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INTRODUCTORY TOPICS IN BINARY SET FUNCTIONS

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Abstract *Let* $X \neq \emptyset$ *an arbitrary set and* $\boldsymbol{U} \subset 2^X$ *a non-empty set of subsets. The function* $\mu : \boldsymbol{U} \to \{0,1\}$ *is called binary set function. If* μ *is countably additive, then it is called a measure. The paper gives some definitions and properties of these functions, its purpose being that of suggesting the reconstruction of the measure theory within this frame, by analogy with* [1], [2].

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1. Set Rings and Function Rings

1.1 We note with $\boldsymbol{B}_2$ the set $\{0,1\}$, called the *binary Boole* (or *Boolean*) algebra, together with the discrete topology, the order $0 \leq 1$ and the laws: the logical complement '$^-$', the reunion '$\cup$', the product '$\cdot$', the modulo 2 sum '$\oplus$', the coincidence '$\otimes$':

a)

$^-$	0	1
	1	0

b)

$\cup$	0	1
0	0	1
1	1	1

c)

$\cdot$	0	1
0	0	0
1	0	1

d)

$\oplus$	0	1
0	0	1
1	1	0

e)

$\otimes$	0	1
0	1	0
1	0	1

table (1)

1.2 Let $X \neq \emptyset$ be an arbitrary set, that we shall call the *total set*. In the set 2^X of the subsets of X, the order is given by the inclusion and the laws are: the complementary relative to X: '$\overline{}$', the reunion '$\vee$', the difference '$-$', the intersection '$\wedge$', the symmetrical difference 'Δ' and the coincidence 'Θ' that is defined like this:

$$A \,\Theta\, B = \overline{A \,\Delta\, B} \tag{1}$$

1.3 **Theorem** Let $\boldsymbol{U} \subset 2^X$ a set of subsets of X. The next statements are equivalent:

a) $A, B \in \boldsymbol{U} \Rightarrow A \vee B,\ A - B \in \boldsymbol{U}$

b) $A, B \in \boldsymbol{U} \Rightarrow A \,\Delta\, B,\ A \wedge B \in \boldsymbol{U}$

and the next statements are equivalent too:

c) $A, B \in \boldsymbol{U} \Rightarrow A \wedge B,\ \overline{B - A} \in \boldsymbol{U}$

d) $A, B \in \boldsymbol{U} \Rightarrow A \,\Theta\, B,\ A \vee B \in \boldsymbol{U}$

1.4 **Remark** In the previous theorem, the conditions a), c); b), d) are dual.

1.5 a) The set $\boldsymbol{U}$ that fulfills one of 1.3 a), b) is called *set ring*, or *ring of subsets of* X (*on* X). N. Bourbaki calls such a set *clan*.

b) Similarly, if $\boldsymbol{U}$ fulfills one of 1.3 c), d), it is called *set ring*, or *ring of subsets of* X (*on* X), the dual structure of the structure from a).

1.6 **Remark** a) $(\boldsymbol{U}, \Delta, \wedge)$ is really a non-unitary, commutative ring. Its neuter element is $\emptyset$.

b) $(\boldsymbol{U}, \Theta, \vee)$ is itself a non-unitary, commutative ring. Its neuter element is X.

1.7 a) If X belongs to the ring $(\boldsymbol{U}, \Delta, \wedge)$, then $(\boldsymbol{U}, \Delta, \wedge)$ is called a *set algebra*.

b) If $\emptyset$ belongs to the ring $(\boldsymbol{U}, \Theta, \vee)$, then $(\boldsymbol{U}, \Theta, \vee)$ is called a *set algebra* too.

1.8 **Remark** a) The condition that $(\boldsymbol{U}, \Delta, \wedge)$ is a set algebra ($(\boldsymbol{U}, \Theta, \vee)$ is a set algebra) implies the one that $\boldsymbol{U}$ is a unitary set ring, because if $X \in \boldsymbol{U}$ (if $\emptyset \in \boldsymbol{U}$), then it is the unit of the ring.

b) Generally speaking, the unit, if it exists, is given by $\bigvee_{A \in \boldsymbol{U}} A$ (by $\bigwedge_{A \in \boldsymbol{U}} A$).

1.9 **Remark** The set algebras are not what is usually meant by the F-algebra structures, where F is a field.

1.10 Let $f : X \to \boldsymbol{B}_2$ a function. Its *support* is by definition the set:

$$supp\ f = \{x \mid x \in X, f(x) = 1\} \tag{1}$$

1.11 If

$$supp\ f = A \tag{1}$$

f will be noted sometimes with χ_A. This function is called the *characteristic function* of the set $A \subset X$.

1.12 Let us define for the set ring $(\boldsymbol{U}, \Delta, \wedge)$, respectively for the set ring $(\boldsymbol{U}, \Theta, \vee)$, the set

$$\boldsymbol{U}' = \{f \mid f : X \to \boldsymbol{B}_2, supp\ f \in \boldsymbol{U}\} \tag{1}$$

1.13 $(\boldsymbol{U}', \oplus, \cdot, \cdot)$ and $(\boldsymbol{U}', \otimes, \cup, \cup)$ are $\boldsymbol{B}_2$-algebras, where '$\cdot$' is the symbol of two laws: the product of the functions and the product of the functions with scalars (both induced from $\boldsymbol{B}_2$), while '$\cup$' is the dual of '$\cdot$'.

1.14 The associations

$$\boldsymbol{U} \ni A \leftrightarrow \chi_A \in \boldsymbol{U}'$$

are ring isomorphisms. They allow us many times to identify the set rings $\boldsymbol{U} \subset 2^X$ and the function rings $\boldsymbol{U}' \subset \boldsymbol{B}_2^X$.

2. Additive and Countably Additive Set Functions

2.1 **Theorem** Let $\boldsymbol{U} \subset 2^X$ a non-empty family of subsets of X and $\mu : \boldsymbol{U} \to \boldsymbol{B}_2$ a function.

a) If $(\boldsymbol{U}, \Delta, \wedge)$ is a set ring, then the next statements are equivalent:

a.1) $\forall A, B \in \boldsymbol{U}, A \wedge B = \emptyset \Rightarrow \mu(A \vee B) = \mu(A) \oplus \mu(B)$ (1)

a.2) $\forall A, B \in \boldsymbol{U}, \mu(A \Delta B) = \mu(A) \oplus \mu(B)$ (2)

b) If $(\boldsymbol{U}, \Theta, \vee)$ is a set ring, then the next statements are equivalent:

b.1) $\forall A, B \in \boldsymbol{U}, A \vee B = X \Rightarrow \mu(A \wedge B) = \mu(A) \otimes \mu(B)$ (3)

b.2) $\forall A, B \in \boldsymbol{U}, \mu(A \Theta B) = \mu(A) \otimes \mu(B)$ (4)

2.2 a) Let $(\boldsymbol{U}, \Delta, \wedge)$ be a set ring. A function $\mu : \boldsymbol{U} \to \boldsymbol{B}_2$ that fulfills one of the equivalent conditions 2.1 a.1), a.2) is called ***additive***, or ***finitely additive***.

b) In a dual manner, let $(\boldsymbol{U}, \Theta, \vee)$ be a set ring. A function $\mu : \boldsymbol{U} \to \boldsymbol{B}_2$ that fulfills one of the equivalent conditions 2.1 b.1), b.2) is called ***additive****, or ***finitely additive****.

2.3 The sets of functions $\boldsymbol{U} \to \boldsymbol{B}_2$ which are additive, respectively additive* are noted with $Ad(\boldsymbol{U})$, respectively $Ad^*(\boldsymbol{U})$. They are naturally organized as $\boldsymbol{B}_2$-linear spaces.

2.4 **Theorem** a) Let $\mu \in Ad(\boldsymbol{U})$. For $A, B \in \boldsymbol{U}$, we have:

a.1) $$\mu(\emptyset) = 0 \quad (1)$$

a.2) $$\mu(A - B) = \mu(A) \oplus \mu(A \wedge B) \quad (2)$$

a.3) $$\mu(A \vee B) \oplus \mu(A \wedge B) \oplus \mu(A \Delta B) = 0 \quad (3)$$

b) If $\mu \in Ad^*(\boldsymbol{U})$, then the next properties are true:

b.1) $$\mu(X) = 1 \quad (4)$$

b.2) $$\mu(\overline{B - A}) = \mu(A) \otimes \mu(A \vee B) \quad (5)$$

b.3) $$\mu(A \wedge B) \otimes \mu(A \vee B) \otimes \mu(A \Theta B) = 1 \quad (6)$$

where $A, B \in \boldsymbol{U}$.

2.5 Let $a : \boldsymbol{N} \to \boldsymbol{B}_2$,

$$a_n \overset{def}{=} a(n), n \in \boldsymbol{N} \quad (1)$$

a binary sequence. If the support of $a : \{n \mid n \in \boldsymbol{N}, a_n = 1\}$ is a finite set, then the summation modulo 2 has sense:

$$\underset{n \in \boldsymbol{N}}{\Xi} a_n = \begin{cases} 1, | \, supp\ a \, | \ is\ odd \\ 0, | \, supp\ a \, | \ is\ even \end{cases} \quad (2)$$

where we have noted with $|\cdot|$ the number of elements of a finite set and where, by definition:

$$|\emptyset| = 0 \quad (3)$$

is even. If the support of a is not finite, then the symbol $\underset{n \in \boldsymbol{N}}{\Xi} a_n$ refers to a divergent series.

