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BINARY SIGNALS: NECESSARY AND SUFFICIENT CONDITIONS OF PERIODICITY OF A POINT

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Abstract. The (binary) signals are the models of the electrical signals from digital electrical engineering. They are presented in two versions, discrete time and real time. The periodicity of the points of the signals is introduced by analogy with the dynamical systems theory. The paper gives necessary and sufficient conditions of periodicity of the points of the signals.

1. Introduction

The asynchronous circuits from the digital electrical engineering are modeled by asynchronous systems. An important special case of such a system consists in a Boolean function $\Phi : \{0,1\}^n \longrightarrow \{0,1\}^n$ that iterates in discrete or real time, and the iterations do not happen on all the coordinates $\Phi_1, ..., \Phi_n$ as in the usual dynamical systems theory (synchronicity), but on some coordinates only (asynchronicity). The functions that the asynchronous systems work with (as inputs, states or outputs) are called (binary) signals and they represent the model of the (two level) electrical signals. In order to study the periodicity of the asynchronous systems, the study of the periodicity of the (values of the) signals is to be made first and it proves to be very interesting by itself.

Roughly speaking, dynamical systems theory refers to periodic points and periodic orbits (=signals in this case) and the paper deals with the adaptation of the first concept, without getting to systems theory. Its aim is to give necessary and sufficient conditions of periodicity.

2. Preliminaries

Definition 2.1. The set $\mathbf{B} = \{0,1\}$ is a field relative to $'\oplus'$, $'\cdot'$, the modulo 2 sum and the product. A linear space structure is induced on $\mathbf{B}^n, n \geq 1$.

Definition 2.2. The topological structure of $\mathbf{B}$ and $\mathbf{B}^n$ is given by the discrete topology.

Notation 2.1. We denote with $\chi_A : \mathbf{R} \to \mathbf{B}$ the characteristic function of the set $A \subset \mathbf{R}$: $\forall t \in \mathbf{R}$,

$$\chi_A(t) = \begin{cases} 1, if\ t \in A, \\ 0, otherwise \end{cases}.$$

Notation 2.2. We use the notation $\mathbf{N}\_ = \{-1, 0, 1, ...\}$.

Definition 2.3. The **discrete time signals** are by definition the functions $\widehat{x} : \mathbf{N}\_ \to \mathbf{B}^n$. Their set is denoted by $\widehat{S}^{(n)}$.

The **continuous time signals** are the functions $x : \mathbf{R} \to \mathbf{B}^n$ of the form $\forall t \in \mathbf{R}$,

$$x(t) = \mu \cdot \chi_{(-\infty,t_0)}(t) \oplus x(t_0) \cdot \chi_{[t_0,t_1)}(t) \oplus ... \oplus x(t_k) \cdot \chi_{[t_k,t_{k+1})}(t) \oplus ... \tag{2.1}$$

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where $\mu \in \mathbf{B}^n$ and $t_k \in \mathbf{R}, k \in \mathbf{N}$ is strictly increasing and unbounded from above. Their set is denoted by $S^{(n)}$. μ is usually denoted by $x(-\infty+0)$ and is called the **initial value** of x.

Remark 2.1. The signal $x \in S^{(n)}$ has the properties of existence $\forall t \in \mathbf{R}$ of the left limit $x(t-0) \in \mathbf{B}^n : \exists \varepsilon > 0, \forall \xi \in (t-\varepsilon, t), x(\xi) = x(t-0)$, of the right limit $x(t+0) \in \mathbf{B}^n : \exists \varepsilon > 0, \forall \xi \in (t, t+\varepsilon), x(\xi) = x(t+0)$ and of right continuity: $x(t) = x(t+0)$. Proving these properties is easy and it was made in [10] for example. We shall use the property of right continuity of x under the form $\forall t \in \mathbf{R}, \exists \varepsilon > 0, \forall \xi \in [t, t+\varepsilon), x(\xi) = x(t)$.

Definition 2.4. The sets

$$\widehat{Or}(\widehat{x}) = \{\widehat{x}(k) | k \in \mathbf{N}_{\text{-}}\},$$
$$Or(x) = \{x(t) | t \in \mathbf{R}\}$$

are called the **orbits** of $\widehat{x}, x$.

Notation 2.3. For $\widehat{x} \in \widehat{S}^{(n)}, x \in S^{(n)}$ and $\mu \in \widehat{Or}(\widehat{x}), \nu \in Or(x)$, we use the notations

$$\widehat{\mathbf{T}}_\mu^{\widehat{x}} = \{k | k \in \mathbf{N}_{\text{-}}, \widehat{x}(k) = \mu\},$$
$$\mathbf{T}_\nu^x = \{t | t \in \mathbf{R}, x(t) = \nu\}.$$

Lemma 2.4. *Let $\mu \in Or(x)$ and $t' \in \mathbf{R}$. If $(-\infty, t'] \subset \mathbf{T}_{x(-\infty+0)}^x$, then $\mathbf{T}_\mu^x \cap [t', \infty) \neq \emptyset$.*

Proof. If $\mu = x(-\infty+0)$, when $t' \in \mathbf{T}_\mu^x$, we have $\mathbf{T}_\mu^x \cap [t', \infty) \neq \emptyset$ true. And if $\mu \neq x(-\infty+0)$, when $\mathbf{T}_\mu^x \cap (-\infty, t'] = \emptyset, \mathbf{T}_\mu^x \neq \emptyset$, we get $\mathbf{T}_\mu^x \subset (t', \infty)$, thus $\mathbf{T}_\mu^x \cap [t', \infty) \neq \emptyset$. □

3. Periodic points

Definition 3.1. We consider the signals $\widehat{x} \in \widehat{S}^{(n)}, x \in S^{(n)}$.

