overline{x(t-0)}\cdot\overline{x(t)}\cup x(t-0)\cdot x(t))\cdot u(t)\cdot\overline{v(t)}=1.
\end{aligned}
\tag{3.5}
$$

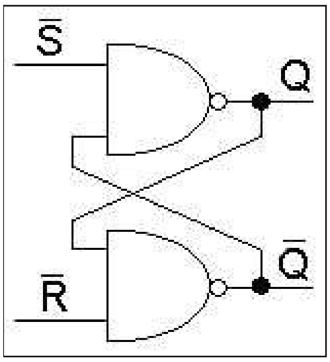

FIGURE 7. The RS latch circuit

4. C-OR element

In [**26**], the following latch, called **COR22**, is indicated:

$$\left\{\begin{array}{l} \overline{x(t-0)}\cdot x(t)=\overline{x(t-0)}\cdot(u(t)\cdot v(t)\cup y(t)\cdot z(t)) \\ x(t-0)\cdot\overline{x(t)}=x(t-0)\cdot\overline{u(t)}\cdot\overline{v(t)}\cdot\overline{y(t)}\cdot\overline{z(t)} \end{array}\right. , \tag{4.1}$$

or, equivalently,

$$x(t)\cdot(u(t)\cdot v(t)\cup y(t)\cdot z(t))\cup\overline{x(t)}\cdot\overline{(u(t)\cup v(t)\cup y(t)\cup z(t))} \tag{4.2}$$

$$\cup(\overline{x(t-0)}\cdot\overline{x(t)}\cup x(t-0)\cdot x(t))\cdot\overline{(u(t)\cdot v(t)\cup y(t)\cdot z(t))}\cdot(u(t)\cup v(t)\cup y(t)\cup z(t))=1.$$

In these equations the inputs are u,v,y,z, while x is the state. We can see that in (1.1), $u(t)$ was replaced by $u(t)\cdot v(t)\cup y(t)\cdot z(t)$, while $v(t)$ was replaced by $\overline{u(t)}\cdot\overline{v(t)}\cdot\overline{y(t)}\cdot\overline{z(t)}=\overline{u(t)\cup v(t)\cup y(t)\cup z(t)}$ and that the admissibility condition

$$(u(t)\cdot v(t)\cup y(t)\cdot z(t))\cdot\overline{u(t)}\cdot\overline{v(t)}\cdot\overline{y(t)}\cdot\overline{z(t)}=0$$

is true. The paper [**26**] does not present the symbol of this circuit.

5. RS latch

The equations of the RS latch are given by

$$\left\{\begin{array}{c} \overline{Q(t-0)}\cdot Q(t)=\overline{Q(t-0)}\cdot S(t) \\ Q(t-0)\cdot\overline{Q(t)}=Q(t-0)\cdot R(t) \\ R(t)\cdot S(t)=0 \end{array}\right. , \tag{5.1}$$

or, equivalently, by

$$Q(t)\cdot\overline{R(t)}\cdot S(t)\cup\overline{Q(t)}\cdot R(t)\cdot\overline{S(t)} \tag{5.2}$$

$$\cup(\overline{Q(t-0)}\cdot\overline{Q(t)}\cup Q(t-0)\cdot Q(t))\cdot\overline{R(t)}\cdot\overline{S(t)}=1.$$

In (5.1), (5.2) R,S,Q are signals. The signals R,S are called **inputs**: the **reset input** and the **set input**, while Q is the state, the indeterminate relative to which the equations are solved. These equations coincide with (1.1) and (1.2), but the notations are different and traditional.We find the things discussed in Section 1 by the following statements related to equation (5.2). At the RS latch, $Q(t)=1$ if $R(t)=0,S(t)=1$; $Q(t)=0$ if $R(t)=1,S(t)=0$; and $Q(t)=Q(t-0),Q$ keeps its previous value if $R(t)=0,S(t)=0$.

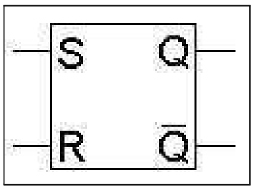

FIGURE 8. The symbol of the RS latch

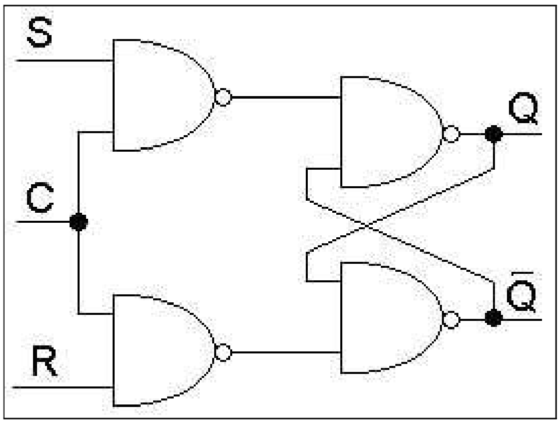

FIGURE 9. The clocked RS latch circuit

6. Clocked RS latch

The equivalent statements

$$\left\{\begin{array}{c} \overline{Q(t-0)} \cdot Q(t) = \overline{Q(t-0)} \cdot S(t) \cdot C(t) \\ Q(t-0) \cdot \overline{Q(t)} = Q(t-0) \cdot R(t) \cdot C(t) \\ R(t) \cdot S(t) \cdot C(t) = 0 \end{array}\right. , \tag{6.1}$$

and

$$\begin{gathered} C(t) \cdot (Q(t) \cdot \overline{R(t)} \cdot S(t) \cup \overline{Q(t)} \cdot R(t) \cdot \overline{S(t)} \\ \cup(\overline{Q(t-0)} \cdot \overline{Q(t)} \cup Q(t-0) \cdot Q(t)) \cdot \overline{R(t)} \cdot \overline{S(t)}) \\ \cup \overline{C(t)} \cdot (\overline{Q(t-0)} \cdot \overline{Q(t)} \cup Q(t-0) \cdot Q(t)) = 1 \end{gathered} \tag{6.2}$$

are called the **equations of the clocked RS latch**, where R, S, C, Q are signals: the **reset**, the **set** and the **clock input**, and the **state** respectively. The equations (6.1), (6.2) follow after some elementary computations from (1.1) and (1.2), where $u(t) = S(t) \cdot C(t)$, $v(t) = R(t) \cdot C(t)$.The clocked RS latch behaves like an RS latch when $C(t) = 1$, while it keeps the state constant $Q(t) = Q(t-0)$ when $C(t) = 0$.

7. D latch

Any of the following equivalent statements

$$\left\{\begin{array}{c} \overline{Q(t-0)} \cdot Q(t) = \overline{Q(t-0)} \cdot D(t) \cdot C(t) \\ Q(t-0) \cdot \overline{Q(t)} = Q(t-0) \cdot \overline{D(t)} \cdot C(t) \end{array}\right. , \tag{7.1}$$

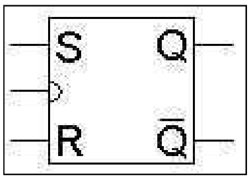

FIGURE 10. The symbol of the clocked RS latch

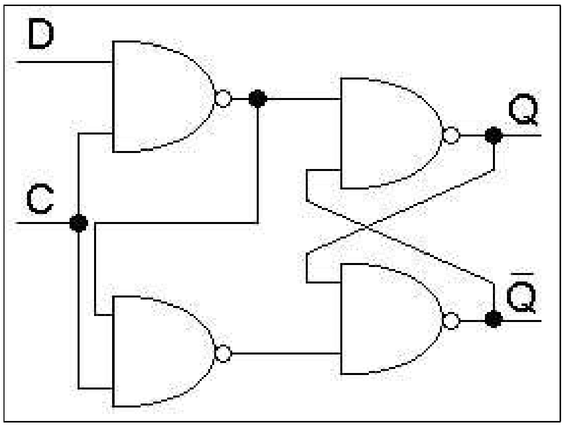

FIGURE 11. The D latch circuit

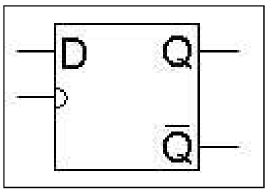

FIGURE 12. The symbol of the D latch

and

$$(7.2)\quad C(t)\cdot(\overline{Q(t)}\cdot\overline{D(t)}\cup Q(t)\cdot D(t))\cup\overline{C(t)}\cdot(\overline{Q(t-0)}\cdot\overline{Q(t)}\cup Q(t-0)\cdot Q(t))=1$$

respectively are called the **equations of the D latch**, where D, C, Q are signals: the **data input** D, the **clock input** C and the **state** Q. On one hand, from (7.1) it is seen the fulfillment of the admissibility condition of the inputs. On the other hand, (7.1), (7.2) are obtained from the equations of the clocked RS latch (6.1), (6.2), where $R = \overline{S\cdot C}$ and the traditional notation D for the data input was used, instead of S. When $C(t) = 1$, the D latch makes $Q(t) = D(t)$, while when $C(t) = 0$, Q is constant.

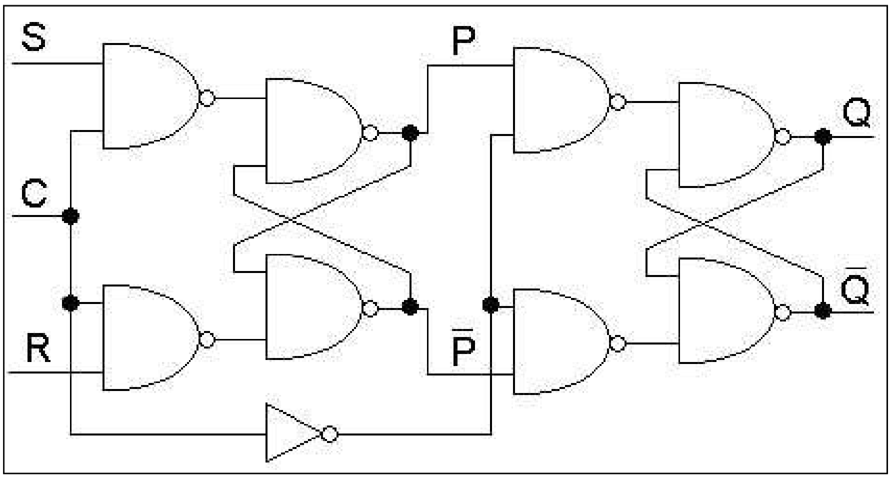

FIGURE 13. The edge triggered RS flip-flop circuit

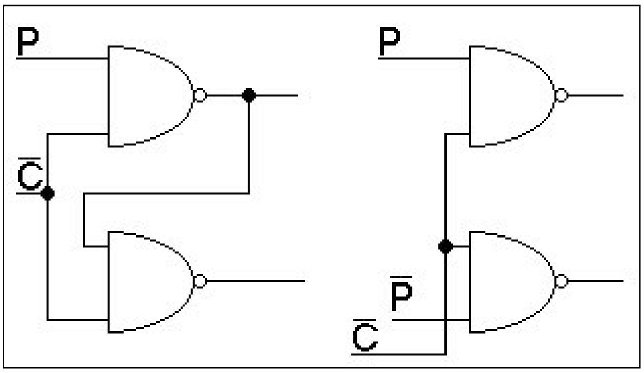

FIGURE 14. Equivalent circuits

8. Edge triggered RS flip-flop

Any of the equivalent statements

$$\left\{\begin{array}{c} \overline{P(t-0)} \cdot P(t) = \overline{P(t-0)} \cdot S(t) \cdot C(t) \\ P(t-0) \cdot \overline{P(t)} = P(t-0) \cdot R(t) \cdot C(t) \\ R(t) \cdot S(t) \cdot C(t) = 0 \\ \overline{Q(t-0)} \cdot Q(t) = \overline{Q(t-0)} \cdot P(t) \cdot \overline{C(t)} \\ Q(t-0) \cdot \overline{Q(t)} = Q(t-0) \cdot \overline{P(t)} \cdot \overline{C(t)} \end{array}\right. , \tag{8.1}$$

and

$$\begin{gathered} C(t) \cdot (\overline{Q(t-0)} \cdot \overline{Q(t)} \cup Q(t-0) \cdot Q(t)) \cdot (P(t) \cdot \overline{R(t)} \cdot S(t) \\ \cup \overline{P(t)} \cdot R(t) \cdot \overline{S(t)} \cup (\overline{P(t-0)} \cdot \overline{P(t)} \cup P(t-0) \cdot P(t)) \cdot \overline{R(t)} \cdot \overline{S(t)}) \\ \cup \overline{C(t)} \cdot (\overline{Q(t)} \cdot \overline{P(t-0)} \cdot \overline{P(t)} \cup Q(t) \cdot P(t-0) \cdot P(t)) = 1, \end{gathered} \tag{8.2}$$

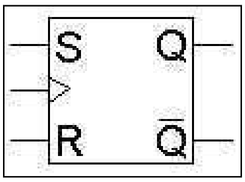

FIGURE 15. The symbol of the edge triggered RS flip-flop

respectively is called the **equation of the edge triggered RS flip-flop**, where R, S, C, P, Q are signals: the **reset input** R, the **set input** S, the **clock input** C, the **next state** P and the **state** Q. In (8.1), (8.2) the signals R, S, C, P and $P, \overline{C}, Q$ satisfy the equations of a clocked RS latch and of a D latch (see Figure 14), while (8.2) represents the term by term product of (6.2) by (7.2), written with these variables. The two latches are called **master** and **slave**.

The analysis of this circuit starts with (8.2). We have

Case 1 $\exists t, C(t) = 1$. Because $Q(t) = Q(t-0), t$ is a point of continuity for Q.

Case 2 $\exists t, C(t) = 0$. We get

$$Q(t) = P(t) = P(t-0). \tag{8.3}$$

Case 2.1 $C(t-0) = 1$. By taking the left limit in (8.2), it implies

$$P(t-0)\cdot\overline{R(t-0)}\cdot S(t-0) \cup \overline{P(t-0)}\cdot R(t-0)\cdot\overline{S(t-0)} \cup \overline{R(t-0)}\cdot\overline{S(t-0)} = 1, \tag{8.4}$$

wherefrom

$$Q(t) = \begin{cases} P(t-0), if\ R(t-0) = 0, S(t-0) = 0 \\ 1, if\ R(t-0) = 0, S(t-0) = 1 \\ 0, if\ R(t-0) = 1, S(t-0) = 0 \end{cases}. \tag{8.5}$$

Case 2.2 $C(t-0) = 0$. For the left limit taken in (8.2), it gives

$$Q(t-0) = P(t-0).$$

If we take into account (8.3), t is a point of continuity for Q.

We conclude that the only time instants t when Q may switch are those when $C(t-0)\cdot\overline{C(t)} = 1$. This is the so-called 'falling edge' of the clock input that gives the name of the edge triggered flip-flop.

9. D flip-flop

Any of the following equivalent conditions:

$$\begin{cases} \overline{P(t-0)}\cdot P(t) = \overline{P(t-0)}\cdot D(t)\cdot C(t) \\ P(t-0)\cdot\overline{P(t)} = P(t-0)\cdot\overline{D(t)}\cdot C(t) \\ \overline{Q(t-0)}\cdot Q(t) = \overline{Q(t-0)}\cdot P(t)\cdot\overline{C(t)} \\ Q(t-0)\cdot\overline{Q(t)} = Q(t-0)\cdot\overline{P(t)}\cdot\overline{C(t)} \end{cases}, \tag{9.1}$$

and

$$\begin{aligned} C(t)\cdot(\overline{Q(t-0)}\cdot\overline{Q(t)} \cup Q(t-0)\cdot Q(t))\cdot(\overline{P(t)}\cdot\overline{D(t)} \cup P(t)\cdot D(t)) \\ \cup\overline{C(t)}\cdot(\overline{Q(t)}\cdot\overline{P(t-0)}\cdot\overline{P(t)} \cup Q(t)\cdot P(t-0)\cdot P(t)) = 1 \end{aligned} \tag{9.2}$$

respectively are called the **equations of the D flip-flop**, where D, C, P, Q are signals, called: the **data input** D, the **clock input** C, the **next state** P and