2.6 Let $A : \boldsymbol{N} \to \boldsymbol{B}_2$,

$$A_n \overset{def}{=} A(n), n \in \boldsymbol{N} \quad (1)$$

a sequence of sets. If for any $x\in X$ the set $\{n\mid n\in \boldsymbol{N}, x\in A_n\}$ is finite, then the symmetrical difference has sense:

$$\underset{n\in \boldsymbol{N}}{\Delta} A_n = \{x \mid x\in X, \mid \{n\mid n\in \boldsymbol{N}, x\in A_n\}\mid \textit{is odd}\} \qquad (2)$$

and if not, the symbol $\underset{n\in \boldsymbol{N}}{\Delta} A_n$ refers to a divergent series of sets.

2.7 **Theorem** Let $(\boldsymbol{U},\Delta,\wedge)\subset 2^X$ be a set ring and $\mu:\boldsymbol{U}\to \boldsymbol{B}_2$ a function. The following statements are equivalent:

a) For any sequence of sets $A_n\in \boldsymbol{U}, n\in \boldsymbol{N}$, the conditions

a.1) $n\neq m \Rightarrow A_n\wedge A_m=\emptyset$

and

a.2) $\underset{n\in \boldsymbol{N}}{\vee} A_n\in \boldsymbol{U}$

imply

a.3) $\{n\mid n\in \boldsymbol{N}, \mu(A_n)=1\}$ is finite

and

a.4) $\mu(\underset{n\in \boldsymbol{N}}{\vee} A_n) = \underset{n\in \boldsymbol{N}}{\Xi} \mu(A_n)$ (1)

b) For any sequence of sets $A_n\in \boldsymbol{U}, n\in \boldsymbol{N}$, the conditions

b.1) $\forall x\in X, \{n\mid n\in \boldsymbol{N}, x\in A_n\}$ is finite

and

b.2) $\underset{n\in \boldsymbol{N}}{\Delta} A_n\in \boldsymbol{U}$

imply

b.3) $\{n\mid n\in \boldsymbol{N}, \mu(A_n)=1\}$ is finite

and

b.4) $\mu(\underset{n\in \boldsymbol{N}}{\Delta} A_n) = \underset{n\in \boldsymbol{N}}{\Xi} \mu(A_n)$ (2)

Proof $a)\Rightarrow b)$ Let $A_n\in \boldsymbol{U}, n\in \boldsymbol{N}$ so that b.1), b.2) are true under the form:

$$\forall x\in X, \{n\mid n\in \boldsymbol{N}, x\in A_n\}\in\{0,1\}$$

$$\underset{n\in \boldsymbol{N}}{\Delta} A_n = \underset{n\in \boldsymbol{N}}{\vee} A_n\in \boldsymbol{U} \qquad (3)$$

a.1), a.2) being fulfilled, a.3), a.4) are also fulfilled, thus b.3), b.4) are fulfilled.

$b)\Rightarrow a)$ If $A_n\in \boldsymbol{U}, n\in \boldsymbol{N}$ satisfies a.1), a.2), then b.1), b.2) are true, thus b.3), b.4) are true resulting that a.3), a.4) are fulfilled.

2.8 a) A function $\mu:\boldsymbol{U}\to \boldsymbol{B}_2$ that satisfies one of the equivalent conditions 2.7 a), b) is called *countably additive*, or *measure*.

b) We take in consideration the duals of 2.5, 2.6, 2.7. A function $\mu:\boldsymbol{U}\to \boldsymbol{B}_2$ that fulfills one of the duals of the previous equivalent conditions is called *countably additive** or *measure**.

2.9 The sets of countably additive, respectively countably additive* $\boldsymbol{U}\to \boldsymbol{B}_2$ functions are noted with $Ad_c(\boldsymbol{U})$, respectively with $Ad_c^*(\boldsymbol{U})$.

These sets are $\boldsymbol{B}_2$-linear spaces.

2.10 The inclusions $Ad_c(\boldsymbol{U}) \subset Ad(\boldsymbol{U})$, $Ad_c^*(\boldsymbol{U}) \subset Ad^*(\boldsymbol{U})$ are easily shown.

2.11 The terminology of additive function, countably additive function and measure is the same if the domain of the function is a $\boldsymbol{B}_2$-algebra $\boldsymbol{U}'$ included in $\boldsymbol{B}_2^X$, via the identification from 1.14.

3. Examples

3.1 Let $X \neq \emptyset$ and $\boldsymbol{U} \subset 2^X$ a set ring. The null function $0:\boldsymbol{U} \to \boldsymbol{B}_2$ is a measure; it is the null element of the linear space $Ad_c(\boldsymbol{U})$.

3.2 Suppose that $\mu:\boldsymbol{U} \to \boldsymbol{B}_2$ is a measure and $A \in \boldsymbol{U}$. The function $\mu_1:\boldsymbol{U} \to \boldsymbol{B}_2$ that is defined by:

$$\mu_1(B) = \mu(A \wedge B),\ B \in \boldsymbol{U} \quad (1)$$

is a measure, called the *restriction of* μ *at* A.

Proof Let $A_n \in \boldsymbol{U}, n \in \boldsymbol{N}$ be disjoint two by two with $\underset{n\in \boldsymbol{N}}{\vee} A_n \in \boldsymbol{U}$, resulting that the sets $A \wedge A_n \in \boldsymbol{U}, n \in \boldsymbol{N}$ are disjoint two by two with

$$\underset{n\in \boldsymbol{N}}{\vee}(A \wedge A_n) = A \wedge \underset{n\in \boldsymbol{N}}{\vee} A_n \in \boldsymbol{U} \quad (2)$$

Because μ is a measure, the set $\{n \mid n \in \boldsymbol{N}, \mu(A \wedge A_n) = 1\}$ is finite and it is true:

$$\mu_1(\underset{n\in \boldsymbol{N}}{\vee} A_n) = \mu(A \wedge \underset{n\in \boldsymbol{N}}{\vee} A_n) = \mu(\underset{n\in \boldsymbol{N}}{\vee}(A \wedge A_n)) = \quad (3)$$
$$= \underset{n\in \boldsymbol{N}}{\Xi} \mu(A \wedge A_n) = \underset{n\in \boldsymbol{N}}{\Xi} \mu_1(A_n)$$

3.3 We fix $x_0 \in X$. The function $\chi^{\{x_0\}}:\boldsymbol{U} \to \boldsymbol{B}_2$ defined by:

$$\chi^{\{x_0\}}(A) = \chi_A(x_0),\ A \in \boldsymbol{U} \quad (1)$$

is a measure. More general, the sum of these functions is a measure too and this means that to each finite set $H \subset X$ it is associated a function $\chi^H:\boldsymbol{U} \to \boldsymbol{B}_2$ defined in the following way:

$$\chi^H(A) = \underset{x\in H}{\Xi} \chi_A(x),\ A \in \boldsymbol{U} \quad (2)$$

When H is the empty set, we find the example 3.1.

3.4 $(S_2, \oplus, \bullet, \cdot)$ is the $\boldsymbol{B}_2$-algebra of the binary sequences $x_n \in \boldsymbol{B}_2, n \in \boldsymbol{N}$, where the sum of the sequences '$\oplus$', the product of the sequences '$\bullet$' and the product of the sequences with scalars '$\cdot$' is made coordinatewise. We mention here that the families of sequences $(x_n^p)_n \in S_2,\ p \in \boldsymbol{N}$ that are disjoint two by two are these that satisfy:

$$p \neq p' \Rightarrow \forall n,\ x_n^p \cdot x_n^{p'} = 0 \quad (1)$$

Let $k \in \boldsymbol{N}$ and we define $\mu_k : S_2 \to \boldsymbol{B}_2$ by:

$$\mu_k((x_n)) = x_k,\ (x_n) \in S_2 \quad (2)$$

- the projection of the vector (x_n) of S_2 on the k-th coordinate. More general, if $H \subset \boldsymbol{N}$ is a finite set

$$H = \{k_1, ..., k_p\} \tag{3}$$

then we have the sum of functions $\mu_H : S_2 \to \boldsymbol{B}_2$,

$$\mu_H = \mu_{k_1} \oplus ... \oplus \mu_{k_p} \tag{4}$$

μ_k and μ_H are countably additive; if H is empty, then μ_H is by definition the null function.