Let $\mu \in \widehat{Or}(\widehat{x})$ and $p \geq 1$. If

$$\forall k \in \widehat{\mathbf{T}}_\mu^{\widehat{x}}, \{k + zp | z \in \mathbf{Z}\} \cap \mathbf{N}_{\text{-}} \subset \widehat{\mathbf{T}}_\mu^{\widehat{x}}, \tag{3.1}$$

we say that μ is **periodic,** with the **period** p. The least p that fulfills (3.1) is called the **prime period** of μ.

Let $\mu \in Or(x)$ and $T > 0$, $t' \in \mathbf{R}$ such that

$$(-\infty, t'] \subset \mathbf{T}_{x(-\infty+0)}^x, \tag{3.2}$$
$$\forall t \in \mathbf{T}_\mu^x \cap [t', \infty), \{t + zT | z \in \mathbf{Z}\} \cap [t', \infty) \subset \mathbf{T}_\mu^x. \tag{3.3}$$

Then μ is called **periodic,** with the **period** T. The least T with the property that t' exists such that (3.2), (3.3) are fulfilled is called the **prime period** of μ.

Lemma 3.1. *a) Let $\widehat{x} \in \widehat{S}^{(n)}$, $\mu \in \widehat{Or}(\widehat{x})$, $p \geq 1$ and $k' \in \mathbf{N}_{\text{-}}$. If*

$$\widehat{\mathbf{T}}_\mu^{\widehat{x}} \cap \{k', k'+1, k'+2, ...\} \neq \emptyset, \tag{3.4}$$

$$\begin{gathered}\forall k \in \widehat{\mathbf{T}}_\mu^{\widehat{x}} \cap \{k', k'+1, k'+2, ...\}, \\ \{k + zp | z \in \mathbf{Z}\} \cap \{k', k'+1, k'+2, ...\} \subset \widehat{\mathbf{T}}_\mu^{\widehat{x}}\end{gathered} \tag{3.5}$$

then for any $k \geq k'$ we have $\widehat{\mathbf{T}}_\mu^{\widehat{x}} \cap \{k, k+1, ..., k+p-1\} \neq \emptyset$.

b) $x \in S^{(n)}$, $\mu \in Or(x)$, $T > 0$, $t' \in \mathbf{R}$ are given. If

$$\mathbf{T}_\mu^x \cap [t', \infty) \neq \emptyset, \tag{3.6}$$

$$\forall t \in \mathbf{T}_\mu^x \cap [t', \infty), \{t + zT | z \in \mathbf{Z}\} \cap [t', \infty) \subset \mathbf{T}_\mu^x, \tag{3.7}$$

then for any $t \geq t'$, we have $\mathbf{T}_\mu^x \cap [t, t+T) \neq \emptyset$.

Proof. a) (3.4) allows us to define $k'' = \min \widehat{\mathbf{T}}_{\mu}^{\widehat{x}} \cap \{k', k'+1, k'+2, ...\}$ and we prove that $k'' \in \widehat{\mathbf{T}}_{\mu}^{\widehat{x}} \cap \{k', k'+1, ..., k'+p-1\}$. If, against all reason, this would not be true, then we would have $k'' \geq k'+p$ and

$$k'' - p \in \{k'' + zp | z \in \mathbf{Z}\} \cap \{k', k'+1, k'+2, ...\} \overset{(3.5)}{\subset} \widehat{\mathbf{T}}_{\mu}^{\widehat{x}},$$

representing a contradiction with the definition of k''.

From (3.5) we infer that $\{k'', k''+p, k''+2p, ...\} \subset \widehat{\mathbf{T}}_{\mu}^{\widehat{x}} \cap \{k', k'+1, k'+2, ...\}$, meaning that $\forall k \geq k'$, $\widehat{\mathbf{T}}_{\mu}^{\widehat{x}} \cap \{k, k+1, ..., k+p-1\} \neq \emptyset$.

b) The request (3.6) allows defining $t'' = \min \mathbf{T}_{\mu}^{x} \cap [t', \infty)$. We show that $t'' \in \mathbf{T}_{\mu}^{x} \cap [t', t'+T)$. If, against all reason, this would not be true, then we would have $t'' \geq t'+T$. This means that $t'' - T \geq t'$, thus

$$t'' - T \in \{t'' + zT | z \in \mathbf{Z}\} \cap [t', \infty) \overset{(3.7)}{\subset} \mathbf{T}_{\mu}^{x},$$

contradiction with the definition of t''.