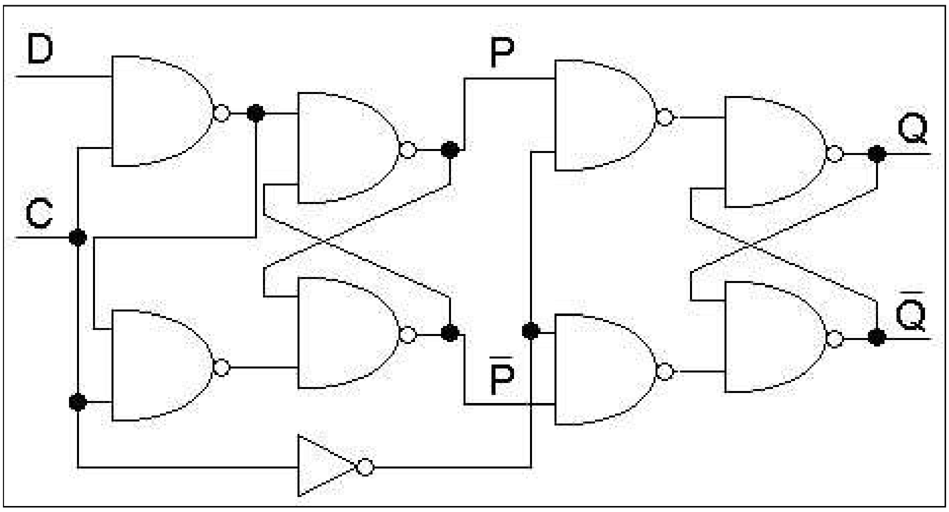

FIGURE 16. The D flip-flop circuit

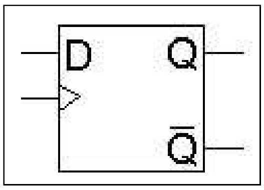

FIGURE 17. The symbol of the D flip-flop

the **state** Q. We note that the equations of the D flip-flop represent the special case of the edge triggered RS flip-flop, where $R = \overline{S \cdot C}$ and S was denoted by D. The D flip-flop has the state Q constant, except for the time instants when $C(t-0) \cdot \overline{C(t)} = 1$; then (8.4) becomes

$$P(t-0) \cdot D(t-0) \cup \overline{P(t-0)} \cdot \overline{D(t-0)} = 1,$$

i.e. $P(t-0) = D(t-0)$, and (8.5) becomes

$$Q(t) = \begin{cases} P(t-0), if\ \overline{D(t-0)} = 0, D(t-0) = 0 \\ 1, if\ \overline{D(t-0)} = 0, D(t-0) = 1 \\ 0, if\ \overline{D(t-0)} = 1, D(t-0) = 0 \end{cases} = \tag{9.3}$$

$$= \begin{cases} 1, if\ D(t-0) = 1 \\ 0, if\ D(t-0) = 0 \end{cases} = D(t-0).$$

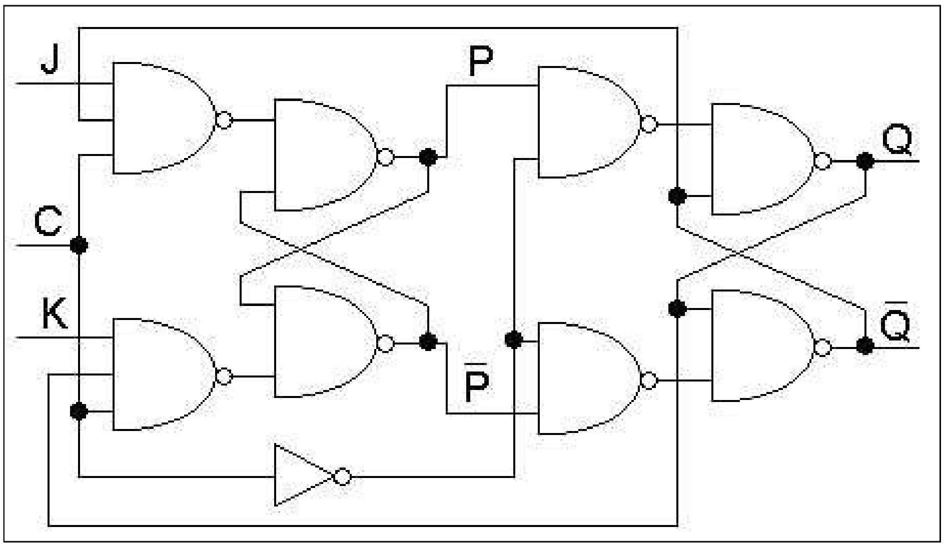

FIGURE 18. The JK flip-flop circuit

10. JK flip-flop

The equivalent statements:

$$\left\{\begin{array}{c} \overline{P(t-0)}\cdot P(t)=\overline{P(t-0)}\cdot J(t)\cdot\overline{Q(t)}\cdot C(t) \\ P(t-0)\cdot\overline{P(t)}=P(t-0)\cdot K(t)\cdot Q(t)\cdot C(t) \\ \overline{Q(t-0)}\cdot Q(t)=\overline{Q(t-0)}\cdot P(t)\cdot\overline{C(t)} \\ Q(t-0)\cdot\overline{Q(t)}=Q(t-0)\cdot\overline{P(t)}\cdot\overline{C(t)} \end{array}\right. \tag{10.1}$$

and

$$\begin{gathered} C(t)\cdot(\overline{Q(t-0)}\cdot\overline{Q(t)}\cup Q(t-0)\cdot Q(t))\cdot(P(t)\cdot J(t)\cdot\overline{Q(t)}\cup\overline{P(t)}\cdot K(t)\cdot Q(t) \\ \cup(\overline{P(t-0)}\cdot\overline{P(t)}\cup P(t-0)\cdot P(t))\cdot(\overline{J(t)}\cdot\overline{K(t)}\cup\overline{J(t)}\cdot\overline{Q(t)}\cup\overline{K(t)}\cdot Q(t))) \\ \cup\overline{C(t)}\cdot(\overline{Q(t)}\cdot\overline{P(t-0)}\cdot\overline{P(t)}\cup Q(t)\cdot P(t-0)\cdot P(t))=1 \end{gathered} \tag{10.2}$$

are called the **equations of the JK flip-flop**, where J, K, C, P, Q are signals: the **J input**, the **K input**, the **clock input** C, the **next state** P and the **state** Q. The first two equations of (10.1) (modeling the master latch) coincide with the first two equations of the edge triggered RS flip-flop, where $S(t) = J(t)\cdot\overline{Q(t)}$, $R(t) = K(t)\cdot Q(t)$, while the last two equations of (10.1) (modeling the slave latch) coincide with the last two equations of the edge triggered RS flip-flop. We note that the conditions of admissibility of the inputs of the master and of the slave latch are fulfilled. Compare (10.2) and (8.2). The JK flip-flop is similar to the edge triggered flip-flop. For example Q changes value only when $C(t-0)\cdot\overline{C(t)} = 1$.

We present a difference relative to the edge triggered flip-flop. For this let be $t_1 < t_2$ two numbers for which $\forall t \in [t_1, t_2), C(t) = 1$. Because in (10.2)

$$\forall t \in [t_1, t_2), Q(t) = Q(t-0) = Q(t_1 - 0),$$

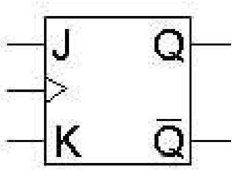

FIGURE 19. The symbol of the JK flip-flop

we have two possibilities: $Q(t_1-0)=0$ and $Q(t_1-0)=1$. The equations

$$P(t)\cdot J(t)\cup(\overline{P(t-0)}\cdot\overline{P(t)}\cup P(t-0)\cdot P(t))\cdot\overline{J(t)}=1,$$
$$\overline{P(t)}\cdot K(t)\cup(\overline{P(t-0)}\cdot\overline{P(t)}\cup P(t-0)\cdot P(t))\cdot\overline{K(t)}=1,$$

obtained in the two situations, show that in the interval $[t_1,t_2)$ P switches at most once: from 0 to 1 in the first case (if $J(t)=1$), and from 1 to 0 in the second case (if $K(t)=1$).

Let us take, in the equations of the D flip-flop, $D(t)=J(t)\cdot\overline{Q(t)}\cup\overline{K(t)}\cdot Q(t)$. By using the fact that

$$(\overline{Q(t-0)}\cdot\overline{Q(t)}\cup Q(t-0)\cdot Q(t))\cdot\overline{P(t)}\cdot(\overline{J(t)}\cdot K(t)\cup\overline{J(t)}\cdot\overline{Q(t)}\cup Q(t)\cdot K(t))=$$
$$=(\overline{Q(t-0)}\cdot\overline{Q(t)}\cup Q(t-0)\cdot Q(t))\cdot\overline{P(t)}\cdot(\overline{J(t)}\cdot\overline{Q(t)}\cup Q(t)\cdot K(t)),$$

we get the equality

$$\begin{aligned}&C(t)\cdot(\overline{Q(t-0)}\cdot\overline{Q(t)}\cup Q(t-0)\cdot Q(t))\cdot(P(t)\cdot J(t)\cdot\overline{Q(t)}\cup\overline{P(t)}\cdot K(t)\cdot Q(t)\cup\\ &\cup\overline{P(t)}\cdot\overline{J(t)}\cdot\overline{Q(t)}\cup P(t)\cdot\overline{K(t)}\cdot Q(t))\cup\\ &\cup\overline{C(t)}\cdot(\overline{Q(t)}\cdot\overline{P(t-0)}\cdot\overline{P(t)}\cup Q(t)\cdot P(t-0)\cdot P(t))=1.\end{aligned}\tag{10.3}$$

Equations (10.2) and (10.3) have similarities and, sometimes, the equation of the JK flip-flop is considered to be (10.3).

11. T flip-flop

The following equivalent statements:

$$\begin{cases}\overline{P(t-0)}\cdot P(t)=\overline{P(t-0)}\cdot\overline{Q(t)}\cdot C(t)\\ P(t-0)\cdot\overline{P(t)}=P(t-0)\cdot Q(t)\cdot C(t)\\ \overline{Q(t-0)}\cdot Q(t)=\overline{Q(t-0)}\cdot P(t)\cdot\overline{C(t)}\\ Q(t-0)\cdot\overline{Q(t)}=Q(t-0)\cdot\overline{P(t)}\cdot\overline{C(t)}\end{cases},\tag{11.1}$$

respectively

$$\begin{aligned}&C(t)\cdot(\overline{Q(t-0)}\cdot\overline{Q(t)}\cdot P(t)\cup Q(t-0)\cdot Q(t)\cdot\overline{P(t)})\\ &\cup\overline{C(t)}\cdot(\overline{Q(t)}\cdot\overline{P(t-0)}\cdot\overline{P(t)}\cup Q(t)\cdot P(t-0)\cdot P(t))=1\end{aligned}\tag{11.2}$$

are called the **equations of the T flip-flop**, where C,P,Q are signals: the **clock input**, the **next state** and the **state**. The conditions of admissibility of the inputs are fulfilled for the first two and for the last two equations from (11.1) (the master and the slave latch). The equations of the T flip-flop represent the following special cases: in the equations of the edge triggered RS flip-flop, $S(t)=\overline{Q(t)},R(t)=Q(t)$; in the equations of the D flip-flop, $D(t)=\overline{Q(t)}$; in the equations of the JK flip-flop (any of (10.2), (10.3)) $J(t)=1,K(t)=1$.

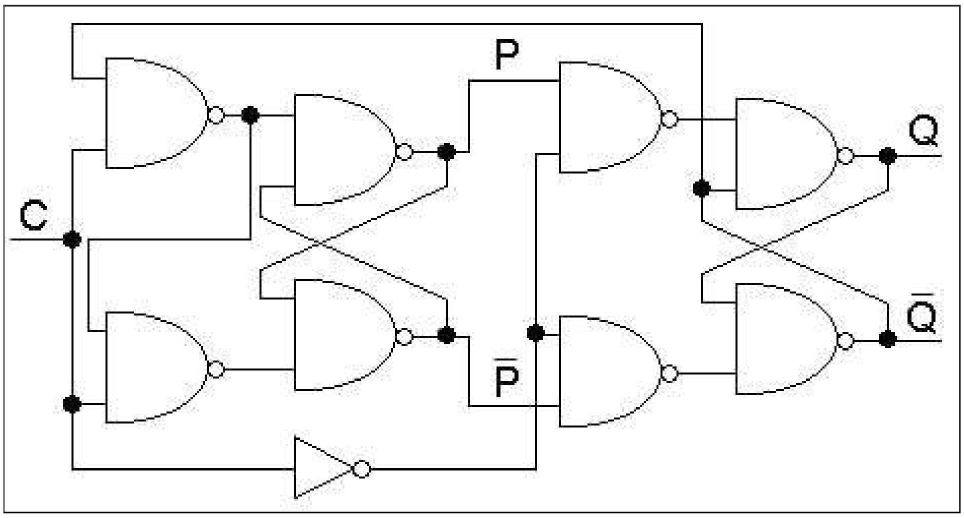

FIGURE 20. The T flip-flop circuit

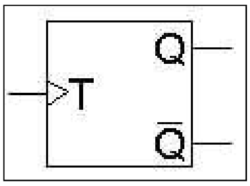

FIGURE 21. The symbol of the T flip-flop

We consider the equation (9.3), showing, for $D(t) = \overline{Q(t)}$, that $C(t-0) \cdot \overline{C(t)} = 1$ implies

$$Q(t) = D(t-0) = \overline{Q(t-0)},$$

i.e. at each falling edge of the clock input, the state Q of the T flip-flop switches (toggles) to its complementary value, while apart from these time instances, Q is constant.

CHAPTER 15

Some applications of the flip-flops

1. A two bit shift register with serial input and parallel output

The circuit is the one in Figure 1.

Suppose the existence of the sequence $t_0 < t_1 < t_2 < ...$ with the property that

$$C(t) = \chi_{[t_0,t_1)}(t) \oplus \chi_{[t_2,t_3)}(t) \oplus \chi_{[t_4,t_5)}(t) \oplus ...$$

In the equations

$$C(t) \cdot (\overline{Q(t-0)} \cdot \overline{Q(t)} \cup Q(t-0) \cdot Q(t)) \cdot (\overline{P(t)} \cdot \overline{D(t)} \cup P(t) \cdot D(t))$$
$$\cup \overline{C(t)} \cdot (\overline{Q(t)} \cdot \overline{P(t-0)} \cdot \overline{P(t)} \cup Q(t) \cdot P(t-0) \cdot P(t)) = 1$$
$$C(t) \cdot (\overline{Q'(t-0)} \cdot \overline{Q'(t)} \cup Q'(t-0) \cdot Q'(t)) \cdot (\overline{P'(t)} \cdot \overline{D'(t)} \cup P'(t) \cdot D'(t))$$
$$\cup \overline{C(t)} \cdot (\overline{Q'(t)} \cdot \overline{P'(t-0)} \cdot \overline{P'(t)} \cup Q'(t) \cdot P'(t-0) \cdot P'(t)) = 1$$

derived from Ch. 14, equation (9.2), we take into account the fact that

$$Q(t) = D'(t)$$

and obtain

$$C(t) \cdot (\overline{P(t)} \cdot \overline{D(t)} \cup P(t) \cdot D(t)) \cdot (\overline{Q(t-0)} \cdot \overline{Q(t)} \cdot \overline{P'(t)} \cup Q(t-0) \cdot Q(t) \cdot P'(t)) \cdot$$
$$\cdot (\overline{Q'(t-0)} \cdot \overline{Q'(t)} \cup Q'(t-0) \cdot Q'(t)) \cup$$
$$\cup \overline{C(t)} \cdot (\overline{Q(t)} \cdot \overline{P(t-0)} \cdot \overline{P(t)} \cup Q(t) \cdot P(t-0) \cdot P(t)) \cdot$$
$$\cdot (\overline{Q'(t)} \cdot \overline{P'(t-0)} \cdot \overline{P'(t)} \cup Q'(t) \cdot P'(t-0) \cdot P'(t)) = 1.$$

We infer:

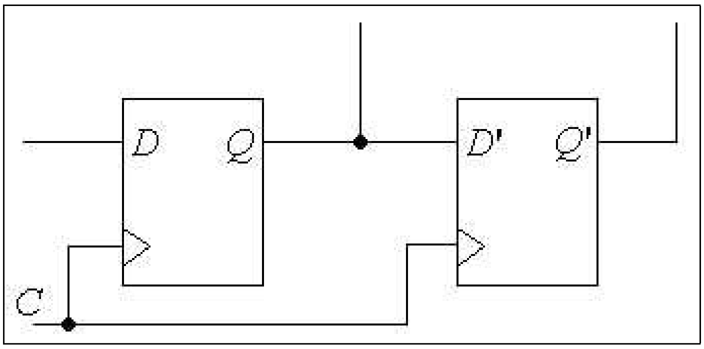

FIGURE 1. Two bit shift register

$t \in (-\infty, t_0)$:

$$\begin{aligned} C(t) &= 0, \\ P(t) &= P(t-0) = Q(t) = x^0, \\ P'(t) &= P'(t-0) = Q'(t) = y^0, \end{aligned}$$

where we have denoted by $x^0, y^0 \in \mathbf{B}$ two parameters, representing the initial conditions. Furthermore:

$t \in [t_0, t_1)$:

$$\begin{aligned} C(t) &= 1, \\ P(t) &= D(t), \\ Q(t) &= Q(t-0) = P'(t) = Q(t_0 - 0) = x^0, \\ Q'(t) &= Q'(t-0) = Q'(t_0 - 0) = y^0. \end{aligned}$$

$t \in [t_1, t_2)$:

$$\begin{aligned} C(t) &= 0, \\ P(t) &= P(t-0) = Q(t) = P(t_1 - 0) = D(t_1 - 0), \\ P'(t) &= P'(t-0) = Q'(t) = P'(t_1 - 0) = x^0. \end{aligned}$$

$t \in [t_2, t_3)$:

$$\begin{aligned} C(t) &= 1, \\ P(t) &= D(t), \\ Q(t) &= Q(t-0) = P'(t) = Q(t_2 - 0) = D(t_1 - 0), \\ Q'(t) &= Q'(t-0) = Q'(t_2 - 0) = x^0. \end{aligned}$$

$t \in [t_3, t_4)$:

$$\begin{aligned} C(t) &= 0, \\ P(t) &= P(t-0) = Q(t) = P(t_3 - 0) = D(t_3 - 0), \\ P'(t) &= P'(t-0) = Q'(t) = P'(t_3 - 0) = D(t_1 - 0). \end{aligned}$$

$t \in [t_4, t_5)$:

$$\begin{aligned} C(t) &= 1, \\ P(t) &= D(t), \\ Q(t) &= Q(t-0) = P'(t) = Q(t_4 - 0) = D(t_3 - 0), \\ Q'(t) &= Q'(t-0) = Q'(t_4 - 0) = D(t_1 - 0). \end{aligned}$$

$t \in [t_5, t_6)$:

$$\begin{aligned} C(t) &= 0, \\ P(t) &= P(t-0) = Q(t) = P(t_5 - 0) = D(t_5 - 0), \\ P'(t) &= P'(t-0) = Q'(t) = P'(t_5 - 0) = D(t_3 - 0) \\ & \quad ... \end{aligned}$$

with the conclusion that

$$Q(t) = x^0 \cdot \chi_{(-\infty, t_1)}(t) \oplus D(t_1 - 0) \cdot \chi_{[t_1, t_3)}(t) \oplus D(t_3 - 0) \cdot \chi_{[t_3, t_5)}(t) \oplus ...$$

$$Q'(t) = y^0 \cdot \chi_{(-\infty, t_1)}(t) \oplus x^0 \cdot \chi_{[t_1, t_3)}(t) \oplus D(t_1 - 0) \cdot \chi_{[t_3, t_5)}(t) \oplus D(t_3 - 0) \cdot \chi_{[t_5, t_7)}(t) \oplus ...$$

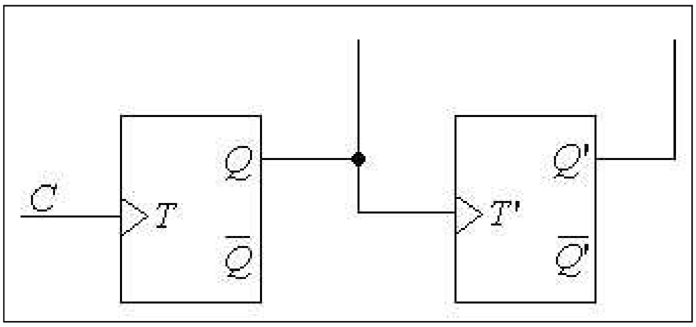

FIGURE 2. Two bit counter

that may be easily generalized to an n-bit shift register with serial input and parallel output.