3.5 a) We say that the sequence $x_n \in \boldsymbol{B}_2, n \in \boldsymbol{N}$ *converges to* $x^0 \in \boldsymbol{B}_2$ if

$$\exists N \in \boldsymbol{N}, \forall n \geq N, x_n = x^0 \tag{1}$$

If so, the unique x^0 with this property (because x is a function) is called the ***limit of*** (x_n). If the previous statement is made under the weaker form: the sequence (x_n) is ***convergent***, this means that such an x^0 like at (1) (uniquely) exists. The limit of the sequence (x_n) has the usual notation $\lim_{n\to\infty} x_n$.

b) $(S_2^0, \oplus, \bullet, \cdot)$ is the $\boldsymbol{B}_2$-algebra of the binary sequences $x_n \in \boldsymbol{B}_2, n \in \boldsymbol{N}$ that converge to 0. We define the measure $\mu : S_2^0 \to \boldsymbol{B}_2$ by

$$\mu((x_n)) = \underset{n \in \boldsymbol{N}}{\Xi} x_n, (x_n) \in S_2^0 \tag{1}$$

3.6 $(S_2^c, \oplus, \bullet, \cdot)$ is the $\boldsymbol{B}_2$-algebra of the convergent binary sequences $x_n \in \boldsymbol{B}_2, n \in \boldsymbol{N}$ and we define $\mu : S_2^c \to \boldsymbol{B}_2$ by:

$$\mu((x_n)) = \lim_{n\to\infty} x_n, (x_n) \in S_2^c \tag{1}$$

μ is additive, but it is not countably additive. In order to see this, we give the example of the sequence of convergent sequences (the canonical base of S_2^c):

$$\varepsilon^n : \boldsymbol{N} \to \boldsymbol{B}_2, \varepsilon^n(m) = \begin{cases} 1, n = m \\ 0, else \end{cases}, m, n \in \boldsymbol{N} \tag{2}$$

$(\varepsilon^n)_n$ are disjoint two by two, their reunion is the constant 1 sequence that is convergent and on the other hand

$$\mu(\bigcup_{n \in \boldsymbol{N}} \varepsilon^n) = 1 \neq 0 = \underset{n \in \boldsymbol{N}}{\Xi} \mu(\varepsilon^n) \tag{3}$$

3.7 A variant of 3.4 is obtained if we take instead of $S_2 = \boldsymbol{B}_2^{\boldsymbol{N}}$ an arbitrary function $\boldsymbol{B}_2$-algebra $\boldsymbol{U} \subset \boldsymbol{B}_2^X$. Let $x_0 \in X$; the function $\mu_{x_0} : \boldsymbol{U} \to \boldsymbol{B}_2$ defined like this:

$$\mu_{x_0}(f) = f(x_0), f \in \boldsymbol{U} \tag{1}$$

is a measure. More general, if $H \subset X$ is a finite set, then the function $\mu_H : \boldsymbol{U} \to \boldsymbol{B}_2$ defined in the following manner:

$$\mu_H(f) = \underset{x\in H}{\Xi} f(x),\ f \in \boldsymbol{U} \quad (2)$$

is a measure. If H is empty, then by definition μ_H is the null function.

We mention the fact that $f^p \in \boldsymbol{U},\ p \in \boldsymbol{N}$ are disjoint two by two if

$$p \neq p' \Rightarrow \forall x \in X,\ f^p(x) \cdot f^{p'}(x) = 0 \quad (3)$$

3.8 We note with $R_f(X)$ the ring - relative to $\Delta, \wedge$ - of the finite subsets of X. The function $\mu_f^X : R_f(X) \to \boldsymbol{B}_2$,

$$\mu_f^X(A) = \begin{cases} 1, |A| \textit{ is odd} \\ 0, |A| \textit{ is even} \end{cases},\ A \in R_f(X) \quad (1)$$

is a measure, called the *finite Boolean measure*.

3.9 We note with Inf_f the ring of the *inferiorly finite sets* $A \subset \boldsymbol{R}$, i.e. the sets with the following property:

$$\forall \alpha \in \boldsymbol{R}, (-\infty, \alpha) \wedge A \text{ is finite}$$

We fix some $\alpha \in \boldsymbol{R}$ and we define $\mu_\alpha : Inf_f \to \boldsymbol{B}_2$ by:

$$\mu_\alpha(A) = \mu_f((-\infty, \alpha) \wedge A),\ A \in Inf_f \quad (1)$$

μ_α is countably additive: for any family $A_n \in Inf_f,\ n \in \boldsymbol{N}$ of two by two disjoint sets so that $\underset{n\in \boldsymbol{N}}{\vee} A_n \in Inf_f$, only a finite number of sets A_n fulfill $(-\infty, \alpha) \wedge A_n \neq \emptyset$ etc.

3.10 a) Let $X \subset \boldsymbol{R}$ and $t \in \boldsymbol{R} \vee \{\infty\}$ a point so that

$$\forall t' < t, (t', t) \wedge X \text{ is infinite}$$

b) We say that the function $f : X \to \boldsymbol{B}_2$ has a *left limit in* t, noted with $f(t-0) \in \boldsymbol{B}_2$, if the next property is true:

$$\exists t' < t, \forall \xi \in (t', t) \wedge X,\ f(\xi) = f(t-0) \quad (1)$$

c) We note with $Lim_X^-(t)$ the $\boldsymbol{B}_2$-algebra of the $X \to \boldsymbol{B}_2$ functions that have a left limit in t.

d) The function $\mu : Lim_X^-(t) \to \boldsymbol{B}_2$;

$$\mu(f) = f(t-0),\ f \in Lim_X^-(t) \quad (1)$$

is a measure, this example being analogue to 3.7.

e) Other examples of measures of the same type with this one may be given.

3.11 a) For $a, b \in \boldsymbol{R} \vee \{\infty\}$, the *symmetrical interval* $[[a,b))$ is defined by:

$$[[a,b)) = \begin{cases} [a,b),\ a < b \\ [b,a),\ b < a \\ \emptyset\ ,\ b = a \end{cases} \quad (1)$$

b) We note with Sym^- the set ring - relative to $\Delta, \wedge$ - generated by the symmetrical intervals $[[a,b))$.

c) We define $\mu : Sym^- \to \boldsymbol{B}_2$ by:

$$\mu(A)=\begin{cases}1, \text{ if } \sup A=\infty\\ 0, \text{ else}\end{cases} \quad (2)$$

where $A\in Sym^{-}$. Because in a sequence of sets $A_n \in Sym^{-}, n\in \boldsymbol{N}$ that are disjoint two by two with $\underset{n\in \boldsymbol{N}}{\vee} A_n \in Sym^{-}$ at most one satisfies the condition $\sup A_n=\infty$, it may be shown that μ is a measure.

3.12 a) We define the next $\boldsymbol{B}_2$-algebras of functions $f:\boldsymbol{R}\to\boldsymbol{B}_2$:

$$I_{[[a,b))}=\{f \mid [[a,b))\wedge supp\ f\ is\ finite\}, a,b\in \boldsymbol{R}\vee\{\infty\} \quad (1)$$

$$I_{\infty}=\{f \mid supp\ f\ is\ finite\} \quad (2)$$

and the integrals

$$\int_a^{b}{}^{-} f=\underset{t\in[[a,b))}{\Xi} f(t),\ f\in I_{[[a,b))} \quad (3)$$

$$\int_{-\infty}^{\infty} f=\underset{t\in \boldsymbol{R}}{\Xi} f(t),\ f\in I_{\infty} \quad (4)$$

b) The next $I_{[[a,b))}\to \boldsymbol{B}_2, I_{\infty}\to \boldsymbol{B}_2$ functions:

$$\mu(f)=\int_a^{b}{}^{-} f,\ f\in I_{[[a,b))} \quad (5)$$

$$\mu(f)=\int_{-\infty}^{\infty} f,\ f\in I_{\infty} \quad (6)$$

are measures.

3.13 a) The set $\boldsymbol{S}\subset 2^{\boldsymbol{R}}$ defined in the next way:

$$\boldsymbol{S}=\{(a_1,b_1)\,\Delta\ldots\Delta\,(a_p,b_p)\,\Delta\,\{c_1,\ldots,c_n\} \mid a_1,b_1,\ \ldots$$
$$\ldots, a_p,b_p,c_1,\ldots,c_n\in \boldsymbol{R},\ p,n\in \boldsymbol{N}\} \quad (1)$$

is a ring of subsets of $\boldsymbol{R}$ and we have supposed that

$$p=0\Rightarrow (a_1,b_1)\,\Delta\ldots\Delta\,(a_p,b_p)=\varnothing \quad (2)$$

$$n=0\Rightarrow \{c_1,\ldots,c_n\}=\varnothing \quad (3)$$

b) The function $\mu:\boldsymbol{S}\to\boldsymbol{B}_2$ given by:

$$\mu\{(a_1,b_1)\,\Delta\ldots\Delta\,(a_p,b_p)\,\Delta\,\{c_1,\ldots,c_n\})=\pi(p+n) \quad (4)$$

where $\pi:\boldsymbol{N}\to\boldsymbol{B}_2$ is the *parity function*:

$$\pi(m)=\begin{cases}1, \text{ if } m \text{ is odd}\\ 0, \text{ if } m \text{ is even}\end{cases}, m\in \boldsymbol{N} \quad (5)$$

- is additive, but it is not countably additive. In order to see this fact, we take the sequence

$$[\frac{1}{n+2},\frac{1}{n+1})=(\frac{1}{n+2},\frac{1}{n+1})\,\Delta\,\{\frac{1}{n+2}\}\in \boldsymbol{S}, n\in \boldsymbol{N} \quad (6)$$

of sets that are disjoint two by two, satisfying

$$\underset{n\in \boldsymbol{N}}{\vee}[\frac{1}{n+2},\frac{1}{n+1})=(0,1)\in \boldsymbol{S} \quad (7)$$

$$\{n \mid \mu([\frac{1}{n+2},\frac{1}{n+1}))=1\}=\varnothing \quad (8)$$

$$\mu((0,1))=1\neq 0=\underset{n\in \boldsymbol{N}}{\Xi}\mu([\frac{1}{n+2},\frac{1}{n+1})) \quad (9)$$

3.14 a) We note with

$$R_f^*(X)=\{H \mid H\subset X, \overline{H}\ is\ finite\} \quad (1)$$

This set is a set ring relative to $\Theta, \vee$ and it is the dual structure of $R_f(X)$.

b) A typical example of measure* is given by the function $\mu_f^{*X}: R_f^*(X)\to \boldsymbol{B}_2$ that is defined in the next manner:

$$\mu_f^{*X}(H)=\begin{cases}0, |\overline{H}|\ is\ odd\\ 1, |\overline{H}|\ is\ even\end{cases}=\overline{\mu_f^X(\overline{H})}, H\in R_f^*(X) \quad (2)$$

(In the equations (1), (2) the superior bar notes two things: the complementary of a set and the logical complement.)