By using (3.7) we get $\{t'', t''+T, t''+2T, ...\} \subset \mathbf{T}_{\mu}^{x} \cap [t', \infty)$. The statement of the Lemma holds. □

4. Necessity conditions of periodicity

Theorem 4.1. *Let $\widehat{x} \in \widehat{S}^{(n)}$ non constant. For $\mu \in \widehat{Or}(\widehat{x}), p \geq 1$ we suppose that*

$$\forall k \in \widehat{\mathbf{T}}_{\mu}^{\widehat{x}}, \{k + zp | z \in \mathbf{Z}\} \cap \mathbf{N}_{\_} \subset \widehat{\mathbf{T}}_{\mu}^{\widehat{x}} \tag{4.1}$$

takes place. Then $n_1, n_2, ..., n_{k_1} \in \{-1, 0, ..., p-2\}$, $k_1 \geq 1$, exist such that

$$\widehat{\mathbf{T}}_{\mu}^{\widehat{x}} = \bigcup_{k \in \mathbf{N}} \{n_1 + kp, n_2 + kp, ..., n_{k_1} + kp\} \tag{4.2}$$

holds.

Proof. We have $\widehat{\mathbf{T}}_{\mu}^{\widehat{x}} \neq \emptyset$. From Lemma 3.1 a), written for $k = k' = -1$, we infer $\widehat{\mathbf{T}}_{\mu}^{\widehat{x}} \cap \{-1, 0, ..., p-2\} \neq \emptyset$, thus the existence of $n_1, n_2, ..., n_{k_1}$, $k_1 \geq 1$ with

$$\{n_1, n_2, ..., n_{k_1}\} = \widehat{\mathbf{T}}_{\mu}^{\widehat{x}} \cap \{-1, 0, ..., p-2\} \tag{4.3}$$

true results.

We prove $\widehat{\mathbf{T}}_{\mu}^{\widehat{x}} \subset \bigcup\limits_{k \in \mathbf{N}} \{n_1 + kp, n_2 + kp, ..., n_{k_1} + kp\}$ and let $k' \in \widehat{\mathbf{T}}_{\mu}^{\widehat{x}}$ arbitrary. A finite sequence $k', k'-p, ..., k'-\overline{k}p \in \widehat{\mathbf{T}}_{\mu}^{\widehat{x}}$ exists from (4.1), $\overline{k} \geq 0$, with the property that $k' - \overline{k}p \in \{-1, 0, ..., p-2\}$, thus, from (4.3), $j \in \{1, ..., k_1\}$ exists with $k' - \overline{k}p = n_j$. We have obtained that $k' = n_j + \overline{k}p \in \bigcup\limits_{k \in \mathbf{N}} \{n_1 + kp, n_2 + kp, ..., n_{k_1} + kp\}$.

We prove $\bigcup\limits_{k \in \mathbf{N}} \{n_1 + kp, n_2 + kp, ..., n_{k_1} + kp\} \subset \widehat{\mathbf{T}}_{\mu}^{\widehat{x}}$ and let for this $k' \in \bigcup\limits_{k \in \mathbf{N}} \{n_1 + kp, n_2 + kp, ..., n_{k_1} + kp\}$ arbitrary. Some $k \in \mathbf{N}$ and some $j \in \{1, ..., k_1\}$ exist such that $k' = n_j + kp$. We conclude, as $n_j \overset{(4.3)}{\in} \widehat{\mathbf{T}}_{\mu}^{\widehat{x}}$, that

$$k' \in \{n_j + zp | z \in \mathbf{Z}\} \cap \mathbf{N}_{\_} \overset{(4.1)}{\subset} \widehat{\mathbf{T}}_{\mu}^{\widehat{x}}.$$

(4.2) holds. □

Remark 4.1. If $\widehat{x}$ is constant, then the previous Theorem takes the form $\widehat{Or}(\widehat{x}) = \{\mu\}, p = 1, k_1 = 1, n_1 = -1$ and (4.2) becomes $\widehat{\mathbf{T}}_{\mu}^{\widehat{x}} = \mathbf{N}_{\_}$.

Theorem 4.2. *The non constant signal $x \in S^{(n)}$ is considered and let the point $\mu = x(-\infty + 0)$ be given, together with $T > 0, t' \in \mathbf{R}$ such that*

$$(-\infty, t'] \subset \mathbf{T}^x_\mu, \tag{4.4}$$

$$\forall t \in \mathbf{T}^x_\mu \cap [t', \infty), \{t + zT | z \in \mathbf{Z}\} \cap [t', \infty) \subset \mathbf{T}^x_\mu \tag{4.5}$$

hold. Then $t_0, a_1, b_1, a_2, b_2, ..., a_{k_1}, b_{k_1} \in \mathbf{R}$, $k_1 \geq 1$ exist such that