2. A two bit counter in cascade

We analyze the circuit in Figure 2.

For some unbounded sequence $t_0 < t_1 < t_2 < ...$ the clock is given by the equation:

$$C(t) = \chi_{[t_0,t_1)}(t) \oplus \chi_{[t_2,t_3)}(t) \oplus \chi_{[t_4,t_5)}(t) \oplus ...$$

In the equations (11.2) from Ch. 14 of the T flip-flops

$$C(t) \cdot (\overline{Q(t-0)} \cdot \overline{Q(t)} \cdot P(t) \cup Q(t-0) \cdot Q(t) \cdot \overline{P(t)}) \cup$$

$$\cup \overline{C(t)} \cdot (\overline{Q(t)} \cdot \overline{P(t-0)} \cdot \overline{P(t)} \cup Q(t) \cdot P(t-0) \cdot P(t)) = 1,$$

$$C'(t) \cdot (\overline{Q'(t-0)} \cdot \overline{Q'(t)} \cdot P'(t) \cup Q'(t-0) \cdot Q'(t) \cdot \overline{P'(t)}) \cup$$

$$\cup \overline{C'(t)} \cdot (\overline{Q'(t)} \cdot \overline{P'(t-0)} \cdot \overline{P'(t)} \cup Q'(t) \cdot P'(t-0) \cdot P'(t)) = 1$$

we ask that

$$Q(t) = C'(t).$$

After using this last equation, by multiplication, we get:

$$C(t) \cdot (\overline{Q(t-0)} \cdot \overline{Q(t)} \cdot P(t) \cdot (\overline{Q'(t)} \cdot \overline{P'(t-0)} \cdot \overline{P'(t)} \cup Q'(t) \cdot P'(t-0) \cdot P'(t)) \cup$$

$$\cup Q(t-0) \cdot Q(t) \cdot \overline{P(t)} \cdot (\overline{Q'(t-0)} \cdot \overline{Q'(t)} \cdot P'(t) \cup Q'(t-0) \cdot Q'(t) \cdot \overline{P'(t)})) \cup$$

$$\cup \overline{C(t)} \cdot (\overline{Q(t)} \cdot \overline{P(t-0)} \cdot \overline{P(t)} \cdot (\overline{Q'(t)} \cdot \overline{P'(t-0)} \cdot \overline{P'(t)} \cup Q'(t) \cdot P'(t-0) \cdot P'(t)) \cup$$

$$\cup Q(t) \cdot P(t-0) \cdot P(t) \cdot (\overline{Q'(t-0)} \cdot \overline{Q'(t)} \cdot P'(t) \cup Q'(t-0) \cdot Q'(t) \cdot \overline{P'(t)})) = 1.$$

Suppose that the following initial conditions are fulfilled:

$t \in (-\infty, t_0)$:

$$P(t) = Q(t) = P'(t) = Q'(t) = 0.$$

The choice of these initial conditions leads to some loss of generality, but is necessary for such a circuit that, in order to count, should start counting with 0.

We get:

$t \in [t_0, t_1)$:

$$C(t) \cdot \overline{Q(t-0)} \cdot \overline{Q(t)} \cdot P(t) \cdot (\overline{Q'(t)} \cdot \overline{P'(t-0)} \cdot \overline{P'(t)} \cup Q'(t) \cdot P'(t-0) \cdot P'(t)) = 1,$$

$$\begin{aligned} C(t) &= 1, \\ Q(t) &= Q(t-0) = Q(t_0-0) = 0, P(t) = 1, \\ P'(t) &= P'(t-0) = Q'(t) = P'(t_0-0) = 0, \end{aligned}$$

$t \in [t_1, t_2)$:

$$\overline{C(t)} \cdot Q(t) \cdot P(t-0) \cdot P(t) \cdot (\overline{Q'(t-0)} \cdot \overline{Q'(t)} \cdot P'(t) \cup Q'(t-0) \cdot Q'(t) \cdot \overline{P'(t)}) = 1,$$

$$\begin{aligned} C(t) &= 0, \\ P(t) &= P(t-0) = Q(t) = P(t_1-0) = 1, \\ Q'(t) &= Q'(t-0) = Q'(t_1-0) = 0, P'(t) = \overline{Q'(t)} = 1, \end{aligned}$$

$t \in [t_2, t_3)$:

$$C(t) \cdot Q(t-0) \cdot Q(t) \cdot \overline{P(t)} \cdot (\overline{Q'(t-0)} \cdot \overline{Q'(t)} \cdot P'(t) \cup Q'(t-0) \cdot Q'(t) \cdot \overline{P'(t)}) = 1,$$

$$\begin{aligned} C(t) &= 1, \\ Q(t) &= Q(t-0) = Q(t_2-0) = 1, P(t) = 0, \\ Q'(t) &= Q'(t-0) = Q'(t_2-0) = 0, P'(t) = \overline{Q'(t)} = 1, \end{aligned}$$

$t \in [t_3, t_4)$:

$$\overline{C(t)} \cdot \overline{Q(t)} \cdot \overline{P(t-0)} \cdot \overline{P(t)} \cdot (\overline{Q'(t)} \cdot \overline{P'(t-0)} \cdot \overline{P'(t)} \cup Q'(t) \cdot P'(t-0) \cdot P'(t)) = 1,$$

$$\begin{aligned} C(t) &= 0, \\ P(t) &= P(t-0) = Q(t) = P(t_3-0) = 0, \\ P'(t) &= P'(t-0) = Q'(t) = P'(t_3-0) = 1, \end{aligned}$$

$t \in [t_4, t_5)$:

$$C(t) \cdot \overline{Q(t-0)} \cdot \overline{Q(t)} \cdot P(t) \cdot (\overline{Q'(t)} \cdot \overline{P'(t-0)} \cdot \overline{P'(t)} \cup Q'(t) \cdot P'(t-0) \cdot P'(t)) = 1,$$

$$\begin{aligned} C(t) &= 1, \\ Q(t) &= Q(t-0) = Q(t_4-0) = 0, P(t) = 1, \\ P'(t) &= P'(t-0) = Q'(t) = P'(t_4-0) = 1, \end{aligned}$$

$t \in [t_5, t_6)$:

$$\overline{C(t)} \cdot Q(t) \cdot P(t-0) \cdot P(t) \cdot (\overline{Q'(t-0)} \cdot \overline{Q'(t)} \cdot P'(t) \cup Q'(t-0) \cdot Q'(t) \cdot \overline{P'(t)}) = 1,$$

$$\begin{aligned} C(t) &= 0, \\ P(t) &= P(t-0) = Q(t) = P(t_5-0) = 1, \\ Q'(t) &= Q'(t-0) = Q'(t_5-0) = 1, P'(t) = \overline{Q'(t)} = 0, \end{aligned}$$

$t \in [t_6, t_7)$:

$$C(t) \cdot Q(t-0) \cdot Q(t) \cdot \overline{P(t)} \cdot (\overline{Q'(t-0)} \cdot \overline{Q'(t)} \cdot P'(t) \cup Q'(t-0) \cdot Q'(t) \cdot \overline{P'(t)}) = 1,$$

$$\begin{aligned} C(t) &= 1, \\ Q(t) &= Q(t-0) = Q(t_6-0) = 1, P(t) = 0, \\ Q'(t) &= Q'(t-0) = Q'(t_6-0) = 1, P'(t) = \overline{Q'(t)} = 0, \end{aligned}$$

$t \in [t_7, t_8)$:

$$\overline{C(t)} \cdot \overline{Q(t)} \cdot \overline{P(t-0)} \cdot \overline{P(t)} \cdot (\overline{Q'(t)} \cdot \overline{P'(t-0)} \cdot \overline{P'(t)} \cup Q'(t) \cdot P'(t-0) \cdot P'(t)) = 1,$$

$$\begin{aligned}
C(t) &= 0, \\
P(t) &= P(t-0) = Q(t) = P(t_7 - 0) = 0, \\
P'(t) &= P'(t-0) = Q'(t) = P'(t_7 - 0) = 0,
\end{aligned}$$

$t \in [t_8, t_9)$:

$$C(t) \cdot \overline{Q(t-0)} \cdot \overline{Q(t)} \cdot P(t) \cdot (\overline{Q'(t)} \cdot \overline{P'(t-0)} \cdot \overline{P'(t)} \cup Q'(t) \cdot P'(t-0) \cdot P'(t)) = 1,$$

$$\begin{aligned}
C(t) &= 1, \\
Q(t) &= Q(t-0) = Q(t_8 - 0) = 0, P(t) = 1, \\
P'(t) &= P'(t-0) = Q'(t) = P'(t_8 - 0) = 0, \\
&\quad ...
\end{aligned}$$

At this point we stop, because the computation becomes repetitive. We summarize the previous facts in the following table:

t	$Q(t)$	$Q'(t)$
$(-\infty, t_0)$	0	0
$[t_0, t_1)$	0	0
$[t_1, t_2)$	1	0
$[t_2, t_3)$	1	0
$[t_3, t_4)$	0	1
$[t_4, t_5)$	0	1
$[t_5, t_6)$	1	1
$[t_6, t_7)$	1	1
$[t_7, t_8)$	0	0
$[t_8, t_9)$	0	0
	...	

Table 1

We have obtained that

$$2^0 \cdot Q(t) + 2^1 \cdot Q'(t) = \sum_{\substack{\mathrm{mod}\ 4 \\ \xi \in (-\infty, t]}} C(\xi - 0) \cdot \overline{C(\xi)}.$$

In writing the last relation, we have supposed the fact that $0, 1 \in \mathbf{B}$ are the same like $0, 1 \in \mathbf{N}$; Q represents the units figure while Q', the multiple of 2 figure. The symbol $\sum\limits_{\mathrm{mod}\ 4}$ sums modulo 4 the number of falling edges of $C(t)$.

The circuit in Figure 1 is generalized to an n-bit counter in cascade, counting modulo 2^n.

3. The Mealy model of the synchronous circuits

The circuit drawn in Figure 3 is called a **Mealy machine**, from the name of G. H. Mealy that has used it for the first time in 1955.

The clock signal satisfies

$$C(t) = \chi_{[t_0,t_1)}(t) \oplus \chi_{[t_2,t_3)}(t) \oplus \chi_{[t_4,t_5)}(t) \oplus ...$$

where the sequence $t_0 < t_1 < t_2 < ...$ is unbounded. The given Boolean functions $F : \mathbf{B}^m \times \mathbf{B}^k \to \mathbf{B}^k$, $G : \mathbf{B}^m \times \mathbf{B}^k \to \mathbf{B}^n$ are called the **next state function**,

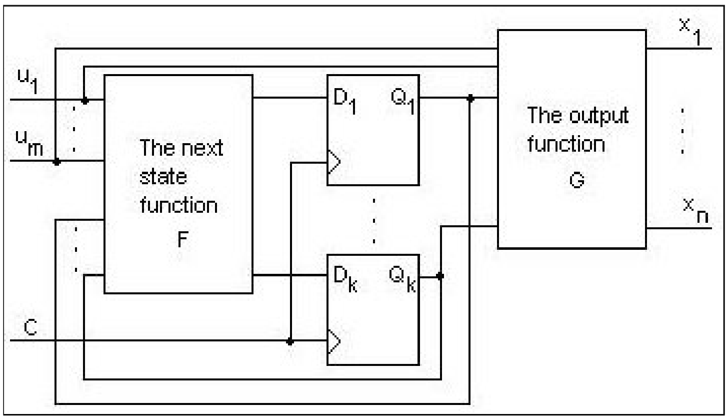

FIGURE 3. The Mealy model

and the **output function** respectively, such that the circuit is described by the following equations:

$$D(t) = D(-\infty + 0) \cdot \chi_{(-\infty,t_0)}(t) \oplus F(u(t), Q(t)) \cdot \chi_{[t_0,\infty)}(t), \tag{3.1}$$

$$Q(t) = Q(-\infty+0)\cdot\chi_{(-\infty,t_1)}(t)\oplus D(t_1-0)\cdot\chi_{[t_1,t_3)}(t)\oplus D(t_3-0)\cdot\chi_{[t_3,t_5)}(t)\oplus ... \tag{3.2}$$

$$x(t) = x(-\infty + 0) \cdot \chi_{(-\infty,t_0)}(t) \oplus G(u(t), Q(t)) \cdot \chi_{[t_0,\infty)}(t), \tag{3.3}$$

with $u \in S^{(m)}$, $D, Q \in S^{(k)}$ and $x \in S^{(n)}$. We substitute (3.2) in (3.1) to get

$$D(t) = D(-\infty + 0) \cdot \chi_{(-\infty,t_0)}(t) \oplus F(u(t), Q(-\infty + 0)) \cdot \chi_{[t_0,t_1)}(t)\oplus \tag{3.4}$$

$$\oplus F(u(t), D(t_1 - 0)) \cdot \chi_{[t_1,t_3)}(t) \oplus F(u(t), D(t_3 - 0)) \cdot \chi_{[t_3,t_5)}(t) \oplus ...$$

and from (3.4) the values

$$D(t_1 - 0) = F(u(t_1 - 0), Q(-\infty + 0)),$$

$$D(t_{2i+1} - 0) = F(u(t_{2i+1} - 0), D(t_{2i-1} - 0)), i \geq 1$$

follow. At this moment we compute the values of $Q(t)$ from (3.2):

$$Q(t_0) = Q(-\infty + 0),$$

$$Q(t_{2i+1}) = D(t_{2i+1} - 0), i \in \mathbf{N},$$

and, taking into account (3.3), we get

$$x(t_0) = G(u(t_0), Q(t_0)) = G(u(t_0), Q(-\infty + 0)),$$

$$x(t_{2i+1}) = G(u(t_{2i+1}), Q(t_{2i+1})) = G(u(t_{2i+1}), D(t_{2i+1} - 0)), i \in \mathbf{N}.$$

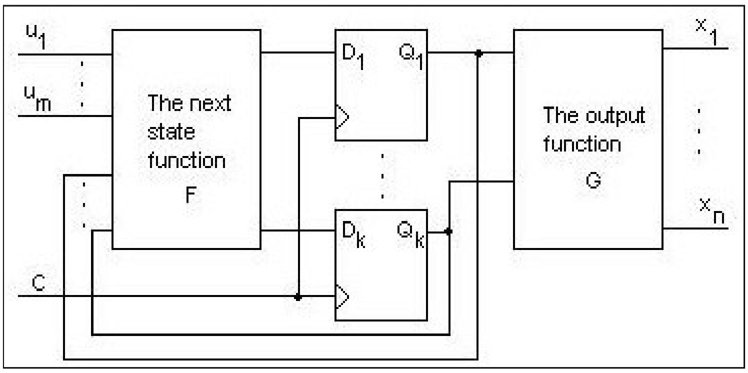

FIGURE 4. The Moore model

4. The Moore model of the synchronous circuits

We have the circuit in Figure 4, called a **Moore machine** by the name of E. F. Moore, who introduced it in 1956.