Let the sequence of sets $A_n\in R_f^*(X), n\in \boldsymbol{N}$ that are disjoint* two by two:

$$n,m\in \boldsymbol{N}, n\neq m\Rightarrow A_n\vee A_m=X \text{ (i.e. } \overline{A_n}\wedge\overline{A_m}=\varnothing) \quad (3)$$

so that $\underset{n\in \boldsymbol{N}}{\wedge}A_n\in R_f^*(X)$. Because from the definition of $R_f^*(X)$, the set

$$\overline{\underset{n\in \boldsymbol{N}}{\wedge}A_n}=\underset{n\in \boldsymbol{N}}{\vee}\overline{A_n} \quad (4)$$

is finite, there results the existence of a rank N with the property that $\overline{A_n}$ are empty for $n>N$. We have:

$$\mu_f^{*X}(\underset{n\in \boldsymbol{N}}{\wedge}A_n)=\overline{\mu_f^X(\overline{\underset{n\in \boldsymbol{N}}{\wedge}A_n})}=\overline{\mu_f^X(\underset{n\in \boldsymbol{N}}{\vee}\overline{A_n})}=\overline{\mu_f^X(\overline{A_0}\vee\overline{A_1}\vee...\vee\overline{A_N})}=$$

$$=\overline{\mu_f^X(\overline{A_0})\oplus\mu_f^X(\overline{A_1})\oplus...\oplus\mu_f^X(\overline{A_N})}=\overline{\underset{n\in \boldsymbol{N}}{\Xi}\mu_f^X(\overline{A_n})}=\overline{\underset{n\in \boldsymbol{N}}{\Xi}\overline{\overline{\mu_f^X(\overline{A_n})}}}=$$

$$=\overline{\underset{n\in \boldsymbol{N}}{\Xi}\overline{\mu_f^{*X}(A_n)}}=\underset{n\in \boldsymbol{N}}{\otimes}\mu_f^{*X}(A_n) \quad (5)$$

4. The Behavior of the Measures Relative to the Monotonous Sequences of Sets

4.1 a) The family $A_n\subset X, n\in \boldsymbol{N}$ is called *ascending sequence* of sets if

$$A_0\subset A_1\subset A_2\subset ... \quad (1)$$

In this case, the reunion $\underset{n\in \boldsymbol{N}}{\vee}A_n$ is called the *limit* of the sequence and is noted sometimes with $\lim\limits_{n\to\infty}A_n$.

b) The family $A_n\subset X, n\in \boldsymbol{N}$ is called *descending sequence* of sets if

$$A_0\supset A_1\supset A_2\supset ... \quad (2)$$

The intersection $\underset{n\in \mathbf{N}}{\wedge} A_n$ is called the *limit* of the sequence and is noted sometimes with $\lim\limits_{n\to\infty} A_n$.

c) If the sequence $A_n \subset X, n\in \mathbf{N}$ is either ascending, or descending, then we say that it is *monotonous*.

4.2 **Theorem** Let $\mathbf{U} \subset 2^X$ a set ring and the function $\mu : \mathbf{U} \to \mathbf{B}_2$.

a) Let $A_n \in \mathbf{U}, n\in \mathbf{N}$ an arbitrary ascending sequence of sets satisfying the property that the set

$$A = \underset{n\in \mathbf{N}}{\vee} A_n \tag{1}$$

belongs to $\mathbf{U}$. If μ is a measure, then the binary sequence $(\mu(A_n))_n$ is convergent (see 3.5 a)) and it is true:

$$\mu(A) = \lim_{n\to\infty} \mu(A_n) \tag{2}$$

b) Suppose that μ is additive and it satisfies the property: for any ascending sequence $A_n \in \mathbf{U}, n\in \mathbf{N}$ of sets so that its reunion A belongs to $\mathbf{U}$, the binary sequence $(\mu(A_n))_n$ is convergent and the relation (2) takes place. Then μ is a measure.

Proof a) We have the disjoint reunion:

$$A = A_0 \vee (A_1 - A_0) \vee ... \vee (A_{n+1} - A_n) \vee ... \tag{3}$$

Because μ is a measure, it results that there exists $N \in \mathbf{N}$ so that

$$n > N \Rightarrow \mu(A_{n+1} - A_n) = 0 \tag{4}$$

thus

$$\begin{aligned}\mu(A) &= \mu(A_0) \oplus \mu(A_1 - A_0) \oplus ... \oplus \mu(A_{N+1} - A_N) = \\ &= \mu(A_0) \oplus \mu(A_1) \oplus \mu(A_0) \oplus ... \oplus \mu(A_{N+1}) \oplus \mu(A_N) = \mu(A_{N+1})\end{aligned} \tag{5}$$

(4) is equivalent with the convergence of the sequence $(\mu(A_n))_n$, as it can be rewritten under the form:

$$n > N \Rightarrow \mu(A_{n+1}) = \mu(A_n) \tag{6}$$

and (5) is equivalent in this situation with (2). In the last equations, we have used 2.4 a.2) under the form:

$$\mu(A_{n+1} - A_n) = \mu(A_{n+1}) \oplus \mu(A_{n+1} \wedge A_n) = \mu(A_{n+1}) \oplus \mu(A_n), n\in \mathbf{N} \tag{7}$$

b) Let $A_n' \in \mathbf{U}, n\in \mathbf{N}$ a sequence of sets that are disjoint two by two and let us suppose that their reunion

$$A = \underset{n\in \mathbf{N}}{\vee} A_n' \tag{8}$$

belongs to $\mathbf{U}$. We define the sequence $A_n \in \mathbf{U}, n\in \mathbf{N}$ by:

$$A_n = A_0' \vee A_1' \vee ... \vee A_n', n\in \mathbf{N} \tag{9}$$

and it is remarked that it is ascending and (1) is satisfied. The hypothesis states the convergence of the sequence with the general term

$$\mu(A_n) = \mu(A_0') \oplus \mu(A_1') \oplus ... \oplus \mu(A_n') \tag{10}$$

in other words there exists an $N \in \mathbf{N}$ for which the implication

$$n > N \Rightarrow \mu(A_n') = 0 \tag{11}$$

is true. The relation (2) becomes

$$\mu(A) = \mu(A_0^{'}) \oplus \mu(A_1^{'}) \oplus ... \oplus \mu(A_N^{'}) = \underset{n \in \boldsymbol{N}}{\Xi} \mu(A_n^{'}) \quad (12)$$

i.e. μ is a measure.

4.3 **Theorem** It is considered the set ring $\boldsymbol{U}$ and the function $\mu : \boldsymbol{U} \to \boldsymbol{B}_2$.

a) We suppose that $A_n \in \boldsymbol{U}, n \in \boldsymbol{N}$ is an arbitrary descending sequence of sets whose intersection

$$A = \underset{n \in \boldsymbol{N}}{\wedge} A_n \quad (1)$$

belongs to $\boldsymbol{U}$ and that μ is a measure. Then the binary sequence $(\mu(A_n))_n$ is convergent and it is true:

$$\mu(A) = \lim_{n \to \infty} \mu(A_n) \quad (2)$$

b) Let us suppose that μ is additive and the next property is satisfied: for any descending sequence $A_n \in \boldsymbol{U}, n \in \boldsymbol{N}$ of sets so that its intersection A belongs to $\boldsymbol{U}$, the binary sequence $(\mu(A_n))_n$ is convergent and the relation (2) is true. In these circumstances μ is a measure.