$$\forall t < t_0, x(t) = \mu, \tag{4.6}$$

$$x(t_0) \neq \mu, \tag{4.7}$$

$$t_0 < a_1 < b_1 < a_2 < b_2 < ... < a_{k_1} < b_{k_1} = t_0 + T, \tag{4.8}$$

$$[a_1, b_1) \cup [a_2, b_2) \cup ... \cup [a_{k_1}, b_{k_1}) = \mathbf{T}^x_\mu \cap [t_0, t_0 + T), \tag{4.9}$$

$$\mathbf{T}^x_\mu = \begin{array}{c} (-\infty, t_0) \cup \bigcup\limits_{k \in \mathbf{N}} ([a_1 + kT, b_1 + kT) \cup [a_2 + kT, b_2 + kT) \cup ... \\ ... \cup [a_{k_1} + kT, b_{k_1} + kT)) \end{array} \tag{4.10}$$

hold.

Proof. A t_0 like at (4.6), (4.7) exists because x is not constant and by comparing (4.4) with (4.6), (4.7) we infer $t' < t_0$. We have from Lemma 2.4 that $\mathbf{T}^x_\mu \cap [t', \infty) \neq \emptyset$, thus the fact that $\mathbf{T}^x_\mu \cap [t_0, t_0 + T) \neq \emptyset$ follows from Lemma 3.1 b).

We have on one hand the existence of $\varepsilon > 0$ with

$$\forall t \in [t_0, t_0 + \varepsilon), x(t) = x(t_0) \overset{(4.7)}{\neq} \mu, \tag{4.11}$$

showing that $a_1 = \min \mathbf{T}^x_\mu \cap [t_0, t_0 + T) > t_0$. On the other hand we must show the existence of b_{k_1} like at (4.8), (4.9). Indeed, we suppose against all reason that such a b_{k_1} does not exist and consequently that $a_{k_1} < b_{k_1} < t_0 + T$, $[a_{k_1}, b_{k_1}) \subset \mathbf{T}^x_\mu$ and $[b_{k_1}, t_0 + T) \cap \mathbf{T}^x_\mu = \emptyset$. Let then $t \in [\max\{b_{k_1}, t' + T\}, t_0 + T)$ arbitrary. We get

$$b_{k_1} \leq \max\{b_{k_1}, t' + T\} \leq t < t_0 + T$$

i.e. $t \notin \mathbf{T}^x_\mu$. We have also $t > t - T \geq t'$ and $t - T \in [t', t_0) \subset \mathbf{T}^x_\mu$, thus

$$t \in \{t - T + zT | z \in \mathbf{Z}\} \cap [t', \infty) \overset{(4.5)}{\subset} \mathbf{T}^x_\mu,$$

contradiction. The existence of $t_0, a_1, b_1, a_2, b_2, ..., a_{k_1}, b_{k_1}$ like at (4.6),..., (4.9) is proved.

We prove $\mathbf{T}^x_\mu \subset (-\infty, t_0) \cup \bigcup\limits_{k \in \mathbf{N}} ([a_1 + kT, b_1 + kT) \cup [a_2 + kT, b_2 + kT) \cup ... \cup [a_{k_1} + kT, b_{k_1} + kT))$ and let $t \in \mathbf{T}^x_\mu$ arbitrary. If $t < t_0$ the inclusion is obvious (from (4.6)), so we can suppose now that $t \geq t_0$. We get from (4.5) the existence of a finite sequence $t, t - T, ..., t - \overline{k}T \in \mathbf{T}^x_\mu, \overline{k} \geq 0$ with the property that $t - \overline{k}T \in [t_0, t_0 + T)$. We infer from (4.9) the existence of $j \in \{1, ..., k_1\}$ with $t - \overline{k}T \in [a_j, b_j)$ and we conclude that $t \in [a_j + \overline{k}T, b_j + \overline{k}T) \in (-\infty, t_0) \cup \bigcup\limits_{k \in \mathbf{N}} ([a_1 + kT, b_1 + kT) \cup [a_2 + kT, b_2 + kT) \cup ... \cup [a_{k_1} + kT, b_{k_1} + kT))$.

We prove $(-\infty, t_0) \cup \bigcup\limits_{k \in \mathbf{N}} ([a_1 + kT, b_1 + kT) \cup [a_2 + kT, b_2 + kT) \cup ... \cup [a_{k_1} + kT, b_{k_1} + kT)) \subset \mathbf{T}^x_\mu$. The fact that $(-\infty, t_0) \subset \mathbf{T}^x_\mu$ coincides with (4.6) and we take an arbitrary $t \in \bigcup\limits_{k \in \mathbf{N}} ([a_1 + kT, b_1 + kT) \cup [a_2 + kT, b_2 + kT) \cup ... \cup [a_{k_1} + kT, b_{k_1} + kT))$. Some $k \in \mathbf{N}$ and some $j \in \{1, ..., k_1\}$ exist with $t \in [a_j + kT, b_j + kT)$, thus $t - kT \in [a_j, b_j) \subset \mathbf{T}^x_\mu \cap [t_0, t_0 + T) \subset \mathbf{T}^x_\mu \cap [t', \infty)$. In particular we can see that $t \geq t - kT \geq t'$. We have

$$t \in \{t - kT + zT | z \in \mathbf{Z}\} \cap [t', \infty) \overset{(4.5)}{\subset} \mathbf{T}^x_\mu.$$