The **clock signal** is, by definition, of the form

$$C(t) = \chi_{[t_0,t_1)}(t) \oplus \chi_{[t_2,t_3)}(t) \oplus \chi_{[t_4,t_5)}(t) \oplus ...,$$

where the sequence $t_0 < t_1 < t_2 < ...$ is unbounded. The circuit makes use of the functions $F : \mathbf{B}^m \times \mathbf{B}^k \to \mathbf{B}^k$, $G : \mathbf{B}^k \to \mathbf{B}^n$ called the **next state function** and the **output function**. The equations are

$$D(t) = D(-\infty + 0) \cdot \chi_{(-\infty,t_0)}(t) \oplus F(u(t), Q(t)) \cdot \chi_{[t_0,\infty)}(t),$$

$$Q(t) = Q(-\infty + 0) \cdot \chi_{(-\infty,t_1)}(t) \oplus D(t_1 - 0) \cdot \chi_{[t_1,t_3)}(t) \oplus D(t_3 - 0) \cdot \chi_{[t_3,t_5)}(t) \oplus ...,$$

$$x(t) = x(-\infty + 0) \cdot \chi_{(-\infty,t_0)}(t) \oplus G(Q(t)) \cdot \chi_{[t_0,\infty)}(t),$$

with $u \in S^{(m)}$, $D, Q \in S^{(k)}$ and $x \in S^{(n)}$. Like in the previous section we obtain

$$D(t_1 - 0) = F(u(t_1 - 0), Q(-\infty + 0)),$$

$$D(t_{2i+1} - 0) = F(u(t_{2i+1} - 0), D(t_{2i-1} - 0)), i \geq 1$$

and eventually

$$x(t_0) = G(Q(t_0)) = G(Q(-\infty + 0)),$$

$$x(t_{2i+1}) = G(Q(t_{2i+1})) = G(D(t_{2i+1} - 0)), i \in \mathbf{N}.$$

CHAPTER 16

Applications at delay theory

1. The delay circuit

The symbol of the delay circuit is given in Figure 1. We mention some possibilities of modeling this circuit that have occurred in Part 2 of the book. In all these examples $u, x \in S$ are the input and the state functions, respectively.

f_{UD} unbounded delays (the universal delay):

$$f_{UD}(u) = \begin{cases} S_c(0), u \in S_c(0) \\ S_c(1), u \in S_c(1) \\ S, u \in S \setminus S_c \end{cases}.$$

$f_{BD}^{m_r,d_r,m_f,d_f}$ bounded delays: $0 \leq m_r \leq d_r, 0 \leq m_f \leq d_f$ and the system is

$$\bigcap_{\xi \in [t-d_r, t-d_r+m_r]} u(\xi) \leq x(t) \leq \bigcup_{\xi \in [t-d_f, t-d_f+m_f]} u(\xi).$$

$f_{BD'}^{d_r,d_f}$ bounded delays, version (upper bounded, lower unbounded delays): $d_r > 0, d_f > 0$ and the following system is satisfied

$$\bigcap_{\xi \in [t-d_r, t)} u(\xi) \leq x(t) \leq \bigcup_{\xi \in [t-d_f, t)} u(\xi).$$

I_d fixed delays (ideal delays): for $d \geq 0$, the relation between u and x is

$$x(t) = u(t-d).$$

The input and the states of $f_{BD}^{m_r,d_r,m_f,d_f}$, $f_{BD'}^{d_r,d_f}$, I_d satisfy $u(-\infty+0) = x(-\infty+0)$.

$f_{AI}^{\delta_r,\delta_f}$ absolute inertia: there are $\delta_r \geq 0, \delta_f \geq 0$ such that x satisfies

$$\overline{x(t-0)} \cdot x(t) \leq \bigcap_{\xi \in [t, t+\delta_r]} x(\xi),$$

$$x(t-0) \cdot \overline{x(t)} \leq \bigcap_{\xi \in [t, t+\delta_f]} \overline{x(\xi)}.$$

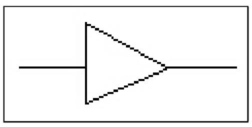

FIGURE 1. The symbol of the delay circuit

$f_{AI'}^{\delta_r,\delta_f}$ absolute inertia, version: there are $\delta_r > 0, \delta_f > 0$ such that

$$\overline{x(t-0)} \cdot x(t) \le \bigcap_{\xi \in [t, t+\delta_r)} x(\xi),$$

$$x(t-0) \cdot \overline{x(t)} \le \bigcap_{\xi \in [t, t+\delta_f)} \overline{x(\xi)}.$$

$f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$ relative inertia: $0 \le \mu_r \le \delta_r, 0 \le \mu_f \le \delta_f$ are given, such that

$$\overline{x(t-0)} \cdot x(t) \le \bigcap_{\xi \in [t-\delta_r, t-\delta_r+\mu_r]} u(\xi),$$

$$x(t-0) \cdot \overline{x(t)} \le \bigcap_{\xi \in [t-\delta_f, t-\delta_f+\mu_f]} \overline{u(\xi)}.$$

$f_{RI'}^{\delta_r,\delta_f}$ relative inertia, version: for $\delta_r > 0, \delta_f > 0$ we have

$$\overline{x(t-0)} \cdot x(t) \le \bigcap_{\xi \in [t-\delta_r, t)} u(\xi),$$

$$x(t-0) \cdot \overline{x(t)} \le \bigcap_{\xi \in [t-\delta_f, t)} \overline{u(\xi)}.$$

The inertia properties $f_{AI}^{\delta_r,\delta_f}$, $f_{AI'}^{\delta_r,\delta_f}$, $f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$, $f_{RI'}^{\delta_r,\delta_f}$ are added to (are intersected with) one of f_{UD}, $f_{BD}^{m_r,d_r,m_f,d_f}$, $f_{BD'}^{d_r,d_f}$, I_d. Here are two special cases of such intersections.

$f_{BD}^{m_r,d_r,m_f,d_f} \cap f_{RI}^{m_r,d_r,m_f,d_f}$ deterministic bounded relative inertial delays: the system takes the form

$$\overline{x(t-0)} \cdot x(t) = \overline{x(t-0)} \cdot \bigcap_{\xi \in [t-d_r, t-d_r+m_r]} u(\xi),$$

$$x(t-0) \cdot \overline{x(t)} = x(t-0) \cdot \bigcap_{\xi \in [t-d_f, t-d_f+m_f]} \overline{u(\xi)}.$$

$f_{BD'}^{d,d} \cap f_{RI'}^{d,d}$ self-dual deterministic upper bounded lower unbounded relative inertial delays, consisting in the equation

$$Dx(t) = (x(t-0) \oplus u(t-0)) \cdot \overline{\bigcup_{\xi \in (t-d, t)} Du(\xi)} \cdot \chi_{[t_0+d, \infty)}(t),$$

where $u(-\infty+0) = u_{|(-\infty,t_0)} = x_{|(-\infty,t_0)} = x(-\infty+0)$ take place.

Some of the previous systems satisfy also supplementary conditions of consistency (i.e. the existence of a solution for any u).

2. Circuit with feedback using a delay circuit

The system was drawn in Figure 2. The wires are ideal and all delays have been concentrated in the delay circuit.

a) f_{UD}

We have

$$\{x | x \in f_{UD}(x)\} = \{x | x \in \begin{cases} S_c(0), x \in S_c(0) \\ S_c(1), x \in S_c(1) \\ S, x \in S \setminus S_c \end{cases} \} = S.$$

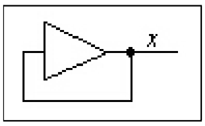

FIGURE 2. Circuit with feedback

b) $f_{BD}^{m_r,d_r,m_f,d_f}$

We suppose that $\exists t_0 \in \mathbf{R}, x_{|(-\infty,t_0)} = 0$. We have the following possibilities.

b.1) $d_f - m_f > 0$

This implies $t_0 - d_f + m_f < t_0$, thus $\bigcup\limits_{\xi\in[t_0-d_f,t_0-d_f+m_f]} x(\xi) = 0$ and, in fact, $\forall t \geq t_0$,

$$x(t) \leq \bigcup_{\xi\in[t-d_f,t-d_f+m_f]} x(\xi) = 0.$$

The unique solution is $x = 0$.

b.2) $d_f - m_f = 0$

Because $x(t) \leq \bigcup\limits_{\xi\in[t-d_f,t]} x(\xi)$ is fulfilled by any x, we get that $x \in f_{BD}^{m_r,d_r,m_f,d_f}(x)$ is equivalent to

$$\bigcap_{\xi\in[t-d_r,t-d_r+m_r]} x(\xi) \leq x(t). \tag{2.1}$$

b.2.1) $d_r > 0$

We analyze the following situations that represent all the possibilities:

i) $x(t) = 0$;

ii) $x(t) = \chi_{[t_1,t_2)}(t) \oplus \chi_{[t_3,t_4)}(t) \oplus ... \oplus \chi_{[t_{2k+1},t_{2k+2})}(t)$,

where $k \in \mathbf{N}$ and $t_0 \leq t_1 < t_2 < t_3 < ... < t_{2k+1} < t_{2k+2}$. We compute

$$\bigcap_{\xi\in[t-d_r,t-d_r+m_r]} x(\xi) = \chi_{[t_1+d_r,t_2+d_r-m_r)}(t)\oplus \tag{2.2}$$
$$\oplus\chi_{[t_3+d_r,t_4+d_r-m_r)}(t) \oplus ... \oplus \chi_{[t_{2k+1}+d_r,t_{2k+2}+d_r-m_r)}(t),$$

where, in the right-hand side of (2.2), a term $\chi_{[t_{2i+1}+d_r,t_{2i+2}+d_r-m_r)}(t)$ is null if $t_{2i+1} + d_r \geq t_{2i+2} + d_r - m_r$, i.e. if $t_{2i+2} - t_{2i+1} \leq m_r, i \in \{0, ..., k\}$. For all $i \in \{0, ..., k\}$ with $t_{2i+2} - t_{2i+1} > m_r$, (2.1) and (2.2) imply the existence of $j \in \{i, ..., k\}$, such that

$$[t_{2i+1} + d_r, t_{2i+2} + d_r - m_r) \subset [t_{2j+1}, t_{2j+2});$$

iii) $x(t) = \chi_{[t_1,\infty)}(t), t_1 \geq t_0$;

iv) $x(t) = \chi_{[t_1,t_2)}(t) \oplus ... \oplus \chi_{[t_{2k+1},t_{2k+2})}(t) \oplus \chi_{[t_{2k+3},\infty)}(t)$,

where $k \in \mathbf{N}$ and $t_0 \leq t_1 < t_2 < t_3 < ... < t_{2k+2} < t_{2k+3}$ and the analysis is made similarly with ii), taking into account that (2.2) is replaced by

$$\bigcap_{\xi\in[t-d_r,t-d_r+m_r]} x(\xi) = \chi_{[t_1+d_r,t_2+d_r-m_r)}(t) \oplus ...\oplus \tag{2.3}$$
$$\oplus\chi_{[t_{2k+1}+d_r,t_{2k+2}+d_r-m_r)}(t) \oplus \chi_{[t_{2k+3}+d_r,\infty)}(t);$$

v) $x(t) = \chi_{[t_1,t_2)}(t) \oplus \chi_{[t_3,t_4)}(t) \oplus ...$,

where $t_0 \leq t_1 < t_2 < t_3 < ...$ is unbounded and

$$(2.4)\qquad \bigcap_{\xi\in[t-d_r,t-d_r+m_r]} x(\xi)=\chi_{[t_1+d_r,t_2+d_r-m_r)}(t)\oplus\chi_{[t_3+d_r,t_4+d_r-m_r)}(t)\oplus...$$

For any $i\in\mathbf{N}$ with $t_{2i+2}-t_{2i+1}>m_r$, there is some $j\geq i$ such that

$$[t_{2i+1}+d_r,t_{2i+2}+d_r-m_r)\subset[t_{2j+1},t_{2j+2}).$$

A special case of b.2.1) is the one when $m_r=0$; (2.1) takes the form

$$(2.5)\qquad x(t-d_r)\leq x(t).$$

Then, for all $i\in\mathbf{N}$, the inclusion

$$[t_{2i+1}+d_r,t_{2i+2}+d_r)\subset supp\ x$$

is fulfilled. For example, the 'periodical' functions

$$x(t)=\chi_{[t_1,t_2)}(t)\oplus\chi_{[t_1+d_r,t_2+d_r)}(t)\oplus...\oplus\chi_{[t_1+kd_r,t_2+kd_r)}(t)\oplus...,$$

where $t_0\leq t_1<t_2\leq t_1+d_r$, satisfy (2.5) because

$$x(t-d_r)=\chi_{[t_1+d_r,t_2+d_r)}(t)\oplus\chi_{[t_1+2d_r,t_2+2d_r)}(t)\oplus...\oplus\chi_{[t_1+(k+1)d_r,t_2+(k+1)d_r)}(t)\oplus...$$

$f_{BD}^{m_r,d_r,m_f,d_f}\cap f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$ In the case b.2.1), $d_f-m_f=0,d_r>0$ adds to the previous requirements the properties

$$(2.6)\qquad \overline{x(t-0)}\cdot x(t)\leq\bigcap_{\xi\in[t-\delta_r,t-\delta_r+\mu_r]} x(\xi),$$

$$(2.7)\qquad x(t-0)\cdot\overline{x(t)}\leq\bigcap_{\xi\in[t-\delta_f,t-\delta_f+\mu_f]}\overline{x(\xi)}$$

implying, if $\delta_r>0$, that $x(t)=0$. For $\delta_r=0$, the inequality (2.6) becomes trivial: $\overline{x(t-0)}\cdot x(t)\leq x(t)$. Then, if $\delta_f>0$, the restrictions corresponding to $f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$ of the solutions x are expressed under the form (see the hypothesis v))

$$\chi_{\{t_2,t_4,...\}}(t)=x(t-0)\cdot\overline{x(t)}\leq$$

$$\leq\bigcap_{\xi\in[t-\delta_f,t-\delta_f+\mu_f]}\overline{x(\xi)}=\chi_{(-\infty,t_1+\delta_f-\mu_f)\cup[t_2+\delta_f,t_3+\delta_f-\mu_f)\cup[t_4+\delta_f,t_5+\delta_f-\mu_f)\cup...}(t)$$

or, equivalently,

$$\{t_2,t_4,...\}\subset(-\infty,t_1+\delta_f-\mu_f)\cup[t_2+\delta_f,t_3+\delta_f-\mu_f)\cup[t_4+\delta_f,t_5+\delta_f-\mu_f)\cup...$$

$\delta_r=\delta_f=0$ means triviality for $f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$.

An interesting situation for $f_{BD}^{m_r,d_r,d_f,d_f}\cap f_{AI}^{\delta_r,\delta_f}$ is the special case $\delta_r\geq m_r,\delta_f=0$, when the inclusion

$$[t_{2i+1}+d_r,t_{2i+2}+d_r-m_r)\subset supp\ x$$

is true for all $i\in\mathbf{N}$ and all solutions x from v), the property $t_{2i+2}-t_{2i+1}>\delta_r\geq m_r$ being satisfied due to $f_{AI}^{\delta_r,\delta_f}$.

b.2.2) $d_r=0$

In this case, (2.1) takes the form $x(t)\leq x(t)$ and all $x\in S$ with $x(-\infty+0)=0$ satisfy $x\in f_{BD}^{m_r,d_r,m_f,d_f}(x)$.

The situation $\exists t_0\in\mathbf{R},x_{|(-\infty,t_0)}=1$ is analyzed in a dual way.

c) $f_{BD'}^{d_r,d_f}$

The system is

$$\bigcap_{\xi\in[t-d_r,t)} x(\xi) \le x(t) \le \bigcup_{\xi\in[t-d_f,t)} x(\xi),$$

with $d_r > 0, d_f > 0$. Let $t_0 \in \mathbf{R}$ be such that $\forall t < t_0, x(t) = 0$. Because $\bigcup\limits_{\xi\in[t_0-d_f,t_0)} x(\xi) = 0$, we get $\forall t \ge t_0, x(t) \le \bigcup\limits_{\xi\in[t-d_f,t)} x(\xi) = 0$. Similarly, let $t_0 \in \mathbf{R}$ be such that $\forall t < t_0, x(t) = 1$. Because $\bigcap\limits_{\xi\in[t_0-d_r,t_0)} x(\xi) = 1$, we obtain $\forall t \ge t_0, 1 = \bigcap\limits_{\xi\in[t-d_r,t)} x(\xi) \le x(t)$. The value t_0 was arbitrary previously, such that the only solutions $x \in f_{BD'}^{d_r,d_f}(x)$ are the constant functions.

On the other hand, the constant functions satisfy trivially any supplementary inertial condition $f_{AI}^{\delta_r,\delta_f}$, $f_{AI'}^{\delta_r,\delta_f}$, $f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$, $f_{RI'}^{\delta_r,\delta_f}$ because $\overline{x(t-0)} \cdot x(t) = x(t-0) \cdot \overline{x(t)} = 0$.

d) I_d

The equation to be solved is

$$x(t) = x(t-d), d \ge 0.$$

If $d > 0$, then the solutions are the two constant functions, while if $d = 0$, then the solutions are all the signals.

e) $f_{BD}^{m_r,d_r,m_f,d_f} \cap f_{RI}^{m_r,d_r,m_f,d_f}$

The system is

$$\overline{x(t-0)} \cdot x(t) = \overline{x(t-0)} \cdot \bigcap_{\xi\in[t-d_r,t-d_r+m_r]} x(\xi), \tag{2.8}$$

$$x(t-0) \cdot \overline{x(t)} = x(t-0) \cdot \bigcap_{\xi\in[t-d_f,t-d_f+m_f]} \overline{x(\xi)} \tag{2.9}$$

and we suppose as before that $\exists t_0 \in \mathbf{R}, x_{|(-\infty,t_0)} = 0$.

e.1) $d_r > 0$

The unique solution is $x(t) = 0$, since $\overline{x(t-0)} \cdot x(t) \le x(t-d_r)$.

e.2) $d_r = 0$

e.2.1) $d_f = m_f > 0$

The switch from 0 to 1 is possible, because (2.8) takes the trivial form $\overline{x(t-0)} \cdot x(t) = \overline{x(t-0)} \cdot x(t)$. From this moment on $\bigcap\limits_{\xi\in[t-d_f,t]} \overline{x(\xi)}$ is null. Thus the solutions have one of the forms $x(t) = 0, x(t) = \chi_{[t_1,\infty)}(t), t_1 \ge t_0$.

e.2.2) $d_f = m_f = 0$

All signals x satisfy the system, because (2.8), (2.9) are both trivial.

e.2.3) $d_f > m_f \ge 0$

The switch from 0 to 1 seems possible and let $t_1 \ge t_0$ be the moment of the first such switch. In other words $\overline{x(t_1-0)} \cdot x(t_1) = 1$. At any time instant $t_2 > t_1$ characterized by $[t_1,t_2) \subset supp\ x$, (2.9) becomes

$$\overline{x(t_2)} = \bigcap_{\xi\in[t_2-d_f,t_2-d_f+m_f]} \overline{x(\xi)}. \tag{2.10}$$

For all $t_2 - d_f + m_f < t_1$, i.e. if $0 < t_2 - t_1 < d_f - m_f$, the right-hand side of (2.10) is 1 and the switch of x from 1 to 0 is necessary. Because $x(t_1) = x(t_1+0)$, we have reached a contradiction showing that (2.8), (2.9) has no solution $x(t) \ne 0$.