Proof a) Let us remark for the beginning that the set

$$\underset{n \in \boldsymbol{N}}{\vee} (A_0 - A_n) = A_0 - \underset{n \in \boldsymbol{N}}{\wedge} A_n = A_0 - A \quad (3)$$

belongs to $\boldsymbol{U}$ and the sequence of sets

$$A_0 - A_0 \subset A_0 - A_1 \subset A_0 - A_2 \subset ... \quad (4)$$

is ascending. We apply 4.2 a) resulting that the binary sequence $(\mu(A_0 - A_n))_n$ is convergent and that it takes place

$$\mu(A_0 - A) = \lim_{n \to \infty} \mu(A_0 - A_n) \quad (5)$$

From (5) it results that

$$\mu(A_0) \oplus \mu(A) = \mu(A_0) \oplus \lim_{n \to \infty} \mu(A_n) \quad (6)$$

and we have the validity of (2).

b) Let $A_n^{'} \in \boldsymbol{U}, n \in \boldsymbol{N}$ a sequence of sets that are disjoint two by two with the property that their reunion

$$A' = \underset{n \in \boldsymbol{N}}{\vee} A_n^{'} \quad (7)$$

belongs to $\boldsymbol{U}$. We define the sequence of sets from $\boldsymbol{U}$:

$$A_n = A' - (A_0^{'} \vee A_1^{'} \vee ... \vee A_n^{'}) = (A' - A_0^{'}) \wedge (A' - A_1^{'}) \wedge ... \wedge (A' - A_n^{'}) \quad (8)$$

where $n \in \boldsymbol{N}$ that proves to be descending and its meet

$$A = \underset{n \in \boldsymbol{N}}{\wedge} A_n = \underset{n \in \boldsymbol{N}}{\wedge} \overset{n}{\underset{k=0}{\wedge}} (A' - A_k^{'}) = A' - \underset{n \in \boldsymbol{N}}{\vee} \overset{n}{\underset{k=0}{\vee}} A_k^{'} = A' - A' = \varnothing \quad (9)$$

belongs to $\boldsymbol{U}$. The hypothesis states that the binary sequence $(\mu(A_n))_n$ is convergent and the relation (2) becomes:

$$0 = \mu(\underset{n \in \boldsymbol{N}}{\wedge} A_n) = \lim_{n \to \infty} \mu(A_n) \quad (10)$$

There exists a rank $N \in \boldsymbol{N}$ so that for any $n > N$ we have:

$$0=\mu(A_n)=\mu(A'-(A_0^{'}\vee A_1^{'}\vee...\vee A_n^{'}))=\mu(A')\oplus\mu(A_0^{'})\oplus\mu(A_1^{'})\oplus...\oplus\mu(A_n^{'}) \quad (11)$$

We have that $(\mu(A_n^{'}))_n$ converges to 0 and if $k>N$ then

$$\mu(\underset{n\in \mathbf{N}}{\vee} A_n^{'})=\mu(A')=\overset{k}{\underset{n=0}{\Xi}}\mu(A_n^{'})=\underset{n\in \mathbf{N}}{\Xi}\mu(A_n^{'}) \quad (12)$$

5. Derivable Measures

5.1 In this paragraph we shall consider that the total space X is equal with $\boldsymbol{R}^n$, $n\geq 1$. The elements $x\in X$ will be consequently $n-$tuples $(x_1,...,x_n)\in \boldsymbol{R}^n$.

5.2 We define the family

$$\boldsymbol{U}_n=\{A\mid A\subset \boldsymbol{R}^n,\ A\ is\ bounded\} \quad (1)$$

It is a set ring (relative to Δ and $\wedge$).

5.3 Let $A\in \boldsymbol{U}_n$ be a bounded set. Its *diameter* is defined to be the real non-negative number

$$d(A)=\sup_{x,y\in A}\sqrt{(x_1-y_1)^2+...+(x_n-y_n)^2} \quad (1)$$

5.4 We define the *locally finite* sets from $\boldsymbol{R}^n$ to be these sets $H\subset \boldsymbol{R}^n$ with the property that

$$\forall A\in \boldsymbol{U}_n,\ A\wedge H\ is\ finite$$

5.5 The set of the locally finite sets from $\boldsymbol{R}^n$ is noted with $Loc_f^{(n)}$ and it is a set ring.

5.6 **Proposition** Let us take a set $H\in Loc_f^{(n)}$. The function $\mu_H:\boldsymbol{U}_n\to \boldsymbol{B}_2$ defined by:

$$\mu_H(A)=\pi(|A\wedge H|),\ A\in \boldsymbol{U}_n \quad (1)$$

is a measure (the function π was defined at 3.13 (5)).

Proof Let $A_p\in \boldsymbol{U}_n$, $p\in \mathbf{N}$ a family of sets that are disjoint two by two with the property that $\underset{p\in \mathbf{N}}{\vee} A_p\in \boldsymbol{U}_n$. Because $\underset{p\in \mathbf{N}}{\vee} A_p\wedge H$ is a finite set, there exists a number $N\in \mathbf{N}$ with:

$$p>N\Rightarrow A_p\wedge H=\emptyset \quad (2)$$

We infer that

$$\{p\mid \mu_H(A_p)=1\}\subset\{0,1,...,N\} \quad (3)$$

$$\mu_H(\underset{p\in \mathbf{N}}{\vee} A_p)=\pi(|\underset{p\in \mathbf{N}}{\vee} A_p\wedge H|)=\pi(|\underset{p\in \mathbf{N}}{\vee}(A_p\wedge H)|)= \quad (4)$$

$$=\pi(|(A_0\wedge H)\vee(A_1\wedge H)\vee...\vee(A_N\wedge H)|)=$$

$$=\pi(|A_0\wedge H|+|A_1\wedge H|+...+|A_N\wedge H|)=$$

$$=\pi(|A_0\wedge H|)\oplus\pi(|A_1\wedge H|)\oplus...\oplus\pi(|A_N\wedge H|)=$$

$$= \underset{p \in \boldsymbol{N}}{\Xi} \pi(| A_p \wedge H |) = \underset{p \in \boldsymbol{N}}{\Xi} \mu_H(A_p)$$

5.7 **Proposition** The function $\mu_H \in Ad_c(\boldsymbol{U}_n)$ that was previously defined fulfills the property that for any $A \in \boldsymbol{U}_n$ and $x \in A$:

$$\exists \varepsilon > 0, \exists a \in \boldsymbol{B}_2, \forall B \in \boldsymbol{U}_n, (x \in B \text{ and } d(B) < \varepsilon) \Rightarrow \mu_H(B) = a \tag{1}$$

Proof We define the real positive number

$$\varepsilon = \begin{cases} \min \{\sqrt{(x_1 - y_1)^2 + ... + (x_n - y_n)^2} \mid y \in H - \{x\}\}, & H - \{x\} \neq \emptyset \\ 1 & , H - \{x\} = \emptyset \end{cases} \tag{2}$$

Such an ε exists, if not there would exist a sphere S_x with the center in x and the property that $S_x \wedge H$ is infinite and this is a contradiction with the hypothesis $H \in Loc_f^{(n)}$.

Any bounded set $B \in \boldsymbol{U}_n$ with the properties that $x \in B$ and $d(B) < \varepsilon$ fulfills the relations:

$$(B - \{x\}) \wedge H = \emptyset \tag{3}$$

$$\mu_H(B - \{x\}) = 0 \tag{4}$$

$$\mu_H(B) = \mu_H((B - \{x\}) \vee \{x\}) = \mu_H(B - \{x\}) \oplus \mu_H(\{x\}) = \tag{5}$$

$$= \mu_H(\{x\}) = \pi(| \{x\} \wedge H |) = \begin{cases} 1, & x \in H \\ 0, & x \notin H \end{cases}$$

5.8 Let now $\mu : \boldsymbol{U}_n \to \boldsymbol{B}_2$ be a measure.

a) We say that it is *derivable in* $x \in A$, where $A \in \boldsymbol{U}_n$, if

$$\exists \varepsilon > 0, \exists a \in \boldsymbol{B}_2, \forall B \in \boldsymbol{U}_n, (x \in B \text{ and } d(B) < \varepsilon) \Rightarrow \mu(B) = a \tag{1}$$

b) In the case that the property of derivability of μ takes place in any $x \in A$, we say that μ is *derivable on* A.

c) If μ is derivable on any set $A \in \boldsymbol{U}_n$, then it is called *derivable*.

5.9 The number $a \in \boldsymbol{B}_2$ depending on $x \in A$ and the function $A \ni x \mapsto a \in \boldsymbol{B}_2$ whose existence is stated in 5.8 are called the ***derivative of*** μ ***in*** x, respectively the ***derivative function of*** μ ***in*** x.

5.10 The derivative of μ in x and the derivative function of μ in x are noted with $d\mu(x)$. Other notations are:

- $\frac{d\mu}{dl}(x)$, if $n = 1$
- $\frac{d\mu}{dS}(x)$, if $n = 2$
- $\frac{d\mu}{dV}(x)$, if $n = 3$

5.11 **Remark** The set $B \in \boldsymbol{U}_n$ formed by one element, $x \in A$

$$B = \{x\} \tag{1}$$

has the property that for any $\varepsilon > 0$,

$$x \in B \text{ and } d(B) = 0 < \varepsilon \tag{2}$$

from where it is inferred that, if μ is derivable in x, then $d\mu(x)$, that generally does not depend on B, is given by:

$$d\mu(x)=\mu(\{x\}) \tag{3}$$

5.12 a) We suppose that μ is a derivable measure on A. The set

$$supp_A\, d\mu=\{x\,|\,x\in A, d\mu(x)=1\}=\{x\,|\,x\in A, \mu(\{x\})=1\} \tag{1}$$

is called the *support of* $d\mu$ *on* A.

b) If μ is derivable (on any set $A\in \boldsymbol{U}_n$), then by definition the set

$$supp\, d\mu=\{x\,|\,x\in \boldsymbol{R}^n, d\mu(x)=1\}=\{x\,|\,x\in \boldsymbol{R}^n, \mu(\{x\})=1\} \tag{2}$$

is called the *support of* $d\mu$ (on $\boldsymbol{R}^n$).