(4.10) is proved. $\square$

Theorem 4.3. *The signal $x \in S^{(n)}$ is not constant and let the point $\mu \in Or(x)$, $\mu \neq x(-\infty+0)$, as well as $T > 0, t' \in \mathbf{R}$ with*

$$(-\infty, t'] \subset \mathbf{T}^x_{x(-\infty+0)}, \tag{4.12}$$

$$\forall t \in \mathbf{T}^x_\mu \cap [t', \infty), \{t + zT | z \in \mathbf{Z}\} \cap [t', \infty) \subset \mathbf{T}^x_\mu. \tag{4.13}$$

Then $t_0, a_1, b_1, a_2, b_2, ..., a_{k_1}, b_{k_1} \in \mathbf{R}$, $k_1 \geq 1$ exist such that

$$\forall t < t_0, x(t) = x(-\infty+0), \tag{4.14}$$

$$x(t_0) \neq x(-\infty+0), \tag{4.15}$$

$$t_0 \leq a_1 < b_1 < a_2 < b_2 < ... < a_{k_1} < b_{k_1} < t_0 + T, \tag{4.16}$$

$$[a_1, b_1) \cup [a_2, b_2) \cup ... \cup [a_{k_1}, b_{k_1}) = \mathbf{T}^x_\mu \cap [t_0, t_0 + T), \tag{4.17}$$

$$\mathbf{T}^x_\mu = \bigcup_{k \in \mathbf{N}} \begin{array}{l}([a_1 + kT, b_1 + kT) \cup [a_2 + kT, b_2 + kT) \cup ... \\ \quad ... \cup [a_{k_1} + kT, b_{k_1} + kT))\end{array} \tag{4.18}$$

are fulfilled.

Proof. As x is not constant we get the existence of t_0 like in (4.14), (4.15) and if we take in consideration (4.12) also, we get $t' < t_0$.

We have from Lemma 2.4 that $\mathbf{T}^x_\mu \cap [t', \infty) \neq \emptyset$, thus the fact that $\mathbf{T}^x_\mu \cap [t_0, t_0 + T) \neq \emptyset$ results from Lemma 3.1 b). We show that $b_{k_1} < t_0 + T$ and for this we suppose against all reason that $b_{k_1} = t_0 + T$. Let $t \in [\max\{a_{k_1}, t' + T\}, t_0 + T)$ arbitrary, fixed. We have $t > t - T \geq t'$ and $t \in [a_{k_1}, t_0 + T) \subset \mathbf{T}^x_\mu$, thus we can apply (4.13):

$$t - T \in \{t + zT | z \in \mathbf{Z}\} \cap [t', \infty) \subset \mathbf{T}^x_\mu.$$

Since $t - T \in [t', t_0)$, we have reached the contradiction

$$\mu = x(t) = x(t - T) = x(-\infty + 0).$$

The fact that $a_1, b_1, a_2, b_2, ..., a_{k_1}, b_{k_1}$ exist making (4.16), (4.17) true is proved.

The proof of the equation (4.18) is made like in the proof of Theorem 4.2. $\square$

Example 4.4. We take $x \in S^{(1)}$,

$$x(t) = \chi_{(-\infty,0)}(t) \oplus \chi_{[1,2)}(t) \oplus \chi_{[3,5)}(t) \oplus \chi_{[6,7)}(t) \oplus \chi_{[8,10)}(t) \oplus \chi_{[11,12)}(t) \oplus ...$$

In this example, see Theorem 4.2, $\mu = 1, t_0 = 0, k_1 = 2, T = 5$ and $t' \in [-2, 0)$.

5. Sufficiency conditions of periodicity

Theorem 5.1. *Let $\widehat{x} \in \widehat{S}^{(n)}$, $\mu \in \widehat{Or}(\widehat{x})$, $p \geq 1$ and $n_1, n_2, ..., n_{k_1} \in \{-1, 0, ..., p-2\}$, $k_1 \geq 1$, such that*

$$\widehat{\mathbf{T}}^{\widehat{x}}_\mu = \bigcup_{k \in \mathbf{N}} \{n_1 + kp, n_2 + kp, ..., n_{k_1} + kp\}. \tag{5.1}$$

We have

$$\forall k \in \widehat{\mathbf{T}}^{\widehat{x}}_\mu, \{k + zp | z \in \mathbf{Z}\} \cap \mathbf{N}_{\_} \subset \widehat{\mathbf{T}}^{\widehat{x}}_\mu. \tag{5.2}$$