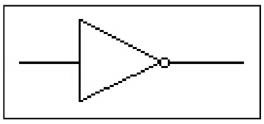

FIGURE 3. The symbol of the logical gate NOT

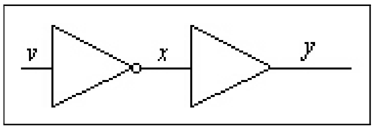

FIGURE 4. The first model of the NOT gate

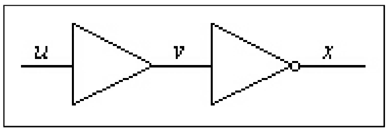

FIGURE 5. The second model of the NOT gate

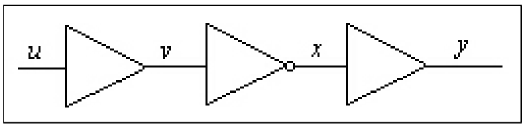

FIGURE 6. The third model of the NOT gate

The analysis of the situation when $\exists t_0 \in \mathbf{R}, x_{|(-\infty,t_0)} = 1$ is similar.

f) $f_{BD'}^{d,d} \cap f_{RI'}^{d,d}$

The solutions of the equation

$$Dx(t) = 0$$

are the constant functions.

3. The logical gate NOT

The logical gate NOT that computes the complement is symbolized like in Figure 3, where the gate and the two wires are characterized by delays. It is modeled by any of the circuits in Figures 4, 5, 6, where the logical gate is ideal

$$x(t) = x(-\infty + 0) \cdot \chi_{(-\infty,t_0)}(t) \oplus \overline{v(t)} \cdot \chi_{[t_0,\infty)}(t), \tag{3.1}$$

as well as the wires and the delays are localized in the delay circuits. The modeling process needs to provide the relationship between x, y and/or between u, v. The last step is the elimination (if possible) of the intermediate variables: x in Figure 4, v in Figure 5, v and x in Figure 6. We give some examples.

a) f_{UD} Figure 6

The fact that u is of the form

$$u(t) = u(t) \cdot \chi_{(-\infty,t_0)}(t) \oplus u(t_0) \cdot \chi_{[t_0,\infty)}(t) \tag{3.2}$$

implies, because $v \in f_{UD}(u)$, that v is of the form

$$v(t) = v(t) \cdot \chi_{(-\infty,t_1)}(t) \oplus u(t_0) \cdot \chi_{[t_1,\infty)}(t). \tag{3.3}$$

From (3.1), there is $t_2 \in \mathbf{R}$ such that x is given by

$$x(t) = x(t) \cdot \chi_{(-\infty,t_2)}(t) \oplus \overline{u(t_0)} \cdot \chi_{[t_2,\infty)}(t) \tag{3.4}$$

and $y \in f_{UD}(x)$ gives the form of y

$$y(t) = y(t) \cdot \chi_{(-\infty,t_3)}(t) \oplus \overline{u(t_0)} \cdot \chi_{[t_3,\infty)}(t). \tag{3.5}$$

b) $f_{BD'}^{d_r,d_f}$ Figure 4

There are $t_0 \in \mathbf{R}, \lambda \in \mathbf{B}, \mu \in \mathbf{B}$ such that $v_{|(-\infty,t_0)} = \lambda, x_{|(-\infty,t_0)} = y_{|(-\infty,t_0)} = \mu$ and we have

$$\bigcap_{\xi \in [t-d_r,t)} x(\xi) \leq y(t) \leq \bigcup_{\xi \in [t-d_f,t)} x(\xi). \tag{3.6}$$

From (3.1) we get:

$$\begin{aligned} &\bigcap_{\xi \in [t-d_r,t)} (\mu \cdot \chi_{(-\infty,t_0)}(\xi) \oplus \overline{v(\xi)} \cdot \chi_{[t_0,\infty)}(\xi)) \leq y(t) \leq \\ &\leq \bigcup_{\xi \in [t-d_f,t)} (\mu \cdot \chi_{(-\infty,t_0)}(\xi) \oplus \overline{v(\xi)} \cdot \chi_{[t_0,\infty)}(\xi)). \end{aligned} \tag{3.7}$$

c) $f_{BD'}^{d_r,d_f}$ Figure 5

$\exists t_0 \in \mathbf{R}, \exists \lambda \in \mathbf{B}, \exists \mu \in \mathbf{B}$ with the property that $u_{|(-\infty,t_0)} = v_{|(-\infty,t_0)} = \lambda, x_{|(-\infty,t_0)} = \mu$. We can write

$$\bigcap_{\xi \in [t-d_r,t)} u(\xi) \leq v(t) \leq \bigcup_{\xi \in [t-d_f,t)} u(\xi), \tag{3.8}$$

$$\bigcap_{\xi \in [t-d_f,t)} \overline{u(\xi)} \leq \overline{v(t)} \leq \bigcup_{\xi \in [t-d_r,t)} \overline{u(\xi)} \quad \text{(from(3.8))}, \tag{3.9}$$

$$\begin{aligned} &\mu \cdot \chi_{(-\infty,t_0)}(t) \oplus \bigcap_{\xi \in [t-d_f,t)} \overline{u(\xi)} \cdot \chi_{[t_0,\infty)}(t) \leq x(t) \leq \\ &\leq \mu \cdot \chi_{(-\infty,t_0)}(t) \oplus \bigcup_{\xi \in [t-d_r,t)} \overline{u(\xi)} \cdot \chi_{[t_0,\infty)}(t) \quad \text{(from (3.1), (3.9))}. \end{aligned} \tag{3.10}$$

d) $f_{BD'}^{d,d} \cap f_{RI'}^{d,d}$ Figure 4

$\exists t_0 \in \mathbf{R}, \exists \lambda \in \mathbf{B}, \exists \mu \in \mathbf{B}, v_{|(-\infty,t_0)} = \lambda, x_{|(-\infty,t_0)} = y_{|(-\infty,t_0)} = \mu$. We obtain

$$x(t-0) = \mu \cdot \chi_{(-\infty,t_0]}(t) \oplus \overline{v(t-0)} \cdot \chi_{(t_0,\infty)}(t) \quad \text{(from(3.1))}, \tag{3.11}$$

$$\begin{aligned} Dx(t) &= x(t-0) \oplus x(t) = \mu \cdot \chi_{(-\infty,t_0]}(t) \oplus \overline{v(t-0)} \cdot \chi_{(t_0,\infty)}(t) \oplus \\ &\oplus \mu \cdot \chi_{(-\infty,t_0)}(t) \oplus \overline{v(t)} \cdot \chi_{[t_0,\infty)}(t) \quad \text{(from (3.1), (3.11))} \\ &= \mu \cdot \chi_{\{t_0\}}(t) \oplus \overline{v(t-0)} \cdot \chi_{(t_0,\infty)}(t) \oplus \overline{v(t_0)} \cdot \chi_{\{t_0\}}(t) \oplus \overline{v(t)} \cdot \chi_{(t_0,\infty)}(t) \\ &= (\mu \oplus \overline{v(t_0)}) \cdot \chi_{\{t_0\}}(t) \oplus (\overline{v(t-0)} \oplus \overline{v(t)}) \cdot \chi_{(t_0,\infty)}(t) \end{aligned} \tag{3.12}$$

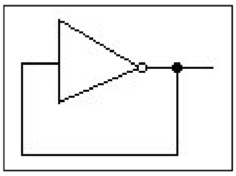

FIGURE 7. Circuit with feedback

$$= (\overline{\mu \oplus v(t_0)}) \cdot \chi_{\{t_0\}}(t) \oplus Dv(t) \cdot \chi_{(t_0,\infty)}(t),$$

(3.13) $$Dy(t) = (y(t-0) \oplus x(t-0)) \cdot \overline{\bigcup_{\xi \in (t-d,t)} Dx(\xi)} \cdot \chi_{[t_0+d,\infty)}(t) \quad \text{(the hypothesis)}$$

$$= (y(t-0) \oplus \mu \cdot \chi_{(-\infty,t_0]}(t) \oplus \overline{v(t-0)} \cdot \chi_{(t_0,\infty)}(t)) \cdot$$

$$\cdot \overline{\bigcup_{\xi \in (t-d,t)} (\overline{\mu \oplus v(t_0)} \cdot \chi_{\{t_0\}}(\xi) \oplus Dv(\xi) \cdot \chi_{(t_0,\infty)}(\xi))} \cdot \chi_{[t_0+d,\infty)}(t) \text{ (from (3.11), (3.12))}$$

$$= \overline{y(t-0) \oplus v(t-0)} \cdot \overline{\bigcup_{\xi \in (t-d,t)} Dv(\xi)} \cdot \chi_{[t_0+d,\infty)}(t).$$

e) $f^{d,d}_{BD'} \cap f^{d,d}_{RI'}$ Figure 5

$\exists t_0 \in \mathbf{R}, \exists \lambda \in \mathbf{B}, \exists \mu \in \mathbf{B}, u_{|(-\infty,t_0)} = v_{|(-\infty,t_0)} = \lambda, x_{|(-\infty,t_0)} = \mu$. The equations are

(3.14) $$v(t) \cdot \chi_{[t_0,\infty)}(t) = \overline{x(t)} \oplus \overline{\mu} \cdot \chi_{(-\infty,t_0)}(t) \quad \text{(from (3.1))},$$

(3.15) $$v(t-0) \cdot \chi_{(t_0,\infty)}(t) = \overline{x(t-0)} \oplus \overline{\mu} \cdot \chi_{(-\infty,t_0]}(t) \quad \text{(from (3.14))}$$

$$= (\overline{x(t-0)} \oplus \overline{\mu}) \cdot \chi_{(-\infty,t_0]}(t) \oplus \overline{x(t-0)} \cdot \chi_{(t_0,\infty)}(t),$$

(3.16) $$Dv(t) = (v(t-0) \oplus u(t-0)) \cdot \overline{\bigcup_{\xi \in (t-d,t)} Du(\xi)} \cdot \chi_{[t_0+d,\infty)}(t) \text{ (the hypothesis)},$$

(3.17) $$Dx(t) = (\overline{\mu \oplus v(t_0)}) \cdot \chi_{\{t_0\}}(t) \oplus Dv(t) \cdot \chi_{(t_0,\infty)}(t) \text{ (see (3.12))}$$

$$= (\overline{\mu \oplus v(t_0)}) \cdot \chi_{\{t_0\}}(t) \oplus (v(t-0) \oplus u(t-0)) \cdot \overline{\bigcup_{\xi \in (t-d,t)} Du(\xi)} \cdot \chi_{[t_0+d,\infty)}(t) \text{ (from (3.16))}$$

$$= (\overline{\mu \oplus v(t_0)}) \cdot \chi_{\{t_0\}}(t) \oplus \overline{x(t-0) \oplus u(t-0)} \cdot \overline{\bigcup_{\xi \in (t-d,t)} Du(\xi)} \cdot \chi_{[t_0+d,\infty)}(t) \text{ (from (3.15))}.$$

A comparison between the forms of (3.13) and (3.17) is interesting.

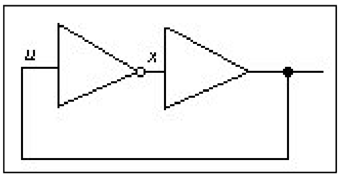

FIGURE 8

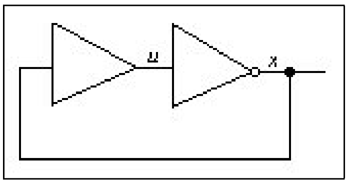

FIGURE 9

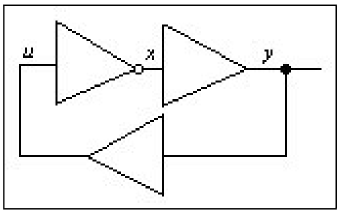

FIGURE 10

4. Circuit with feedback using a logical gate NOT

The circuit is the one in Figure 7, where the logical gate NOT and the wires have delays. The way that we model this circuit is described in Figures 8, 9, 10, where the logical gate and the wires do not have delays and the delays have been concentrated in the delay circuits. In Figures 8, 9 we have the existence of some $t_0 \in \mathbf{R}$ and $\mu \in \mathbf{B}$ with the property that

$$u(t) = \mu \cdot \chi_{(-\infty,t_0)}(t) \oplus u(t) \cdot \chi_{[t_0,\infty)}(t), \tag{4.1}$$

$$x(t) = \mu \cdot \chi_{(-\infty,t_0)}(t) \oplus \overline{u(t)} \cdot \chi_{[t_0,\infty)}(t), \tag{4.2}$$

while in Figure 10 we have, in addition to (4.1), (4.2), the truth of

$$y(t) = \mu \cdot \chi_{(-\infty,t_0)}(t) \oplus y(t) \cdot \chi_{[t_0,\infty)}(t). \tag{4.3}$$

a) f_{UD} Figure 8

Suppose that x is of the form

$$x(t) = x(t) \cdot \chi_{(-\infty,t_1)}(t) \oplus x(t_1) \cdot \chi_{[t_1,\infty)}(t). \tag{4.4}$$

From $u \in f_{UD}(x)$, this implies that u is of the form

$$u(t) = u(t) \cdot \chi_{(-\infty,t_2)}(t) \oplus u(t_2) \cdot \chi_{[t_2,\infty)}(t), \tag{4.5}$$

where

$$x(t_1) = u(t_2), \tag{4.6}$$

for some $t_1 \in \mathbf{R}, t_2 \in \mathbf{R}$. On the other hand, (4.2),(4.4) and (4.5) show that

$$x(t_1) = \overline{u(t_2)}. \tag{4.7}$$

(4.6) and (4.7) are contradictory, meaning the falsity of the hypothesis (4.4). We conclude that the circuit is unstable and, instead of (4.4), we can write

$$x(t) = \mu \cdot \chi_{(-\infty,t_0)}(t) \oplus \overline{\mu} \cdot \chi_{[t_0,t_1)}(t) \oplus \mu \cdot \chi_{[t_1,t_2)}(t) \oplus \overline{\mu} \cdot \chi_{[t_2,t_3)}(t) \oplus ..., \tag{4.8}$$

where $t_0 < t_1 < t_2 < ...$ is unbounded.

b) $f_{BD'}^{d_r,d_f}$ Figure 8

$$\bigcap_{\xi \in [t-d_r,t)} x(\xi) \le u(t) \le \bigcup_{\xi \in [t-d_f,t)} x(\xi) \tag{4.9}$$

and if we substitute (4.2) in (4.9) for $\mu = 0$ we obtain

$$\bigcap_{\xi \in [t-d_r,t)} \overline{u(\xi)} \cdot \chi_{[t_0,\infty)}(\xi) \le u(t) \le \bigcup_{\xi \in [t-d_f,t)} \overline{u(\xi)} \cdot \chi_{[t_0,\infty)}(\xi). \tag{4.10}$$

Suppose that the solution of (4.10) is of the form

$$u(t) = \chi_{[t_1,t_2)}(t) \oplus \chi_{[t_3,t_4)}(t) \oplus ..., \tag{4.11}$$

where $t_0 \le t_1 < t_2 < t_3 < t_4 < ...$ is an unbounded sequence that we try to characterize in the following. We have

$$\overline{u(t)} \cdot \chi_{[t_0,\infty)}(t) = \chi_{[t_0,t_1)}(t) \oplus \chi_{[t_2,t_3)}(t) \oplus ..., \tag{4.12}$$

$$\bigcap_{\xi \in [t-d_r,t)} \overline{u(\xi)} \cdot \chi_{[t_0,\infty)}(\xi) = \chi_{[t_0+d_r,t_1]}(t) \oplus \chi_{[t_2+d_r,t_3]}(t) \oplus ..., \tag{4.13}$$

$$\bigcup_{\xi \in [t-d_f,t)} \overline{u(\xi)} \cdot \chi_{[t_0,\infty)}(\xi) = \chi_{(t_0,t_1+d_f)}(t) \cup \chi_{(t_2,t_3+d_f)}(t) \cup ... \tag{4.14}$$

and, taking into account (4.11), (4.13), (4.14), the inequality (4.10) becomes

$$\begin{aligned}[t_0+d_r,t_1] \cup [t_2+d_r,t_3] \cup ... \subset [t_1,t_2) \cup [t_3,t_4) \cup ... \subset \\ \subset (t_0,t_1+d_f) \cup (t_2,t_3+d_f) \cup ...\end{aligned} \tag{4.15}$$