5.13 **Theorem** We consider the derivable measure $\mu:\boldsymbol{U}_n\to \boldsymbol{B}_2$ on the closed set $A\in \boldsymbol{U}_n$ (A is compact). Then the set $supp_A\, d\mu$ is finite.

Proof Let us suppose that $supp_A\, d\mu$ is infinite, in contradiction with the conclusion of the theorem. Because A is bounded, there exists (Cesaro) a convergent sequence $x^p\in supp_A\, d\mu,\ p\in \boldsymbol{N}$ and the fact that A is closed implies that

$$x=\lim_{p\to\infty} x^p \tag{1}$$

belongs to A. We apply the hypothesis of derivability of μ in x:

$$\exists\varepsilon>0,\ \forall B\in \boldsymbol{U}_n, (x\in B \text{ and } d(B)<\varepsilon)\Rightarrow \mu(B)=\mu(\{x\}) \tag{2}$$

We fix ε, B like above so that for some $x^p\neq x$ it is true in addition $x^p\in B$. The set $B-\{x^p\}$ satisfies the same hypothesis like B, that is:

$$x\in B-\{x^p\} \text{ and } d(B-\{x^p\})\le d(B)<\varepsilon \tag{3}$$

and the conclusion must be the same:

$$\mu(B-\{x^p\})=\mu(\{x\}) \tag{4}$$

It is inferred that:

$$\mu(\{x\})=\mu(B)=\mu((B-\{x^p\})\vee\{x^p\})= \tag{5}$$

$$=\mu(B-\{x^p\})\oplus\mu(\{x^p\})=\mu(\{x\})\oplus 1$$

The last equation is a contradiction, having its origin in our supposition that $supp_A\, d\mu$ is infinite.

5.14 **Corollary** Let the measure $\mu:\boldsymbol{U}_n\to \boldsymbol{B}_2$.

a) If μ is derivable on the topological closure $\overline{A}$ of the set $A\in \boldsymbol{U}_n$, then:

a.1) the set $supp_A\, d\mu$ is finite

a.2) $$\forall x\in A, \exists\varepsilon>0,\ \forall B\in \boldsymbol{U}_n, (x\in B \text{ and } d(B)<\varepsilon)\Rightarrow \mu(B-\{x\})=0 \tag{1}$$

a.3) $$\mu(A)=\underset{x\in A}{\Xi}\,\mu(\{x\}) \tag{2}$$

a.4) For any partition $A_i\subset A, i\in I$, we have

$$\mu(A)=\underset{i\in I}{\Xi}\,\mu(A_i) \tag{3}$$

b) If the measure μ is derivable, then the set $supp\, d\mu$ is locally finite.

Proof a.3) If $supp_A\, d\mu$ is empty, then for any $x \in A$ we have that $x \notin supp_A\, d\mu$ and by replacing in 5.8 (1) B with A and a with $\mu(\{x\})$, it results

$$\mu(A) = \mu(\{x\}) = 0 \tag{4}$$

making the statement of the theorem obvious.

We suppose now that

$$supp_A\, d\mu = \{x^1,..., x^p\},\ p \geq 1 \tag{5}$$

There exists a partition $A_1,..., A_p \in \boldsymbol{U}_n$ of A with the property that $x^i \in A_i, i = \overline{1, p}$ and moreover

$$\mu(A) = \mu(\bigvee_{i=1}^{p} A_i) = \mathop{\Xi}_{i=1}^{p} \mu(A_i) = \mathop{\Xi}_{i=1}^{p} \mu(\{x^i\}) = \mathop{\Xi}_{x \in A} \mu(\{x\})(= \pi(p)) \tag{6}$$

b) μ is derivable on the compacts $\overline{A} \in \boldsymbol{U}_n$ and from a.1) all the sets

$$A \wedge supp\ d\mu = supp_A\, d\mu \tag{7}$$

are finite.

5.15 Let us suppose that $\mu : \boldsymbol{U}_n \to \boldsymbol{B}_2$ is derivable on the topological closure $\overline{A}$ of $A \in \boldsymbol{U}_n$. The binary number

$$\mu(A) = \mathop{\Xi}_{x \in A} \mu(\{x\}) = \mathop{\Xi}_{x \in A} d\mu(x) = \mathop{\Xi}_{x \in \boldsymbol{R}^n} f(x) \cdot d\mu(x) \tag{1}$$

is noted with $\int_A d\mu$, $\int_{\boldsymbol{R}^n} f \cdot d\mu$ or $\int f \cdot d\mu$ and is called the *integral of* $f : \boldsymbol{R}^n \to \boldsymbol{B}_2$ *relative to* μ, where the relation between f and A is by definition the following:

$$A = \{x \mid x \in \boldsymbol{R}^n, f(x) = 1\} = supp\ f \tag{2}$$

5.16 We note with

$$I_{Loc}^{(n)} = \{ f \mid f : \boldsymbol{R}^n \to \boldsymbol{B}_2,\ supp\ f \in Loc_f^{(n)} \} \tag{1}$$

the $\boldsymbol{B}_2$-algebra of the functions with locally finite support, that are called *locally integrable functions*.

5.17 **Theorem** a) The function $g \in I_{Loc}^{(n)}$ defines a derivable measure $\mu^g : \boldsymbol{U}_n \to \boldsymbol{B}_2$ by the formula:

$$\mu^g(A) = \pi(| A \wedge supp\ g |),\ A \in \boldsymbol{U}_n \tag{1}$$

It is true the relation

$$d\mu^g(x) = g(x),\ x \in \boldsymbol{R}^n \tag{2}$$

b) Conversely, if $\mu : \boldsymbol{U}_n \to \boldsymbol{B}_2$ is a derivable measure, then there exists in a unique manner the function $g \in I_{Loc}^{(n)}$ so that

$$\mu(A) = \pi(| A \wedge supp\ g |),\ A \in \boldsymbol{U}_n \tag{3}$$

being also true the relation

$$d\mu(x) = g(x),\ x \in \boldsymbol{R}^n \tag{4}$$

Proof a) The fact that μ^g is a measure was already proved at 5.6, if we put

$$\mu^g(A) = \mu_{supp\ g}(A),\ A \in \boldsymbol{U}_n \tag{5}$$

and (2) results from

$$d\mu^g(x) = \mu^g(\{x\}) = \pi(|\{x\} \wedge supp\ g|) = \begin{cases} \pi(1) = 1, & if\ x \in supp\ g \\ \pi(0) = 0, & if\ x \notin supp\ g \end{cases} = g(x) \tag{6}$$

b) If μ is derivable, then $supp\ d\mu \in Loc_f^{(n)}$ from 5.14 b) and the function $g : \boldsymbol{R}^n \to \boldsymbol{B}_2$ defined by:

$$g(x) = d\mu(x) = \mu(\{x\}),\ x \in \boldsymbol{R}^n \tag{7}$$

is locally integrable. As (4) was proved at (7), we prove (3) by taking into account 5.14 a.3):

$$\mu(A) = \underset{x \in A}{\Xi}\ \mu(\{x\}) = \pi(|A \wedge supp\ d\mu|) = \pi(|A \wedge supp\ g|),\ A \in \boldsymbol{U}_n \tag{8}$$

5.18 **Corollary** For $g \in I_{Loc}^{(n)}$ and $A \in \boldsymbol{U}_n$ it is defined the integral

$$\int_A g = \int_A d\mu^g = \mu^g(A) = \pi(|A \wedge supp\ g|) \tag{1}$$

6. The Lebesgue-Stieltjes Measure

6.1 We say about the function $f : \boldsymbol{R} \to \boldsymbol{B}_2$ that

a) it has a *left limit* in any point $t \in \boldsymbol{R} \vee \{\infty\}$, if (see 3.10)

$$\forall t \in \boldsymbol{R} \vee \{\infty\}, \exists t' < t, \exists f(t-0) \in \boldsymbol{B}_2, \forall \xi \in (t', t),\ f(\xi) = f(t-0) \tag{1}$$

b) it is *left continuous* in any point $t \in \boldsymbol{R}$, if a) is true in any $t \in \boldsymbol{R}$ and moreover:

$$\forall t \in \boldsymbol{R},\ f(t) = f(t-0) \tag{2}$$

6.2 We fix a function f satisfying the properties from 6.1. We prolong f to $\boldsymbol{R} \vee \{\infty\}$ by left continuity in the point ∞:

$$f(\infty) = f(\infty - 0) \tag{1}$$

and we note this new function with f too.

6.3 The relation

$$\mu([[a_1, b_1)) \Delta \ldots \Delta [[a_n, b_n))) = f(a_1) \oplus f(b_1) \oplus \ \ldots\ \oplus f(a_n) \oplus f(b_n) \tag{1}$$

where $a_1, \ldots, a_n, b_1, \ldots, b_n \in \boldsymbol{R} \vee \{\infty\}$ obviously defines an additive function $\mu : Sym^- \to \boldsymbol{B}_2$ (see 3.11 for the definition of the symmetrical intervals and of Sym^-). Our purpose is that of proving the next:

6.4 **Theorem** μ is a measure.