Proof. We take an arbitrary $k' \in \widehat{\mathbf{T}}^{\widehat{x}}_\mu$, for which $j \in \{1, ..., k_1\}$ and $k \in \mathbf{N}$ exist with $k' = n_j + kp$. We have

$$\{n_j + kp + zp | z \in \mathbf{Z}\} \cap \mathbf{N}_{\_} = \{n_j, n_j + p, n_j + 2p, ...\} \overset{(5.1)}{\subset} \widehat{\mathbf{T}}^{\widehat{x}}_\mu.$$

$\square$

Theorem 5.2. *The signal $x \in S^{(n)}$ is given with $\mu = x(-\infty+0)$, $T > 0$ and the numbers $t_0, a_1, b_1, a_2, b_2, ..., a_{k_1}, b_{k_1} \in \mathbf{R}$, $k_1 \geq 1$ that fulfill*

$$t_0 < a_1 < b_1 < ... < a_{k_1} < b_{k_1} = t_0 + T, \tag{5.3}$$

$$\mathbf{T}^x_\mu = (-\infty, t_0) \cup \bigcup_{k \in \mathbf{N}} ([a_1 + kT, b_1 + kT) \cup ... \cup [a_{k_1} + kT, b_{k_1} + kT)). \tag{5.4}$$

For any $t' \in [a_{k_1} - T, t_0)$, the properties

$$(-\infty, t'] \subset \mathbf{T}^x_\mu, \tag{5.5}$$

$$\forall t \in \mathbf{T}^x_\mu \cap [t', \infty), \{t + zT | z \in \mathbf{Z}\} \cap [t', \infty) \subset \mathbf{T}^x_\mu \tag{5.6}$$

hold.

Proof. Let $t' \in [a_{k_1} - T, t_0)$ arbitrary. From

$$(-\infty, t'] \subset (-\infty, t_0) \subset \mathbf{T}^x_\mu$$

we infer (5.5).

We have

$$\mathbf{T}^x_\mu \cap [t', \infty) = [t', t_0) \cup [a_1, b_1) \cup ... \cup [a_{k_1}, b_{k_1}) \cup [a_1 + T, b_1 + T) \cup ...$$

and we take an arbitrary $t \in \mathbf{T}^x_\mu \cap [t', \infty)$. We have several possibilities.

a) Case $t \in [t', t_0)$, when

$$\{t + zT | z \in \mathbf{Z}\} \cap [t', \infty) = \{t, t + T, t + 2T, ...\} \subset$$
$$\subset [t', t_0) \cup [t' + T, b_{k_1}) \cup [t' + 2T, b_{k_1} + T) \cup ... \subset$$
$$\subset (-\infty, t_0) \cup [a_{k_1}, b_{k_1}) \cup [a_{k_1} + T, b_{k_1} + T) \cup ... \subset \mathbf{T}^x_\mu.$$

We have used the fact that

$$t - T < t_0 - T < a_{k_1} - T \leq t' \leq t < t_0,$$

thus the least term in $\{t + zT | z \in \mathbf{Z}\} \cap [t', \infty)$ is t indeed.

b) Case $t \in [a_j + kT, b_j + kT), k \geq 0, j \in \{1, 2, ..., k_1 - 1\}$,

$$\{t + zT | z \in \mathbf{Z}\} \cap [t', \infty) = \{t + (-k)T, t + (-k + 1)T, t + (-k + 2)T, ...\} \subset$$
$$\subset [a_j, b_j) \cup [a_j + T, b_j + T) \cup [a_j + 2T, b_j + 2T) \cup ... \subset \mathbf{T}^x_\mu$$

and we have used

$$t + (-k - 1)T < t' < t_0 < a_j \leq t + (-k)T < b_j < t' + T.$$

c) Case $t \in [a_{k_1} + kT, b_{k_1} + kT)$, $k \geq 0$ when there are two subcases,

c.1) Case $t \in [t' + (k + 1)T, b_{k_1} + kT)$,

$$\{t + zT | z \in \mathbf{Z}\} \cap [t', \infty) = \{t + (-k - 1)T, t + (-k)T, t + (-k + 1)T, ...\} \subset$$
$$\subset [t', t_0) \cup [t' + T, b_{k_1}) \cup [t' + 2T, b_{k_1} + T) \cup ... \subset$$
$$\subset (-\infty, t_0) \cup [a_{k_1}, b_{k_1}) \cup [a_{k_1} + T, b_{k_1} + T) \cup ... \subset \mathbf{T}^x_\mu$$

and we have used the fact that

$$t + (-k - 2)T < t_0 - T < a_{k_1} - T \leq t' \leq t + (-k - 1)T < t_0.$$

c.2) Case $t \in [a_{k_1} + kT, t' + (k + 1)T)$,

$$\{t + zT | z \in \mathbf{Z}\} \cap [t', \infty) = \{t + (-k)T, t + (-k + 1)T, t + (-k + 2)T, ...\} \subset$$
$$\subset [a_{k_1}, t' + T) \cup [a_{k_1} + T, t' + 2T) \cup [a_{k_1} + 2T, t' + 3T) \cup ... \subset$$
$$\subset [a_{k_1}, b_{k_1}) \cup [a_{k_1} + T, b_{k_1} + T) \cup [a_{k_1} + 2T, b_{k_1} + 2T) \cup ... \subset \mathbf{T}^x_\mu$$