In (4.15) any of the sets $[t_0+d_r,t_1]$, $[t_2+d_r,t_3], ...$ may be empty if $t_0+d_r > t_1, t_2+d_r > t_3, ...$ and any of the sets (t_0,t_1+d_f), $(t_2,t_3+d_f), ...$ may mutually overlap if $t_1+d_f > t_2, t_3+d_f > t_4, ...$

The left inclusion of (4.15) is satisfied if the following properties are fulfilled:

- $t_0+d_r > t_1$ ($[t_0+d_r,t_1] = \emptyset$) or $t_0+d_r = t_1$ ($[t_0+d_r,t_1] = \{t_1\} \subset [t_1,t_2)$),
- $t_2+d_r > t_3$ ($[t_2+d_r,t_3] = \emptyset$) or $t_2+d_r = t_3$ ($[t_2+d_r,t_3] = \{t_3\} \subset [t_3,t_4)$),

...

while the right inclusion of (4.15) is satisfied if the following statements are true:

- $t_1 > t_0$;
- $t_1+d_f \le t_2$ and $t_2 \le t_1+d_f$ ($(t_0,t_1+d_f) \cap (t_2,t_3+d_f) = \emptyset$ and $[t_1,t_2) \subset (t_0,t_1+d_f)$) or $t_1+d_f > t_2$ ($(t_0,t_1+d_f) \cap (t_2,t_3+d_f) \neq \emptyset$ and $[t_1,t_2) \subset (t_0,t_1+d_f) \cup (t_2,t_3+d_f)$);

- $t_3 + d_f \leq t_4$ and $t_4 \leq t_3 + d_f$ $((t_2, t_3 + d_f) \cap (t_4, t_5 + d_f) = \emptyset$ and $[t_3, t_4) \subset (t_2, t_3 + d_f))$ or $t_3 + d_f > t_4$ $((t_2, t_3 + d_f) \cap (t_4, t_5 + d_f) \neq \emptyset$ and $[t_3, t_4) \subset (t_2, t_3 + d_f) \cup (t_4, t_5 + d_f))$,

...

i.e. in the unbounded sequence $t_0 < t_1 < t_2 < ...$ we have

$$\forall k \in \mathbf{N}, t_{2k+1} - t_{2k} \leq d_r, t_{2k+2} - t_{2k+1} \leq d_f \tag{4.16}$$

The substitution of (4.2) in (4.9), for $\mu = 1$, gives

$$\begin{aligned} \bigcap_{\xi \in [t-d_r,t)} \chi_{(-\infty,t_0)}(\xi) \oplus \overline{u(\xi)} \cdot \chi_{[t_0,\infty)}(\xi) \leq u(t) \leq \\ \leq \bigcup_{\xi \in [t-d_f,t)} \chi_{(-\infty,t_0)}(\xi) \oplus \overline{u(\xi)} \cdot \chi_{[t_0,\infty)}(\xi), \end{aligned} \tag{4.17}$$

with a solution of the form

$$u(t) = \chi_{(-\infty,t_0)}(t) \oplus \chi_{[t_1,t_2)}(t) \oplus \chi_{[t_3,t_4)}(t) \oplus ..., \tag{4.18}$$

where the sequence $t_0 \leq t_1 < t_2 < t_3 < ...$ is unbounded. Eventually we get that $t_0 = t_1$ and (4.16) is still true.

Adding $f_{AI}^{\delta_r,\delta_f}$ to the upper bounded, lower unbounded delay model gives the minimal length of the 0-pulses, and of the 1-pulses respectively; in (4.11): $\forall k \in \mathbf{N}$

$$\begin{aligned} \delta_f &< t_{2k+3} - t_{2k+2}, \\ \delta_r &< t_{2k+2} - t_{2k+1}, \end{aligned} \tag{4.19}$$

while in (4.18), with $t_0 = t_1$, $\forall k \in \mathbf{N}$

$$\begin{aligned} \delta_f &< t_{2k+3} - t_{2k+2}, \\ \delta_r &< t_{2k+4} - t_{2k+3}. \end{aligned} \tag{4.20}$$

$f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$ adds to (4.9) the two requirements

$$\overline{u(t-0)} \cdot u(t) \leq \bigcap_{\xi \in [t-\delta_r, t-\delta_r+\mu_r]} x(\xi), \tag{4.21}$$

$$u(t-0) \cdot \overline{u(t)} \leq \bigcap_{\xi \in [t-\delta_f, t-\delta_f+\mu_f]} \overline{x(\xi)}. \tag{4.22}$$

(4.2) with $\mu = 0$ and (4.11) give

$$\begin{aligned} \bigcap_{\xi \in [t-\delta_r, t-\delta_r+\mu_r]} x(\xi) &= \bigcap_{\xi \in [t-\delta_r, t-\delta_r+\mu_r]} \overline{u(\xi)} \cdot \chi_{[t_0,\infty)}(\xi) \\ &= \bigcap_{\xi \in [t-\delta_r, t-\delta_r+\mu_r]} (\chi_{[t_0,t_1)}(\xi) \oplus \chi_{[t_2,t_3)}(\xi) \oplus \chi_{[t_4,t_5)}(\xi) \oplus ...) \\ &= \chi_{[t_0+\delta_r, t_1+\delta_r-\mu_r)}(t) \oplus \chi_{[t_2+\delta_r, t_3+\delta_r-\mu_r)}(t) \oplus \chi_{[t_4+\delta_r, t_5+\delta_r-\mu_r)}(t) \oplus ... \end{aligned} \tag{4.23}$$

while inequality (4.21)

$$\begin{aligned} \overline{u(t-0)} \cdot u(t) = \chi_{\{t_1,t_3,t_5,...\}}(t) \leq \\ \leq \chi_{[t_0+\delta_r, t_1+\delta_r-\mu_r)}(t) \oplus \chi_{[t_2+\delta_r, t_3+\delta_r-\mu_r)}(t) \oplus \chi_{[t_4+\delta_r, t_5+\delta_r-\mu_r)}(t) \oplus ... \end{aligned}$$

is equivalent to

$$\{t_1, t_3, t_5, ...\} \subset [t_0+\delta_r, t_1+\delta_r-\mu_r) \cup [t_2+\delta_r, t_3+\delta_r-\mu_r) \cup [t_4+\delta_r, t_5+\delta_r-\mu_r) \cup ...$$

Similarly, from (4.2) with $\mu = 0$, (4.11) and (4.22), we obtain

$\{t_2, t_4, t_6, ...\} \subset (-\infty, t_0+\delta_f-\mu_f)\cup[t_1+\delta_f, t_2+\delta_f-\mu_f)\cup[t_3+\delta_f, t_4+\delta_f-\mu_f)\cup...$

We note that in order for the last inclusions to be true two necessary conditions are $\delta_r - \mu_r > 0$ and $\delta_f - \mu_f > 0$ respectively.

Equation (4.18), representing the case $\mu = 1$ combined with $f_{RI}^{\mu_r,\delta_r,\mu_f,\delta_f}$, leads to conclusions of the same nature.

c) I_d Figure 10

(4.1), (4.2), (4.3) together with

$$y(t) = x(t-d_1), \tag{4.24}$$

$$u(t) = y(t-d_2) \tag{4.25}$$

are true where $d_1 \geq 0, d_2 \geq 0$. By eliminating u, x, we obtain

$$\begin{aligned} y(t) = \mu\cdot\chi_{(-\infty,t_0+d_1)}(t)\oplus\overline{\mu}\cdot\chi_{[t_0+d_1,t_0+d_1+d_2)}(t)\oplus \\ \oplus\overline{y(t-d_1-d_2)}\cdot\chi_{[t_0+d_1+d_2,\infty)}(t). \end{aligned} \tag{4.26}$$

c.1) $d_1 + d_2 = 0$

Equation (4.26) is inconsistent.

c.2) $d_1 + d_2 > 0$

The solution of (4.26) is

$$\begin{aligned} y(t) = \mu\cdot\chi_{(-\infty,t_0+d_1)}(t)\oplus\overline{\mu}\cdot\chi_{[t_0+d_1,t_0+2d_1+d_2)}(t)\oplus \\ \oplus\mu\cdot\chi_{[t_0+2d_1+d_2,t_0+3d_1+2d_2)}(t)\oplus\overline{\mu}\cdot\chi_{[t_0+3d_1+2d_2,t_0+4d_1+3d_2)}(t)\oplus... \end{aligned} \tag{4.27}$$

d) $f_{BD}^{m_r,d_r,m_f,d_f}\cap f_{RI}^{m_r,d_r,m_f,d_f}$ Figure 9

(4.1), (4.2) together with

$$\overline{u(t-0)}\cdot u(t) = \overline{u(t-0)}\cdot\bigcap_{\xi\in[t-d_r,t-d_r+m_r]} x(\xi), \tag{4.28}$$

$$u(t-0)\cdot\overline{u(t)} = u(t-0)\cdot\bigcap_{\xi\in[t-d_f,t-d_f+m_f]} \overline{x(\xi)} \tag{4.29}$$

are true, where $0 \leq m_r \leq d_r$, $0 \leq m_f \leq d_f$. By eliminating x, we get

$$\overline{u(t-0)}\cdot u(t) = \overline{u(t-0)}\cdot\bigcap_{\xi\in[t-d_r,t-d_r+m_r]}(\mu\cdot\chi_{(-\infty,t_0)}(\xi)\oplus\overline{u(\xi)}\cdot\chi_{[t_0,\infty)}(\xi)), \tag{4.30}$$

$$u(t-0)\cdot\overline{u(t)} = u(t-0)\cdot\bigcap_{\xi\in[t-d_f,t-d_f+m_f]}(\overline{\mu}\cdot\chi_{(-\infty,t_0)}(\xi)\oplus u(\xi)\cdot\chi_{[t_0,\infty)}(\xi)). \tag{4.31}$$

We suppose that $\mu = 0$.

d.1) $d_r - m_r > 0, d_f - m_f > 0$

Because, in (4.30), we have $\forall\xi \in [t_0, t_0 + d_r), u(\xi) = 0$, the implication is $u(t_0 + d_r) = 1$. Because in (4.31) we have $\forall\xi \in [t_0 + d_r, t_0 + d_r + d_f), u(\xi) = 1$, we infer that $u(t_0 + d_r + d_f) = 0$ etc. The solution is

$$\begin{aligned} u(t) = \chi_{[t_0+d_r,t_0+d_r+d_f)}(t)\oplus\chi_{[t_0+2d_r+d_f,t_0+2d_r+2d_f)}(t)\oplus \\ \oplus\chi_{[t_0+3d_r+2d_f,t_0+3d_r+3d_f)}(t)\oplus..., \end{aligned} \tag{4.32}$$

i.e. from (4.2)

$$x(t) = \overline{u(t)} \cdot \chi_{[t_0,\infty)}(t) = \chi_{[t_0,t_0+d_r)}(t) \oplus \chi_{[t_0+d_r+d_f,t_0+2d_r+d_f)}(t) \oplus \tag{4.33}$$

$$\oplus \chi_{[t_0+2d_r+2d_f,t_0+3d_r+2d_f)}(t) \oplus ...$$

d.2) $d_r - m_r = 0$ *or* $d_f - m_f = 0$

We suppose that $d_r = m_r > 0$ is true. In this situation, (4.30) is

$$\begin{aligned} \overline{u(t-0)} \cdot u(t) &= \overline{u(t-0)} \cdot \bigcap_{\xi \in [t-d_r,t]} \overline{u(\xi)} \cdot \chi_{[t_0,\infty)}(\xi) = \\ &= \overline{u(t-0)} \cdot \bigcap_{\xi \in [t-d_r,t)} \overline{u(\xi)} \cdot \chi_{[t_0,\infty)}(\xi) \cdot \overline{u(t)}. \end{aligned} \tag{4.34}$$

For $t < t_0 + d_r, u(t) = 0$ and at $t = t_0 + d_r$, we get the contradiction $u(t_0 + d_r) = \overline{u(t_0 + d_r)}$. The system is inconsistent. The possibilities $d_r = m_r = 0, d_f = m_f > 0, d_f = m_f = 0$ give inconsistent systems too.

The situation $\mu = 1$ is to be treated similarly.

e) $f_{BD'}^{d,d} \cap f_{RI'}^{d,d}$ Figure 10

(4.1), (4.2), (4.3) together with

$$Dy(t) = (y(t-0) \oplus x(t-0)) \cdot \overline{\bigcup_{\xi \in (t-d_1,t)} Dx(\xi)} \cdot \chi_{[t_0+d_1,\infty)}(t), \tag{4.35}$$

$$Du(t) = (u(t-0) \oplus y(t-0)) \cdot \overline{\bigcup_{\xi \in (t-d_2,t)} Dy(\xi)} \cdot \chi_{[t_0+d_2,\infty)}(t) \tag{4.36}$$

are true, where $d_1 > 0, d_2 > 0$.

Suppose that $\mu = 0$. Then (4.36) gives $Du(t_0) = 0$, i.e. $u(t_0) = 0$.

From (4.2), $x(t_0) = 1$. From (4.35), y becomes 1 at the time instant $t_0 + d_1$. From (4.36), u becomes 1 at the time instant $t_0 + d_1 + d_2$, when in (4.2) x becomes 0. The conclusion is:

$$x(t) = \chi_{[t_0,t_0+d_1+d_2)}(t) \oplus \chi_{[t_0+2d_1+2d_2,t_0+3d_1+3d_2)}(t) \oplus ..., \tag{4.37}$$

$$y(t) = \chi_{[t_0+d_1,t_0+2d_1+d_2)}(t) \oplus \chi_{[t_0+3d_1+2d_2,t_0+4d_1+3d_2)}(t) \oplus ..., \tag{4.38}$$

$$u(t) = \chi_{[t_0+d_1+d_2,t_0+2d_1+2d_2)}(t) \oplus \chi_{[t_0+3d_1+3d_2,t_0+4d_1+4d_2)}(t) \oplus ... \tag{4.39}$$

The situation $\mu = 1$ is similar.

In this case, the solutions are the same as at c), the model I_d.

5. A delay line for the falling transitions only

The circuit proposed in Figure 11, reproduced from [**14**], has the gates and the wires governed by delays. The model is offered by the circuit in Figure 12, where all variables that occur are signals, the gates and the wires have no delays and the delays are concentrated in the delay circuits.

From the static point of view, if we would have had $u, x_1, y_1, ..., x_5, y_5, z, w \in \mathbf{B}$, we note that

$$x_3 = y_3 = \overline{x_2} = \overline{y_2} = x_1, \tag{5.1}$$

$$x_5 = y_5 = \overline{x_4} = \overline{y_4} = \overline{\overline{x_1 \cdot x_3}} = x_1 \cdot x_3, \tag{5.2}$$

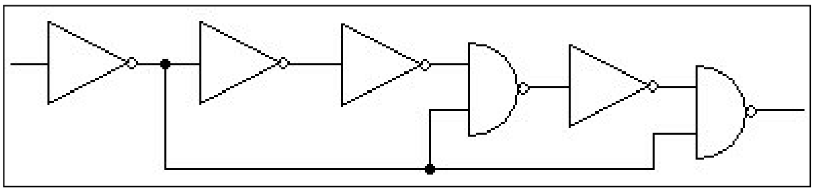

FIGURE 11. A delay line

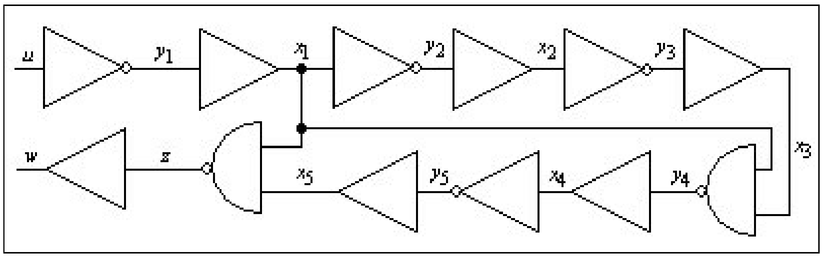

FIGURE 12. The model of the delay line

$$w = z = \overline{x_1 \cdot x_5} \overset{(5.2)}{=} \overline{x_1 \cdot x_3} \overset{(5.1)}{=} \overline{x_1} = \overline{y_1} = u, \tag{5.3}$$

i.e. the Boolean function that this circuit computes is the identity.