Proof Let $A_n \in Sym^-, n \in \boldsymbol{N}$ a sequence of sets that are disjoint two by two with the property that the reunion

$$A = \bigvee_{n \in \boldsymbol{N}} A_n \tag{1}$$

belongs to Sym^-. We can suppose without loss that all the sets A_n are non-empty.

Case a) $(t_n)_n$ is a real strictly increasing sequence that converges to b and we have:

$$-\infty < a = t_0 < t_1 < t_2 < ... < b \le \infty \quad (2)$$
$$A_n = [t_n, t_{n+1}),\ n \in \boldsymbol{N} \quad (3)$$
$$A = [a, b) \quad (4)$$

There exists an $N \in \boldsymbol{N}$ with

$$n > N \Rightarrow \mu(A_n) = f(t_n) \oplus f(t_{n+1}) = f(b-0) \oplus f(b-0) = 0 \quad (5)$$

and we can write that

$$\underset{n \in \boldsymbol{N}}{\Xi} \mu(A_n) = \overset{N}{\underset{n=0}{\Xi}} \mu(A_n) = f(t_0) \oplus f(t_1) \oplus f(t_1) \oplus f(t_2) \oplus ... \oplus f(t_N) \oplus f(t_{N+1}) \quad (6)$$
$$= f(t_0) \oplus f(t_{N+1}) = f(a) \oplus f(b-0) = f(a) \oplus f(b) = \mu(A)$$

Case b) A is of the general form

$$A = [a_1, b_1) \vee [a_2, b_2) \vee ... \vee [a_k, b_k) \quad (7)$$

where

$$-\infty < a_1 < b_1 \le a_2 < b_2 \le ... \le a_k < b_k \le \infty \quad (8)$$

We note

$$A_{n,i} = A_n \wedge [a_i, b_i) \quad (9)$$

where $n \in \boldsymbol{N}$ and $i = \overline{1,k}$; we have:

$$\underset{n \in \boldsymbol{N}}{\Xi} \mu(A_n) = \underset{n \in \boldsymbol{N}}{\Xi} (\mu(A_{n,1} \vee A_{n,2} \vee ... \vee A_{n,k})) = \quad (10)$$
$$= \underset{n \in \boldsymbol{N}}{\Xi} (\mu(A_{n,1} \oplus \mu(A_{n,2}) \oplus ... \oplus \mu(A_{n,k})) =$$
$$= \underset{n \in \boldsymbol{N}}{\Xi} \mu(A_{n,1} \oplus \underset{n \in \boldsymbol{N}}{\Xi} \mu(A_{n,2}) \oplus ... \oplus \underset{n \in \boldsymbol{N}}{\Xi} \mu(A_{n,k}) =$$
$$= \mu(\underset{n \in \boldsymbol{N}}{\vee} A_{n,1}) \oplus \mu(\underset{n \in \boldsymbol{N}}{\vee} A_{n,2}) \oplus ... \oplus \mu(\underset{n \in \boldsymbol{N}}{\vee} A_{n,k}) = \quad \text{(from a))}$$
$$= \mu([a_1, b_1)) \oplus \mu([a_2, b_2)) \oplus ... \oplus \mu([a_k, b_k)) =$$
$$= \mu([a_1, b_1) \vee [a_2, b_2) \vee ... \vee [a_k, b_k)) = \mu(A)$$

6.5 The measure μ that was defined at 6.3 is called the (*left*) *Lebesgue-Stieltjes measure associated to* f .

6.6 The right dual construction is made starting from an $\boldsymbol{R} \to \boldsymbol{B}_2$ function (see 6.1) with a right limit in any $t \in \{-\infty\} \vee \boldsymbol{R}$, right continuous in any $t \in \boldsymbol{R}$ that is prolonged (see 6.2) to $\{-\infty\} \vee \boldsymbol{R}$ by right continuity in the point $-\infty$. It is defined then (see 6.3) a measure $Sym^+ \to \boldsymbol{B}_2$, where Sym^+, the dual of Sym^-, is the set ring generated by the symmetrical intervals

$$((a,b]] = \begin{cases} (a,b],\ a < b \\ (b,a],\ b < a \\ \varnothing\ ,\ a = b \end{cases} \quad (1)$$

where $a, b \in \{-\infty\} \vee \boldsymbol{R}$.

6.7 **Theorem** Let $\mu_1 : Sym^- \to \boldsymbol{B}_2$ an arbitrary measure.

a) The function

$$g(t) = \mu_1([[a,t))) \quad (1)$$

where $a, t \in \boldsymbol{R} \vee \{\infty\}$ is left continuous on $\boldsymbol{R} \vee \{\infty\}$.

b) μ_1 is the left Lebesgue-Stieltjes measure associated to g.

Proof a) It is considered the sequence $(t_n)_n$

$$-\infty < a = t_0 < t_1 < t_2 < ... < t \le \infty \tag{2}$$

that is strictly increasing and convergent to t. The sets

$$A_n = [t_n, t_{n+1}), n \in \boldsymbol{N} \tag{3}$$

belong to Sym^- and are disjoint two by two and their reunion

$$\underset{n \in \boldsymbol{N}}{\vee} A_n = [a, t) \tag{4}$$

is an element from Sym^- too. It results that there exists $N \in \boldsymbol{N}$ with

$$n > N \Rightarrow \mu_1(A_n) = \mu_1([t_n, t_{n+1})) = \mu_1([a, t_n)) \oplus \mu_1([a, t_{n+1})) = g(t_n) \oplus g(t_{n+1}) = 0 \tag{5}$$

showing the existence of $g(t-0)$. But

$$g(t) = \mu_1([a,t)) = \mu_1(\underset{n \in \boldsymbol{N}}{\vee}[t_n, t_{n+1})) = \underset{n \in \boldsymbol{N}}{\Xi} \mu_1([t_n, t_{n+1})) = \tag{6}$$

$$= \overset{N}{\underset{n=0}{\Xi}} \mu_1([t_n, t_{n+1})) = \overset{N}{\underset{n=0}{\Xi}} (g(t_n) \oplus g(t_{n+1})) = g(t_0) \oplus g(t_{N+1})$$

From the fact that

$$g(t_0) = \mu_1([a,a)) = \mu_1(\emptyset) = 0 \tag{7}$$

$$g(t_{N+1}) = g(t-0) \tag{8}$$

it results, as t is arbitrary, the statement of the theorem.

b) We have that

$$\mu_1([[a_1, b_1)) \Delta ... \Delta [[a_n, b_n))) = \tag{9}$$

$$= \mu_1([[a, a_1)) \Delta [[a, b_1)) \Delta ... \Delta [[a, a_n)) \Delta [[a, b_n))) =$$

$$= \mu_1([[a, a_1))) \oplus \mu_1([[a, b_1))) \oplus ... \oplus \mu_1([[a, a_n))) \oplus \mu_1([[a, b_n))) =$$

$$= g(a_1) \oplus g(b_1) \oplus ... \oplus g(a_n) \oplus g(b_n)$$

is true for any $a_1, ..., a_n, b_1, ..., b_n \in \boldsymbol{R} \vee \{\infty\}$.

7. Measurable Spaces and Measurable Functions. The Integration of the Binary Functions Relative to a Measure

7.1 It is called *measurable space* a pair $(X, \boldsymbol{U})$ where X is a set and $\boldsymbol{U} \subset 2^X$ is a ring of subsets of X. The sets $A \in \boldsymbol{U}$ are called *measurable*.

7.2 We say that we have defined a *measurable*, or *integrable function* $f : (X, \boldsymbol{U}) \to \boldsymbol{B}_2$ where $(X, \boldsymbol{U})$ is a measurable space, if it is given the function $f : X \to \boldsymbol{B}_2$ with $supp\ f \in \boldsymbol{U}$, i.e. the support of f is a measurable set.

7.3 Recall that the set of the binary functions

$$\boldsymbol{U}' = \{f \mid f : (X, \boldsymbol{U}) \to \boldsymbol{B}_2,\ f \textit{ is measurable}\} \tag{1}$$

(see 1.13, 1.14) is a $\boldsymbol{B}_2$-algebra relative to the obvious laws. As ring, it is isomorphic with $\boldsymbol{U}$.

7.4 Let $(X, \boldsymbol{U})$ a measurable space and $M \subset X$. Because the set

$$\boldsymbol{U} \wedge M = \{A \wedge M \mid A \in \boldsymbol{U}\} \tag{1}$$

is a set ring, the pair $(M, \boldsymbol{U} \wedge M)$ is a measurable space, called *measurable subspace* of $(X, \boldsymbol{U})$.

7.5 **Proposition** a) If $f : (X, \boldsymbol{U}) \to \boldsymbol{B}_2$ is measurable, then its restriction $f_{|M} : (M, \boldsymbol{U} \wedge M) \to \boldsymbol{B}_2$ is measurable.

b) If $g : (M, \boldsymbol{U} \wedge M) \to \boldsymbol{B}_2$ is measurable, then it can be prolonged to a measurable function $f : (X, \boldsymbol{U}) \to \boldsymbol{B}_2$.

7.6 Let us suppose that $(X, \boldsymbol{U})$ is a measurable space, $f : (X, \boldsymbol{U}) \to \boldsymbol{B}_2$ is a measurable function and $\mu : \boldsymbol{U} \to \boldsymbol{B}_2$ is a measure. The number $\mu(supp\ f)$ is called the *integral of* f *relative to* μ and is noted with $\int f \cdot d\mu$.