and we have used

$$t+(-k-1)T<t'<t_0<a_{k_1}\leq t+(-k)T<t'+T.$$

(5.6) holds. □

Theorem 5.3. *Let* x, $\mu\in Or(x)$, $\mu\neq x(-\infty+0)$, $T>0$ *and the numbers* $t_0,a_1,b_1,a_2,b_2,\ldots,a_{k_1},b_{k_1}\in\mathbf{R}$, $k_1\geq 1$, *with the property that*

$$\forall t<t_0, x(t)=x(-\infty+0), \tag{5.7}$$

$$x(t_0)\neq x(-\infty+0), \tag{5.8}$$

$$b_{k_1}-T<t_0\leq a_1<b_1<\ldots<a_{k_1}<b_{k_1}, \tag{5.9}$$

$$\mathbf{T}^x_\mu=\bigcup_{k\in\mathbf{N}}([a_1+kT,b_1+kT)\cup\ldots\cup[a_{k_1}+kT,b_{k_1}+kT)). \tag{5.10}$$

For any $t'\in[b_{k_1}-T,t_0)$, *we have*

$$(-\infty,t']\subset\mathbf{T}^x_{x(-\infty+0)}, \tag{5.11}$$

$$\forall t\in\mathbf{T}^x_\mu\cap[t',\infty),\{t+zT|z\in\mathbf{Z}\}\cap[t',\infty)\subset\mathbf{T}^x_\mu. \tag{5.12}$$

Proof. Let $t'\in[b_{k_1}-T,t_0)$ be arbitrary, for which

$$(-\infty,t']\subset(-\infty,t_0)\subset\mathbf{T}^x_{x(-\infty+0)},$$

thus (5.11) is true.

We get $\mathbf{T}^x_\mu\cap[t',\infty)=\mathbf{T}^x_\mu$ and we take an arbitrary $t\in\mathbf{T}^x_\mu\cap[t',\infty)$. Then $k\geq 0$ and $j\in\{1,2,\ldots,k_1\}$ exist such that $t\in[a_j+kT,b_j+kT)$. We have:

$$\{t+zT|z\in\mathbf{Z}\}\cap[t',\infty)=\{t+(-k)T,t+(-k+1)T,t+(-k+2)T,\ldots\}\subset$$

$$\subset[a_j,b_j)\cup[a_j+T,b_j+T)\cup[a_j+2T,b_j+2T)\cup\ldots\subset\mathbf{T}^x_\mu,$$

where

$$t+(-k-1)T<t'<t_0\leq a_j\leq t+(-k)T<b_j\leq t'+T.$$

(5.12) holds. □

6. A special case

Theorem 6.1. *Let* $\widehat{x}\in\widehat{S}^{(n)}$, $\mu\in\widehat{Or}(\widehat{x})$, $p\geq 1$ *and* $n_1\in\{-1,0,\ldots,p-2\}$ *such that*

$$\widehat{\mathbf{T}}^{\widehat{x}}_\mu=\{n_1,n_1+p,n_1+2p,\ldots\}. \tag{6.1}$$

Then

a) μ *is a periodic point of* $\widehat{x}$ *with the period* p *:*

$$\forall k\in\widehat{\mathbf{T}}^{\widehat{x}}_\mu,\{k+zp|z\in\mathbf{Z}\}\cap\mathbf{N}_{\text{-}}\subset\widehat{\mathbf{T}}^{\widehat{x}}_\mu; \tag{6.2}$$

b) p *is the prime period of* μ *: for any* $p'\geq 1$ *with*

$$\forall k\in\widehat{\mathbf{T}}^{\widehat{x}}_\mu,\{k+zp'|z\in\mathbf{Z}\}\cap\mathbf{N}_{\text{-}}\subset\widehat{\mathbf{T}}^{\widehat{x}}_\mu, \tag{6.3}$$

we infer $p'\geq p$.

Proof. a) This is a special case of Theorem 5.1, written for $k_1=1$.

b) We suppose against all reason that $p'\geq 1$ exists with $p'<p$ and (6.3) is true. As $n_1\in\widehat{\mathbf{T}}^{\widehat{x}}_\mu$, we obtain that $n_1+p'\in\widehat{\mathbf{T}}^{\widehat{x}}_\mu$, contradiction with (6.1). □

Theorem 6.2. *Let $x \in S^{(n)}$, $\mu = x(-\infty + 0)$, $T > 0$ and the points $t_0, a_1, b_1 \in \mathbf{R}$ having the property that*

$$t_0 < a_1 < b_1 = t_0 + T, \tag{6.4}$$

$$\mathbf{T}_\mu^x = (-\infty, t_0) \cup [a_1, b_1) \cup [a_1 + T, b_1 + T) \cup [a_1 + 2T, b_1 + 2T) \cup ... \tag{6.5}$$

hold.