We get back to the general situation when all variables are signals. There are some $t_0 \in \mathbf{R}$ and $\mu_0, ..., \mu_6 \in \mathbf{B}$ such that $u_{|(-\infty,t_0)} = \mu_0, y_{i|(-\infty,t_0)} = x_{i|(-\infty,t_0)} = \mu_i, i = \overline{1,5}, z_{|(-\infty,t_0)} = w_{|(-\infty,t_0)} = \mu_6$ and the system of equations and inequalities is

$$y_1(t) = \mu_1 \cdot \chi_{(-\infty,t_0)}(t) \oplus \overline{u(t)} \cdot \chi_{[t_0,\infty)}(t), \tag{5.4}$$

$$y_2(t) = \mu_2 \cdot \chi_{(-\infty,t_0)}(t) \oplus \overline{x_1(t)} \cdot \chi_{[t_0,\infty)}(t), \tag{5.5}$$

$$y_3(t) = \mu_3 \cdot \chi_{(-\infty,t_0)}(t) \oplus \overline{x_2(t)} \cdot \chi_{[t_0,\infty)}(t), \tag{5.6}$$

$$y_4(t) = \mu_4 \cdot \chi_{(-\infty,t_0)}(t) \oplus \overline{x_1(t) \cdot x_3(t)} \cdot \chi_{[t_0,\infty)}(t), \tag{5.7}$$

$$y_5(t) = \mu_5 \cdot \chi_{(-\infty,t_0)}(t) \oplus \overline{x_4(t)} \cdot \chi_{[t_0,\infty)}(t), \tag{5.8}$$

$$z(t) = \mu_6 \cdot \chi_{(-\infty,t_0)}(t) \oplus \overline{x_1(t) \cdot x_5(t)} \cdot \chi_{[t_0,\infty)}(t), \tag{5.9}$$

$$\bigcap_{\xi \in [t-d_r,t)} y_i(\xi) \leq x_i(t) \leq \bigcup_{\xi \in [t-d_f,t)} y_i(\xi), i = \overline{1,5}, \tag{5.10}$$

$$\bigcap_{\xi \in [t-d_r,t)} z(\xi) \leq w(t) \leq \bigcup_{\xi \in [t-d_f,t)} z(\xi). \tag{5.11}$$

Thus we use the model $f_{BD'}^{d_r,d_f}$ with $d_r > 0, d_f > 0$, the parameters that characterize all six delay circuits. For an easier analysis of the circuit, we make the simplifying hypothesis $\exists\mu \in \mathbf{B}, \mu_0 = \mu_2 = \mu_4 = \mu_6 = \mu,\ \mu_1 = \mu_3 = \mu_5 = \overline{\mu}$, with which (5.4),...,(5.9) become

$$y_1(t) = \overline{u(t)}, \tag{5.12}$$

$$y_2(t) = \overline{x_1(t)}, \tag{5.13}$$

$$y_3(t) = \overline{x_2(t)}, \tag{5.14}$$

$$y_4(t) = \overline{x_3(t) \cdot x_1(t)}, \tag{5.15}$$

$$y_5(t) = \overline{x_4(t)}, \tag{5.16}$$

$$z(t) = \overline{x_5(t) \cdot x_1(t)}. \tag{5.17}$$

We have

$$\bigcap_{\xi\in[t-d_r,t)} y_1(\xi) \le x_1(t) \le \bigcup_{\xi\in[t-d_f,t)} y_1(\xi) \quad \text{(equation (5.10))}, \tag{5.18}$$

$$\bigcap_{\xi\in[t-d_r,t)} \overline{u(\xi)} \le x_1(t) \le \bigcup_{\xi\in[t-d_f,t)} \overline{u(\xi)} \quad \text{(from (5.12) and (5.18))}, \tag{5.19}$$

$$\bigcap_{\xi\in[t-d_r,t)} y_3(\xi) \le x_3(t) \le \bigcup_{\xi\in[t-d_f,t)} y_3(\xi) \quad \text{(equation (5.10))}, \tag{5.20}$$

$$\bigcap_{\xi\in[t-d_r,t)} \overline{x_2(\xi)} \le x_3(t) \le \bigcup_{\xi\in[t-d_f,t)} \overline{x_2(\xi)} \quad \text{(from (5.14) and (5.20))}, \tag{5.21}$$

$$\bigcap_{\xi\in[t-d_r,t)} y_2(\xi) \le x_2(t) \le \bigcup_{\xi\in[t-d_f,t)} y_2(\xi) \quad \text{(equation (5.10))}, \tag{5.22}$$

$$\bigcap_{\xi\in[t-d_f,t)} \overline{y_2(\xi)} \le \overline{x_2(t)} \le \bigcup_{\xi\in[t-d_r,t)} \overline{y_2(\xi)} \quad \text{(from (5.22))}, \tag{5.23}$$

$$\bigcap_{\xi\in[t-d_f,t)} x_1(\xi) \le \overline{x_2(t)} \le \bigcup_{\xi\in[t-d_r,t)} x_1(\xi) \quad \text{(from (5.13) and (5.23))}, \tag{5.24}$$

$$\bigcap_{\xi\in[t-d_r-d_f,t)} x_1(\xi) \le x_3(t) \le \bigcup_{\xi\in[t-d_r-d_f,t)} x_1(\xi) \quad \text{(from (5.21) and (5.24))}, \tag{5.25}$$

$$\bigcap_{\xi\in[t-2d_r-d_f,t)} \overline{u(\xi)} \le x_3(t) \le \bigcup_{\xi\in[t-d_r-2d_f,t)} \overline{u(\xi)} \quad \text{(from (5.19) and (5.25))}, \tag{5.26}$$

$$\overline{y_4(t)} = x_3(t) \cdot x_1(t) \quad \text{(from (5.15))}, \tag{5.27}$$

$$\bigcap_{\xi\in[t-2d_r-d_f,t)} \overline{u(\xi)} \cdot \bigcap_{\xi\in[t-d_r,t)} \overline{u(\xi)} \le \overline{y_4(t)} \le \bigcup_{\xi\in[t-d_r-2d_f,t)} \overline{u(\xi)} \cdot \bigcup_{\xi\in[t-d_f,t)} \overline{u(\xi)} \tag{5.28}$$

from (5.26), (5.19) and (5.27). But $[t-2d_r-d_f,t)\supset[t-d_r,t)$, $[t-d_r-2d_f,t)\supset[t-d_f,t)$ imply

$$\bigcap_{\xi\in[t-2d_r-d_f,t)}\overline{u(\xi)}\leq\bigcap_{\xi\in[t-d_r,t)}\overline{u(\xi)},$$

$$\bigcup_{\xi\in[t-d_r-2d_f,t)}\overline{u(\xi)}\geq\bigcup_{\xi\in[t-d_f,t)}\overline{u(\xi)},$$

$$\bigcap_{\xi\in[t-2d_r-d_f,t)}\overline{u(\xi)}\cdot\bigcap_{\xi\in[t-d_r,t)}\overline{u(\xi)}=\bigcap_{\xi\in[t-2d_r-d_f,t)}\overline{u(\xi)},$$

$$\bigcup_{\xi\in[t-d_r-2d_f,t)}\overline{u(\xi)}\cdot\bigcup_{\xi\in[t-d_f,t)}\overline{u(\xi)}=\bigcup_{\xi\in[t-d_f,t)}\overline{u(\xi)},$$

wherefrom (5.28) becomes

$$\bigcap_{\xi\in[t-2d_r-d_f,t)}\overline{u(\xi)}\leq\overline{y_4(t)}\leq\bigcup_{\xi\in[t-d_f,t)}\overline{u(\xi)}.\tag{5.29}$$

Furthermore:

$$\bigcap_{\xi\in[t-d_r,t)}y_5(\xi)\leq x_5(t)\leq\bigcup_{\xi\in[t-d_f,t)}y_5(\xi)\quad\text{(equation (5.10))},\tag{5.30}$$

$$\bigcap_{\xi\in[t-d_r,t)}\overline{x_4(\xi)}\leq x_5(t)\leq\bigcup_{\xi\in[t-d_f,t)}\overline{x_4(\xi)}\quad\text{(from (5.16) and (5.30))},\tag{5.31}$$

$$\bigcap_{\xi\in[t-d_f,t)}\overline{y_4(\xi)}\leq\overline{x_4(t)}\leq\bigcup_{\xi\in[t-d_r,t)}\overline{y_4(\xi)}\quad\text{(similar with (5.23))},\tag{5.32}$$

$$\bigcap_{\xi\in[t-d_r-d_f,t)}\overline{y_4(\xi)}\leq x_5(t)\leq\bigcup_{\xi\in[t-d_r-d_f,t)}\overline{y_4(\xi)}\quad\text{(from (5.31) and (5.32))},\tag{5.33}$$

$$\bigcap_{\xi\in[t-3d_r-2d_f,t)}\overline{u(\xi)}\leq x_5(t)\leq\bigcup_{\xi\in[t-d_r-2d_f,t)}\overline{u(\xi)}\quad\text{(from (5.29) and (5.33))},\tag{5.34}$$

$$\overline{z(t)}=x_5(t)\cdot x_1(t)\quad\text{(from (5.17))},\tag{5.35}$$

$$\bigcap_{\xi\in[t-3d_r-2d_f,t)}\overline{u(\xi)}\cdot\bigcap_{\xi\in[t-d_r,t)}\overline{u(\xi)}\leq\overline{z(t)}\leq\bigcup_{\xi\in[t-d_r-2d_f,t)}\overline{u(\xi)}\cdot\bigcup_{\xi\in[t-d_f,t)}\overline{u(\xi)}\tag{5.36}$$

from (5.35), (5.34) and (5.19). With arguments like those for (5.28) from (5.36) we infer

$$\bigcap_{\xi\in[t-3d_r-2d_f,t)}\overline{u(\xi)}\leq\overline{z(t)}\leq\bigcup_{\xi\in[t-d_f,t)}\overline{u(\xi)}.\tag{5.37}$$

Thus

$$\bigcap_{\xi\in[t-d_f,t)}u(\xi)\leq z(t)\leq\bigcup_{\xi\in[t-3d_r-2d_f,t)}u(\xi).\tag{5.38}$$

From (5.11) and (5.38) we get

$$\bigcap_{\xi\in[t-d_r-d_f,t)}u(\xi)\leq w(t)\leq\bigcup_{\xi\in[t-3d_r-3d_f,t)}u(\xi).\tag{5.39}$$

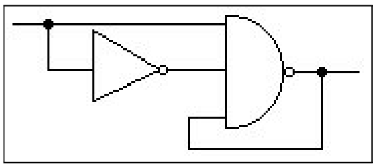

FIGURE 13. Circuit with transient oscillations

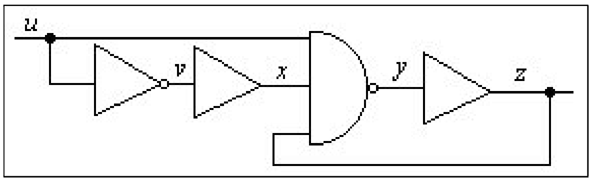

FIGURE 14. The model of the circuit

The conclusion expressed by (5.39) is that the circuit increases the one gate upper bound of the rising delay from d_r to $d_r + d_f$ and the one gate upper bound of the falling delay from d_f to $3d_r + 3d_f$ respectively, i.e. the growth of the falling delay is bigger than the growth of the rising delay. This justifies the title of the section.

6. Circuit with transient oscillations

In Figure 13 we reproduce an example of circuit from [**35**]. Its model is drawn in Figure 14. Like before, in the first figure the two logical gates and the wires have delays, while in the second, all the delays are concentrated in the delay circuits. Even if the static analysis of such a circuit, when $u, v, x, y, z \in \mathbf{B}$, is not appropriate due to the feedback loop, we remark that the proposed circuit computes the constant 1 Boolean function because

$$z = y = \overline{u \cdot x \cdot z} = \overline{u \cdot \overline{v} \cdot z} = \overline{u \cdot \overline{u} \cdot z} = \overline{0} = 1.$$

The conclusion is that, when $u, v, x, y, z \in S$, after solving the system, we must obtain $\lim\limits_{t \to \infty} z(t) = 1$ independently of the choice of u, of the choice of the initial conditions and of the choice of the type of delays.

We choose fixed delays and we suppose the existence of $t_0 \in \mathbf{R}, \mu_0, \mu_1, \mu_2 \in \mathbf{B}$, such that $u_{|(-\infty,t_0)} = \mu_0, v_{|(-\infty,t_0)} = x_{|(-\infty,t_0)} = \mu_1, y_{|(-\infty,t_0)} = z_{|(-\infty,t_0)} = \mu_2$. The equations are

$$v(t) = \mu_1 \cdot \chi_{(-\infty,t_0)}(t) \oplus \overline{u(t)} \cdot \chi_{[t_0,\infty)}(t), \tag{6.1}$$

$$x(t) = v(t - d), \tag{6.2}$$

$$y(t) = \mu_2 \cdot \chi_{(-\infty,t_0)}(t) \oplus \overline{u(t) \cdot x(t) \cdot z(t)} \cdot \chi_{[t_0,\infty)}(t), \tag{6.3}$$

$$z(t) = y(t - d'), \tag{6.4}$$

wherefrom

$$\begin{aligned} x(t) &= \mu_1 \cdot \chi_{(-\infty,t_0)}(t - d) \oplus \overline{u(t - d)} \cdot \chi_{[t_0,\infty)}(t - d) = \\ &= \mu_1 \cdot \chi_{(-\infty,t_0+d)}(t) \oplus \overline{u(t - d)} \cdot \chi_{[t_0+d,\infty)}(t) \quad \text{(from (6.1), (6.2))}, \end{aligned} \tag{6.5}$$

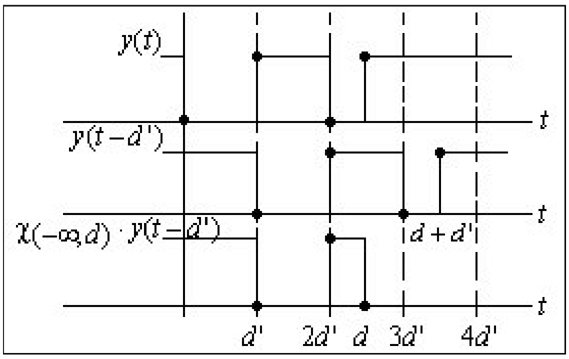

FIGURE 15. The solution, case $2kd' < d \leq (2k+1)d'$

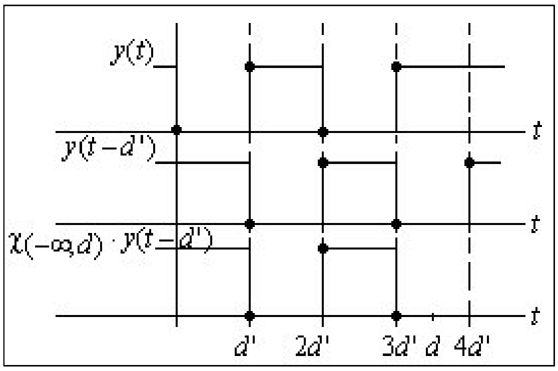

FIGURE 16. The solution, case $(2k+1)d' < d \leq (2k+2)d'$

$$y(t) = \mu_2 \cdot \chi_{(-\infty,t_0)}(t) \oplus \qquad \text{(from (6.3), (6.4), (6.5))} \tag{6.6}$$

$$\oplus \overline{u(t) \cdot (\mu_1 \cdot \chi_{(-\infty,t_0+d)}(t) \oplus \overline{u(t-d)} \cdot \chi_{[t_0+d,\infty)}(t)) \cdot y(t-d')} \cdot \chi_{[t_0,\infty)}(t).$$

We solve (6.6) in the special case $\mu_1 = \mu_2 = 1$ and $u(t) = 1$. The equation becomes

$$y(t) = \chi_{(-\infty,t_0)}(t) \oplus \overline{\chi_{(-\infty,t_0+d)}(t) \cdot y(t-d')} \cdot \chi_{[t_0,\infty)}(t). \tag{6.7}$$

The solution of (6.7) is the following

$2kd' < d \leq (2k+1)d'$ implies

$$y(t) = \begin{cases} \chi_{(-\infty,t_0)}(t) \oplus \chi_{[t_0+d',t_0+2d')}(t) \oplus ... \\ ... \oplus \chi_{[t_0+(2k-1)d',t_0+2kd')}(t) \oplus \chi_{[t_0+d,\infty)}(t), k \geq 1 \\ \chi_{(-\infty,t_0)}(t) \oplus \chi_{[t_0+d,\infty)}(t), k = 0 \end{cases},$$

$(2k+1)d' < d \leq (2k+2)d'$ implies

$$y(t) = \begin{cases} \chi_{(-\infty,t_0)}(t) \oplus \chi_{[t_0+d',t_0+2d')}(t) \oplus ... \\ ... \oplus \chi_{[t_0+(2k-1)d',t_0+2kd')}(t) \oplus \chi_{[t_0+(2k+1)d',\infty)}(t), k \geq 1 \\ \chi_{(-\infty,t_0)}(t) \oplus \chi_{[t_0+d',\infty)}(t), k = 0 \end{cases},$$

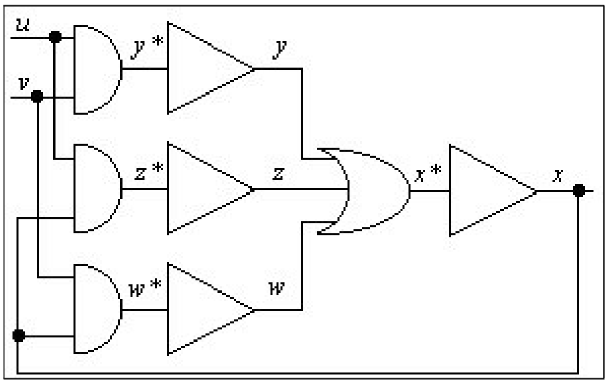

FIGURE 17. The model of the circuit from Figure 1, page 227

where $k \in \mathbf{N}$. In Figures 15 and 16 we have drawn these two functions for $t_0 = 0$ and $k = 1$.

The output $z(t)$ of the circuit is obtained from (6.4).

The idea of solving equation (6.6) in other cases, as well as the behavior of the circuit in Figure 13, are obvious now.