7.7 Let $f_n, f \in \boldsymbol{U}', n \in \boldsymbol{N}$.

a) If

$$supp\ f_0 \subset supp\ f_1 \subset supp\ f_2 \subset ... \tag{1}$$

$$\bigvee_{n \in \boldsymbol{N}} supp\ f_n = supp\ f \tag{2}$$

then we say that f_n *converges*, or *tends increasingly to* f and this fact is noted with $f_n \uparrow f$.

b) If

$$supp\ f_0 \supset supp\ f_1 \supset supp\ f_2 \supset ... \tag{1}$$

$$\bigwedge_{n \in \boldsymbol{N}} supp\ f_n = supp\ f \tag{2}$$

then we say that f_n *converges*, or *tends decreasingly to* f and this fact is noted with $f_n \downarrow f$.

c) In one of the situations from a), b) we say that f_n *converges*, or *tends monotonously to* f and the notation is $f_n\ \mathbf{b}\ f$.

7.8 Let us suppose that $f, g : (X, \boldsymbol{U}) \to \boldsymbol{B}_2$ are measurable and that $\mu : \boldsymbol{U} \to \boldsymbol{B}_2$ is a measure. We say that f and g are *equal almost everywhere* and we write this fact with

$$f = g \quad \text{a.e.} \tag{1}$$

if

$$\mu(\{x \mid f(x) \neq g(x)\} = \mu(supp\ f\ \Delta\ supp\ g) = 0 \tag{2}$$

or, in an equivalent manner, if

$$\mu(supp\ f) = \mu(supp\ g) \tag{3}$$

7.9 **Proposition** The function $\boldsymbol{U}' \ni f\ \mathbf{a}\ \int f \cdot d\mu \in \boldsymbol{B}_2$ satisfies the following properties:

a) it is linear

b) $f_n\ \mathbf{b}\ f \Rightarrow \int f_n \cdot d\mu \to \int f \cdot d\mu$

c) $f = g$ a.e. $\Leftrightarrow \int f \cdot d\mu = \int g \cdot d\mu$

where $f_n, f, g \in \boldsymbol{U}', n \in \boldsymbol{N}$.

Proof b) is a restatement of 4.2 a) and 4.3 a).

7.10 **Corollary** If $f_n \in \boldsymbol{U}', n \in \boldsymbol{N}$ converges to 0 decreasingly, then

$$\int f_n \cdot d\mu \to 0 \qquad (1)$$

7.11 Let $f_n, f : (X, \boldsymbol{U}) \to \boldsymbol{B}_2, n \in \boldsymbol{N}$ measurable and $\mu : \boldsymbol{U} \to \boldsymbol{B}_2$ a measure. We say that f_n *tends to* f *in measure* and we note this property with $f_n \underset{\mu}{\to} f$ if

$$\int f_n \cdot d\mu \to \int f \cdot d\mu \qquad (2)$$

7.12 Let $f, \chi_A : X \to \boldsymbol{B}_2$ two functions, where f is arbitrary and χ_A is the characteristic function of the set $A \subset X$. If $A \wedge supp\ f \in \boldsymbol{U}$ - condition that is called ***of integrability***- then the number

$$\int_A f \cdot d\mu \overset{def}{=} \int (\chi_A \cdot f) \cdot d\mu \qquad (1)$$

is called the ***integral of*** f ***, on*** A ***, relative to*** μ.

7.13 The function $f \cdot \mu : \boldsymbol{U} \to \boldsymbol{B}_2$ defined by

$$(f \cdot \mu)(A) = \int_A f \cdot d\mu \qquad (1)$$

where $A, supp\ f \in \boldsymbol{U}$ is a measure, that coincides with the restriction of μ at $supp\ f$.

8. Riemann Integrals

8.1 We end the paper with a short paragraph that introduces the Riemann integrals of the $f : \boldsymbol{R} \to \boldsymbol{B}_2$ functions (generalizations are possible to $f : \boldsymbol{R}^n \to \boldsymbol{B}_2$ functions). The main feature for this type of integral is considering the set ring $R_f(\boldsymbol{R})$ and the finite Boolean measure (see 3.8) $\mu_f^{\boldsymbol{R}} : R_f(\boldsymbol{R}) \to \boldsymbol{B}_2$.

8.2 For the set $A \subset \boldsymbol{R}$, the property $A \wedge supp\ f \in R_f(\boldsymbol{R})$ (see 7.12) is called the condition of ***Riemann integrability of*** f ***on*** A. If it is fulfilled, we say that f is ***Riemann integrable***, or ***integrable in the sense of Riemann, on*** A .

8.3 **Special cases** for 8.2. a) $[[a,b)) \wedge supp\ f \in R_f(\boldsymbol{R})$ (see 3.12 (1) for the definition of $I_{[[a,b))}$), $a, b \in \boldsymbol{R} \vee \{\infty\}$. These functions are called ***left integrable*** (in the sense of Riemann) ***from*** a ***to*** b .

b) $\boldsymbol{R} \wedge supp\ f \in R_f(\boldsymbol{R})$ (see 3.12 (2) for the definition of I_∞). These functions are called ***integrable*** (in the sense of Riemann).

c) $\forall a, b \in \boldsymbol{R}, (a,b) \wedge supp\ f \in R_f(\boldsymbol{R})$ (see 5.4, 5.5, 5.16 for the definition of $I_{Loc}^{(1)}$). These functions are called ***locally integrable*** (in the sense of Riemann) and they have a locally finite support.

d) $\forall a, b \in \boldsymbol{R} \vee \{\infty\}, (a,b) \wedge supp\ f \in R_f(\boldsymbol{R})$ defines the $\boldsymbol{B}_2$-algebra of functions I_{Sup}. We say about these functions that they are ***left integrable*** (in the sense of Riemann) and that they have the support ***superiorly finite***, dual notion to that of inferiorly finite set that was defined at 3.9.

8.4 If f is Riemann integrable on A, then the number (see 7.12 (1))

$$\int_A f \cdot d\mu_f^{\boldsymbol{R}} = \mu_f^{\boldsymbol{R}}(A \wedge supp\ f) = \underset{x \in A}{\Xi} f(x) \quad (1)$$

is called the ***integral, in the sense of Riemann, of*** f ***, on*** A.

8.5 **Special cases** for 8.4 a) $f \in I_{[[a,b))}$, $a, b \in \boldsymbol{R} \vee \{\infty\}$; the integral $\int_{[[a,b))} f \cdot d\mu_f^{\boldsymbol{R}}$ is noted with $\int_a^{b} {}^{-} f$ and is called the ***left integral*** (in the sense of Riemann) ***of*** f ***from*** a ***to*** b.

b) $f \in I_\infty$; the integral $\int_{\boldsymbol{R}} f \cdot d\mu_f^{\boldsymbol{R}}$ is usually noted with $\int_{-\infty}^{\infty} f$ and is called the ***integral*** (in the sense of Riemann) ***of*** f.

8.6 The cases 8.3 a) and 8.5 a) have right duals, that refer to symmetrical intervals of the form $((a,b]]$, $a, b \in \{-\infty\} \vee \boldsymbol{R}$ (see 6.6).

8.7 We define the subring of sets $Sym' \subset Sym^-$ to be the one that is generated by the symmetrical intervals $[[a,b))$, $a, b \in \boldsymbol{R}$ (at Sym^- we had $[[a,b))$, $a, b \in \boldsymbol{R} \vee \{\infty\}$).

8.8 a) Let us suppose that $f \in I_{Loc}^{(1)}$. Then the measure $f \cdot \mu_f^{\boldsymbol{R}} : Sym' \to \boldsymbol{B}_2$ (see 7.13) is called the ***indefinite integral*** of f.

b) The function $F^- : \boldsymbol{R} \to \boldsymbol{B}_2$, which is defined in the next manner:

$$F^-(t) = f \cdot \mu_f^{\boldsymbol{R}}([[a,t))), t \in \boldsymbol{R} \quad (1)$$

where $a \in \boldsymbol{R}$ is a parameter is called the ***left primitive*** of f.

c) The left primitive $F^-(t)$ has a left limit and it is left continuous in any $t \in \boldsymbol{R}$.

8.9 If at 8.8 $f \in I_{Sup}$ (where $I_{Sup} \subset I_{Loc}^{(1)}$), then $f \cdot \mu_f^{\boldsymbol{R}}$ is extended to Sym^- and F^- is extended to $\boldsymbol{R} \vee \{\infty\}$, by left continuity in the point ∞. $f \cdot \mu_f^{\boldsymbol{R}}$ is in this situation the left Lebesgue-Stieltjes measure associated to F^- (see 6.3).

8.10 Together with the duals in the left-right sense that have appeared having their origin in the order of $\boldsymbol{R}$, the previous notions have also another type of duality, so called in the algebraical sense, resulting by the replacement of 0 with 1 and viceversa, to be compared, from the table 1.1, the laws $'\oplus'$ and $'\otimes'$. For example, the algebraical dual of $\int_a^{b} {}^{-} f$ is defined like this:

$$\int_a^{b} {}^{*-} f = \bigotimes_{x \in [[a,b))} f(x) \tag{1}$$

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