a) For any $t' \in [a_1 - T, t_0)$, the properties

$$(-\infty, t'] \subset \mathbf{T}_\mu^x, \tag{6.6}$$

$$\forall t \in \mathbf{T}_\mu^x \cap [t', \infty), \{t + zT | z \in \mathbf{Z}\} \cap [t', \infty) \subset \mathbf{T}_\mu^x \tag{6.7}$$

are fulfilled.

b) For any $T' > 0, t'' \in [a_1 - T, t_0)$ such that

$$(-\infty, t''] \subset \mathbf{T}_\mu^x, \tag{6.8}$$

$$\forall t \in \mathbf{T}_\mu^x \cap [t'', \infty), \{t + zT' | z \in \mathbf{Z}\} \cap [t'', \infty) \subset \mathbf{T}_\mu^x, \tag{6.9}$$

we have $T' \geq T$.

Proof. a) This is a special case of Theorem 5.2, written for $k_1 = 1$.

b) We suppose against all reason that $T' < T$. Let us note in the beginning that

$$\max\{a_1, b_1 - T'\} < \min\{b_1, a_1 + T - T'\}$$

is true, since all of $a_1 < b_1, a_1 < a_1 + T - T', b_1 - T' < b_1, b_1 - T' < a_1 + T - T'$ hold. We infer that any $t \in [\max\{a_1, b_1 - T'\}, \min\{b_1, a_1 + T - T'\})$ fulfills $t \in [a_1, b_1) \subset \mathbf{T}_\mu^x \cap [t'', \infty)$ and

$$t + T' \in \{t + zT' | z \in \mathbf{Z}\} \cap [t'', \infty) \overset{(6.9)}{\subset} \mathbf{T}_\mu^x,$$

and on the other hand we have

$$b_1 \leq \max\{a_1 + T', b_1\} \leq t + T' < \min\{b_1 + T', a_1 + T\} \leq a_1 + T,$$

meaning that $t + T' \notin \mathbf{T}_\mu^x$, contradiction. We conclude that $T' \geq T$. □

Theorem 6.3. *Let $x \in S^{(n)}$, $\mu \in Or(x)$, $\mu \neq x(-\infty + 0)$, $T > 0$ and the points $t_0, a_1, b_1 \in \mathbf{R}$ with the property that*

$$\forall t < t_0, x(t) = x(-\infty + 0), \tag{6.10}$$

$$x(t_0) \neq x(-\infty + 0), \tag{6.11}$$

$$b_1 - T < t_0 \leq a_1 < b_1, \tag{6.12}$$

$$\mathbf{T}_\mu^x = [a_1, b_1) \cup [a_1 + T, b_1 + T) \cup [a_1 + 2T, b_1 + 2T) \cup ... \tag{6.13}$$

hold.

a) For any $t' \in [b_1 - T, t_0)$, the following properties

$$(-\infty, t'] \subset \mathbf{T}_{x(-\infty+0)}^x, \tag{6.14}$$

$$\forall t \in \mathbf{T}_\mu^x \cap [t', \infty), \{t + zT | z \in \mathbf{Z}\} \cap [t', \infty) \subset \mathbf{T}_\mu^x \tag{6.15}$$

are fulfilled.

b) For any $T' > 0, t'' \in [b_1 - T, t_0)$ such that

$$(-\infty, t''] \subset \mathbf{T}_{x(-\infty+0)}^x, \tag{6.16}$$

$$\forall t \in \mathbf{T}_\mu^x \cap [t'', \infty), \{t + zT' | z \in \mathbf{Z}\} \cap [t'', \infty) \subset \mathbf{T}_\mu^x, \tag{6.17}$$

we have $T' \geq T$.

Proof. a) This is a special case of Theorem 5.3, written for $k_1 = 1$.

b) We suppose against all reason now that $T' < T$. Let us notice the truth of

$$\max\{a_1, b_1 - T'\} < \min\{b_1, a_1 + T - T'\}.$$

We infer that $t \in [\max\{a_1, b_1 - T'\}, \min\{b_1, a_1 + T - T'\})$ satisfies $t \in [a_1, b_1) \subset \mathbf{T}^x_\mu \cap [t'', \infty)$ and

$$t + T' \in \{t + zT' | z \in \mathbf{Z}\} \cap [t'', \infty) \overset{(6.17)}{\subset} \mathbf{T}^x_\mu,$$

thus $t + T' \in \mathbf{T}^x_\mu$; on the other hand

$$b_1 \leq \max\{a_1 + T', b_1\} \leq t + T' < \min\{b_1 + T', a_1 + T\} \leq a_1 + T,$$

wherefrom $t + T' \notin \mathbf{T}^x_\mu$. We have obtained a contradiction proving that $T' \geq T$. □

Remark 6.1. Theorems 6.2, 6.3 represent the same phenomenon and their proof is formally the same: when $\mathbf{T}^x_\mu$ has one of the forms (6.5), (6.13), the prime period of μ is T. The difference between the Theorems is given by the fact that $\mu = x(-\infty + 0)$ in the first case and $\mu \neq x(-\infty + 0)$ in the second case.

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