7. Example of C gate

The circuit that we analyze is drawn in Figure 1, page 227, where the logical gates and the wires have delays and its model is the one in Figure 17, where the gates and the wires are ideal. There are $t_0 \in \mathbf{R}$ and $\mu_0, ..., \mu_5 \in \mathbf{B}$ such that $u_{|(-\infty,t_0)} = \mu_0$, $v_{|(-\infty,t_0)} = \mu_1$, $y^*_{|(-\infty,t_0)} = y_{|(-\infty,t_0)} = \mu_2$, $z^*_{|(-\infty,t_0)} = z_{|(-\infty,t_0)} = \mu_3$, $w^*_{|(-\infty,t_0)} = w_{|(-\infty,t_0)} = \mu_4$, $x^*_{|(-\infty,t_0)} = x_{|(-\infty,t_0)} = \mu_5$ and the following equations

$$y^*(t) = \mu_2 \cdot \chi_{(-\infty,t_0)}(t) \oplus u(t) \cdot v(t) \cdot \chi_{[t_0,\infty)}(t), \tag{7.1}$$

$$z^*(t) = \mu_3 \cdot \chi_{(-\infty,t_0)}(t) \oplus u(t) \cdot x(t) \cdot \chi_{[t_0,\infty)}(t), \tag{7.2}$$

$$w^*(t) = \mu_4 \cdot \chi_{(-\infty,t_0)}(t) \oplus v(t) \cdot x(t) \cdot \chi_{[t_0,\infty)}(t), \tag{7.3}$$

$$x^*(t) = \mu_5 \cdot \chi_{(-\infty,t_0)}(t) \oplus (y(t) \cup z(t) \cup w(t)) \cdot \chi_{[t_0,\infty)}(t) \tag{7.4}$$

are fulfilled. In order to simplify the analysis, we suppose that $\mu_0 = \mu_1 = ... = \mu_5$. In this case (7.1),...,(7.4) become

$$y^*(t) = u(t) \cdot v(t), \tag{7.5}$$

$$z^*(t) = u(t) \cdot x(t), \tag{7.6}$$

$$w^*(t) = v(t) \cdot x(t), \tag{7.7}$$

$$x^*(t) = y(t) \cup z(t) \cup w(t). \tag{7.8}$$

a) The bounded delay model

$$\bigcap_{\xi \in [t-d_r, t-d_r+m_r]} y^*(\xi) \leq y(t) \leq \bigcup_{\xi \in [t-d_f, t-d_f+m_f]} y^*(\xi), \tag{7.9}$$

$$\bigcap_{\xi\in[t-d_r,t-d_r+m_r]} z^*(\xi) \leq z(t) \leq \bigcup_{\xi\in[t-d_f,t-d_f+m_f]} z^*(\xi), \tag{7.10}$$

$$\bigcap_{\xi\in[t-d_r,t-d_r+m_r]} w^*(\xi) \leq w(t) \leq \bigcup_{\xi\in[t-d_f,t-d_f+m_f]} w^*(\xi), \tag{7.11}$$

$$\bigcap_{\xi\in[t-D_r,t-D_r+M_r]} x^*(\xi) \leq x(t) \leq \bigcup_{\xi\in[t-D_f,t-D_f+M_f]} x^*(\xi), \tag{7.12}$$

with $0 \leq m_r \leq d_r$, $0 \leq m_f \leq d_f$, $0 \leq M_r \leq D_r$, $0 \leq M_f \leq D_f$ while the consistency conditions are fulfilled under the form: $d_r \geq d_f - m_f$, $d_f \geq d_r - m_r$ and $D_r \geq D_f - M_f$, $D_f \geq D_r - M_r$ respectively. We have considered that the three AND gates are identical. We eliminate the intermediate variables $y^*, z^*, w^*, y, z, w, x^*$

$$x(t) \overset{(7.8),(7.12)}{\geq} \bigcap_{\xi\in[t-D_r,t-D_r+M_r]} (y(\xi)\cup z(\xi)\cup w(\xi)) \geq \tag{7.13}$$

$$\geq \bigcap_{\xi\in[t-D_r,t-D_r+M_r]} y(\xi) \overset{(7.9)}{\geq} \bigcap_{\xi\in[t-D_r,t-D_r+M_r]} \bigcap_{\omega\in[\xi-d_r,\xi-d_r+m_r]} y^*(\omega) =$$

$$= \bigcap_{\xi\in[t-d_r-D_r,t-d_r-D_r+m_r+M_r]} y^*(\xi) \overset{(7.5)}{=} \bigcap_{\xi\in[t-d_r-D_r,t-d_r-D_r+m_r+M_r]} (u(\xi)\cdot v(\xi)),$$

$$x(t) \overset{(7.8),(7.12)}{\leq} \bigcup_{\xi\in[t-D_f,t-D_f+M_f]} (y(\xi)\cup z(\xi)\cup w(\xi)) = \tag{7.14}$$

$$= \bigcup_{\xi\in[t-D_f,t-D_f+M_f]} y(\xi) \cup \bigcup_{\xi\in[t-D_f,t-D_f+M_f]} z(\xi) \cup \bigcup_{\xi\in[t-D_f,t-D_f+M_f]} w(\xi) \leq$$

$$\overset{(7.9),(7.10),(7.11)}{\leq} \bigcup_{\xi\in[t-D_f,t-D_f+M_f]} \bigcup_{\omega\in[\xi-d_f,\xi-d_f+m_f]} y^*(\omega)\cup$$

$$\cup \bigcup_{\xi\in[t-D_f,t-D_f+M_f]} \bigcup_{\omega\in[\xi-d_f,\xi-d_f+m_f]} z^*(\omega)\cup$$

$$\cup \bigcup_{\xi\in[t-D_f,t-D_f+M_f]} \bigcup_{\omega\in[\xi-d_f,\xi-d_f+m_f]} w^*(\omega) =$$

$$= \bigcup_{\xi\in[t-d_f-D_f,t-d_f-D_f+m_f+M_f]} y^*(\xi) \cup \bigcup_{\xi\in[t-d_f-D_f,t-d_f-D_f+m_f+M_f]} z^*(\xi)\cup$$

$$\cup \bigcup_{\xi\in[t-d_f-D_f,t-d_f-D_f+m_f+M_f]} w^*(\xi) =$$

$$\overset{(7.5),(7.6),(7.7)}{=} \bigcup_{\xi\in[t-d_f-D_f,t-d_f-D_f+m_f+M_f]} u(\xi)\cdot v(\xi)\cup$$

$$\cup \bigcup_{\xi\in[t-d_f-D_f,t-d_f-D_f+m_f+M_f]} u(\xi)\cdot x(\xi)\cup$$

$$\cup \bigcup_{\xi\in[t-d_f-D_f,t-d_f-D_f+m_f+M_f]} v(\xi)\cdot x(\xi) =$$

$$= \bigcup_{\xi\in[t-d_f-D_f,t-d_f-D_f+m_f+M_f]} (u(\xi)\cdot v(\xi)\cup(u(\xi)\cup v(\xi))\cdot x(\xi)) \leq$$

$$\leq \bigcup_{\xi\in[t-d_f-D_f,t-d_f-D_f+m_f+M_f]} (u(\xi)\cdot v(\xi)\cup u(\xi)\cup v(\xi)) =$$
$$= \bigcup_{\xi\in[t-d_f-D_f,t-d_f-D_f+m_f+M_f]} (u(\xi)\cup v(\xi)).$$

Thus, by cumulating (7.13) and (7.14),

$$\bigcap_{\xi\in[t-d_r-D_r,t-d_r-D_r+m_r+M_r]} (u(\xi)\cdot v(\xi)) \leq x(t) \leq$$
$$\leq \bigcup_{\xi\in[t-d_f-D_f,t-d_f-D_f+m_f+M_f]} (u(\xi)\cup v(\xi)),$$

we have obtained a system that is very much similar to $f_{BD}^{m_r+M_r,d_r+D_r,m_f+M_f,d_f+D_f}$.

b) The deterministic model

We ask that (7.13) and (7.14) be fulfilled together with

$$\overline{x(t-0)}\cdot x(t) \leq \bigcap_{\xi\in[t-d_r-D_r,t-d_r-D_r+m_r+M_r]} (u(\xi)\cdot v(\xi)), \tag{7.15}$$

$$x(t-0)\cdot \overline{x(t)} \leq \bigcap_{\xi\in[t-d_f-D_f,t-d_f-D_f+m_f+M_f]} \overline{u(\xi)}\cdot\overline{v(\xi)}. \tag{7.16}$$

The system (7.13), (7.14), (7.15), (7.16) represents a deterministic model, similar to $f_{BD}^{m_r,d_r,m_f,d_f}\cap f_{RI}^{m_r,d_r,m_f,d_f}$ and it is equivalent to

$$\overline{x(t-0)}\cdot x(t) = \overline{x(t-0)}\cdot \bigcap_{\xi\in[t-d_r-D_r,t-d_r-D_r+m_r+M_r]} (u(\xi)\cdot v(\xi)),$$
$$x(t-0)\cdot\overline{x(t)} = x(t-0)\cdot \bigcap_{\xi\in[t-d_f-D_f,t-d_f-D_f+m_f+M_f]} \overline{u(\xi)}\cdot\overline{v(\xi)},$$

that is similar to the system (13.1), (13.2) from Ch. 13.

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APPENDIX A

Intersections with temporal logic

The language of the classical logic of the propositions CLP contains the following atoms:

- the individual constants $0, 1 \in \mathbf{B}$, also called **antilogy**, respectively **tautology**;

- the Boolean variables $\lambda_1, ..., \lambda_m, ..., \mu_1, ..., \mu_n, ... \in \mathbf{B}$, also called **propositional variables**.

Intuitively, they represent statements which are either false, or true. So is the case of the Boolean functions $H : \mathbf{B}^m \to \mathbf{B}, \mathbf{B}^m \ni (\lambda_1, ..., \lambda_m) \longmapsto H(\lambda_1, ..., \lambda_m) \in \mathbf{B}$, also called **formulae** of CLP. The **connectors** of CLP are the laws of $\mathbf{B}$: $\overline{}, \cdot, \cup, \oplus, ...$ etc.

The semantics of CLP answers the question: in the interpretation I that assigns to the $n-$tuple of variables $\lambda = (\lambda_1, ..., \lambda_m)$ the value $\lambda^0 = (\lambda_1^0, ..., \lambda_m^0)$, do we have $H(\lambda^0) = 1$? If yes, we say that H is satisfied (or that it is valid, or that it holds true, or that it is true) in I. We denote this fact by $\lambda^0 \models H$. The constant function $H = 1$ is identified to the tautology and we write $\models H$ (H is satisfied in any interpretation).

Temporal logic TL, also known as tense logic, uses, in the variant from our work, the 'frame' (i.e. couple) $(\mathbf{R}, <)$, where $\mathbf{R}$=the set of the real numbers=the 'set of the possible worlds'=the time set and $<$ that is the order of $\mathbf{R}$. When we debate what conditions this frame fulfills, it is necessary, or perhaps useful, to use the algebraic structure of field of $\mathbf{R}$, consistent with $<$. Due to the axiom of Archimedes, $\mathbf{R}$ satisfies the properties of seriality

$$\forall t, \exists t', t < t'$$

and of density

$$\forall t, \forall t'', t < t'' \Longrightarrow \exists t', t < t' < t''.$$

In addition, $\mathbf{R}$ satisfies the property of completeness (in the sense of the upper bound): any subset bounded from above has a least upper bound.

In our TL we have the atoms:

- the individual constants $0, 1 \in S$;

- the pseudo-Boolean variables $u_1, ..., u_m, ..., x_1, ..., x_n, ... \in S$,

called as before **antilogy**, **tautology** and **propositional variables** respectively, that are intuitively considered as statements whose truth varies against time. The formulae of TL are the functions $\Phi : S^{(m)} \times S^{(n)} \to S$, $S^{(m)} \times S^{(n)} \ni (u_1, ..., u_m, x_1, ..., x_n) \longmapsto \Phi(u_1, ..., u_m, x_1, ..., x_n) \in S$. The first m coordinates of the argument $u = (u_1, ..., u_m)$, grouped under the name of input, have the role of stating conditions while the last n coordinates of the argument $x = (x_1, ..., x_n)$, grouped under the name of state, have the role of obeying the conditions that were stated by u. The manner in which this obedience takes place is described against time by the function $\Phi(u,x)(t)$.

The semantics of TL answers the question: in the interpretation $\widetilde{I}$ that assigns the value $\widetilde{\Lambda} = (\widetilde{u_1},...,\widetilde{u_m},\widetilde{x_1},...,\widetilde{x_n})$ to the $m+n-$tuple of variables $\Lambda = (u_1,...,u_m,x_1,...,x_n)$, do we have $\forall t \in \mathbf{R}, \Phi(\widetilde{\Lambda})(t) = 1$? If the answer is positive, we use to say that Φ is satisfied in $\widetilde{I}$ and we denote $\widetilde{\Lambda} \models \Phi$. If there is an interpretation $\widetilde{I}$ in which Φ is satisfied, Φ is called the **asynchronous system given under the implicit form**. Like in the case of CLP, the constant function $\Phi = 1$ is identified to the tautology and to the autonomous system $S^{(n)}$ and we denote $\models \Phi$ (Φ is satisfied in any interpretation).

The connectors of TL are the laws induced by those of $\mathbf{B}$ in S and those defined with $\bigcap$ and $\bigcup$. Let us mention here the traditional connectors of temporal logic consisting in the deterministic systems (given under the closed form):

$$G : S \to S, \forall u \in S, G(u)(t) = \bigcap_{\xi \in [t,\infty)} u(\xi)$$

read: 'it is and will always be the case that u';

$$F : S \to S, \forall u \in S, F(u)(t) = \bigcup_{\xi \in [t,\infty)} u(\xi)$$

read: 'it is or will be the case that u';

$$H : S \to S, \forall u \in S, H(u)(t) = \bigcap_{\xi \in (-\infty,t]} u(\xi)$$

read: 'it is and has always been the case that u';

$$P : S \to S, \forall u \in S, P(u)(t) = \bigcup_{\xi \in (-\infty,t]} u(\xi)$$

read: 'it is or has been the case that u';

$$\mathbf{S} : S^{(2)} \to S, \forall u \in S^{(2)}, \mathbf{S}(u)(t) = \bigcup_{t' \in (-\infty,t]} u_1(t') \cdot \bigcap_{\xi \in [t',t]} u_2(\xi)$$

read: 'u_2 has been true since a time when u_1 was true';

$$\mathbf{U} : S^{(2)} \to S, \forall u \in S^{(2)}, \mathbf{U}(u)(t) = \bigcup_{t' \in [t,\infty)} u_1(t') \cdot \bigcap_{\xi \in [t,t']} u_2(\xi)$$

read: 'u_2 will be true until a time when u_1 will be true'.

In the following 'tempting' variants of G, F, H, P:

$$G_1(u)(t) = \bigcap_{\xi \in (t,\infty)} u(\xi),$$

$$F_1(u)(t) = \bigcup_{\xi \in (t,\infty)} u(\xi),$$

$$H_1(u)(t) = \bigcap_{\xi \in (-\infty,t)} u(\xi),$$

$$P_1(u)(t) = \bigcup_{\xi \in (-\infty,t)} u(\xi),$$

$G_1(u), F_1(u)$ are signals, while $H_1(u), P_1(u)$ are just differentiable. From the variants of $\mathbf{S}, \mathbf{U}$

$$\mathbf{S}_1(u)(t) = \bigcup_{t' \in (-\infty, t)} u_1(t') \cdot \bigcap_{\xi \in [t', t]} u_2(\xi),$$
$$\mathbf{U}_1(u)(t) = \bigcup_{t' \in (t, \infty)} u_1(t') \cdot \bigcap_{\xi \in [t, t']} u_2(\xi),$$

the first is differentiable and the last is a signal.

We have added the left limit and the right limit connectors:

$$L : S \to Diff, \forall u \in S, L(u)(t) = u(t-0)$$

read: 'in the recent past, it was always the case that u';

$$R : S \to Diff, \forall u \in S, R(u)(t) = u(t+0)$$

read: 'in the next future, it will always be the case that u', and the semi-derivatives and the derivatives defined with them.

Other connectors that we have used are:

$$f : S \to S, \forall u \in S, f(u)(t) = \bigcap_{\xi \in [t-d_r, t-d_r+m_r]} u(\xi),$$
$$f : S \to S, \forall u \in S, f(u)(t) = \bigcup_{\xi \in [t-d_f, t-d_f+m_f]} u(\xi),$$
$$f : S \to Diff, \forall u \in S, f(u)(t) = \bigcup_{\xi \in (t-d, t)} Du(\xi)$$

etc.

A consistency exists between the propositional variables and the conditions on frames expressed by the fact that: for any $u \in S$ there is an unbounded sequence $t_0 < t_1 < t_2 < ...$ such that u is constant in the intervals $(-\infty, t_0)$, $[t_0, t_1)$, $[t_1, t_2)$, ... This was the definition of the signals, whose purpose is, for example, to indicate the existence of an initial time instant and of L, R. We conclude that even if $\mathbf{R}$ is unbounded from bellow, this property is not necessary in TL. On the other hand, even if $\mathbf{R}$ is dense and complete, for each u time is discrete. The strong possibilities offered by $\mathbf{R}$ are used when choosing (t_k).

An open question is: what is gained and what is lost when considering the two possibilities, real time versus discrete time (the time set equal to $\mathbf{N}$). This problem is not trivial at all. Even if things do not look attractive, we have asked sometimes if instead of $\mathbf{R}$, why can't $\mathbf{Q}$ be used as time set and which is exactly the role of the completeness of $\mathbf{R}$ (making the difference between the two sets) in this work.

APPENDIX B

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APPENDIX C

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