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Binary Valued Functions

Serban E. Vlad
str. Zimbrului, Nr.3, Bl.PB68, Et.2, Ap.11, 3700, Oradea, Romania
E-mail: serban_e_vlad@yahoo.com

In the paper 'Pseudo-Boolean Field Lines' presented by the author at the National Conference on Geometry and Topology in Timisoara, October, 1989, there were introduced the differential equations of the asynchronous automata, i.e. a model for the behavior of the Boolean circuits. The present work enlarges the mathematical frame of the context by studying the pseudo-boolean derivatives and integrals.

1. Preliminaries

1.1 $\boldsymbol{B}_2 = \{0,1,\oplus,\cdot\}$ is the binary Boole algebra, where $'\oplus'$ is the modulo 2 sum and $'\cdot'$ is the product. It has the discrete topology, i.e. each subset is open.

1.2 For some function $x:\boldsymbol{R}\to\boldsymbol{B}_2$, the *support* set of x is defined by

$$supp\ x = \{t \mid x(t) = 1\}$$

1.3 A *real set ring* is a set R of subsets of $\boldsymbol{R}$ which is closed under the symmetrical difference $'\Delta'$ and under the intersection $'\wedge'$. The neuter element is $\emptyset \in R$.

For example, the set R_f of the finite subsets of $\boldsymbol{R}$ is a real set ring.

1.4 There are defined the elementary functions $\delta, \eta, \eta^* : \boldsymbol{R}\to\boldsymbol{B}_2$

$$\delta(t) = \begin{cases} 1, t = 0 \\ 0, t \neq 0 \end{cases}, t \in \boldsymbol{R}$$

$$\eta(t) = \begin{cases} 1, t \geq 0 \\ 0, t < 0 \end{cases}, t \in \boldsymbol{R}$$

$$\eta^*(t) = \begin{cases} 1, t \leq 0 \\ 0, t > 0 \end{cases}, t \in \boldsymbol{R}$$

the function $\mu_f : R_f \to \boldsymbol{B}_2$

$$\mu_f(A) = \begin{cases} 1, if\ \mid A \mid is\ odd \\ 0, if\ \mid A \mid is\ even \end{cases}, A \in R_f$$

and, for some $E \subset \boldsymbol{R}$

$$\bigcup_{\xi\in E} x(\xi) = \begin{cases} 1, if\ \exists \xi \in E, x(\xi) = 1 \\ 0, else \end{cases} \text{ (the reunion)}$$

$$\bigcap_{\xi\in E} x(\xi) = \begin{cases} 0, if\ \exists \xi \in E, x(\xi) = 0 \\ 1, else \end{cases} \text{ (the intersection)}$$

If $E \wedge supp\ x \in R_f$, then it is also defined

$$\underset{\xi\in E}{\Xi} x(\xi)=\mu_f(E\wedge supp\ x) \text{ (the modulo 2 summation)}$$

1.5 For the sequence $x:\boldsymbol{N}\to\boldsymbol{B}_2$, $x_n=x(n)$ and $x^0\in\boldsymbol{B}_2$, we have the convergence

$$\lim_{n\to\infty} x_n=x^0 \Leftrightarrow \exists N\in\boldsymbol{N},\forall n\geq N, x_n=x^0$$

1.6 For $x:\boldsymbol{R}\to\boldsymbol{B}_2$, there are defined the *left* $x(t-0)$ and the *right* $x(t+0)$ *limits* of x in $t\in\boldsymbol{R}$ by

$$\exists\varepsilon>0,\forall\xi\in(t-\varepsilon,t),x(\xi)=x(t-0)$$
$$\exists\varepsilon>0,\forall\xi\in(t,t+\varepsilon),x(\xi)=x(t+0)$$

1.7 For any $x:\boldsymbol{R}\to\boldsymbol{B}_2$, there are defined the *inferior* $\underline{x(t-0)}$ and the *superior* $\overline{x(t-0)}$ *left limit* of x in t by

$$\underline{x(t-0)}=\bigcap_{\xi\in(t-0,t)} x(\xi)$$
$$\overline{x(t-0)}=\bigcup_{\xi\in(t-0,t)} x(\xi)$$

(the previous left limits from the right hand terms always exist). We have:

$$\underline{x(t-0)}=\overline{x(t-0)}\Rightarrow\underline{x(t-0)}=\overline{x(t-0)}=x(t-0)$$

and similarly for the right limits.

There are obviously defined the functions with left limit, respectively with right limit (in t) and the (inferior, superior) left, respectively right limit functions.

1.8 Let $\chi_A:\boldsymbol{R}\to\boldsymbol{B}_2$ be the characteristic function of $A\subset\boldsymbol{R}$. Then for $x,y:\boldsymbol{R}\to\boldsymbol{B}_2$ there are true:

$$x=\chi_{supp\ x}$$
$$x\oplus y=\chi_{supp\ x\ \Delta\ supp\ y}$$
$$x\cdot y=\chi_{supp\ x\wedge supp\ y}$$

1.9 There are defined the *symmetrical intervals*:

$$[[a,b))=[a,b)\vee[b,a)$$
$$((a,b]]=(a,b]\vee(b,a],\ a,b\in\boldsymbol{R}$$

2. Derivatives

2.1 For $x:\boldsymbol{R}\to\boldsymbol{B}_2$ there are defined the *inferior left derivative* $\underline{D}x(t)$, the *superior left derivative* $\overline{D}x(t)$ and the *left derivative* $Dx(t)$ of x in t by:

$$\underline{D}x(t)=\underline{x(t-0)}\oplus x(t)$$
$$\overline{D}x(t)=\overline{x(t-0)}\oplus x(t)$$
$$Dx(t)=x(t-0)\oplus x(t)$$

There are obviously defined the left and the right derivable functions (in t), respectively the (inferior, superior) left and right derivative functions.

2.2 **Remark** We have two kind of dualities here: one coming from $\boldsymbol{B}_2,\bigcap,\bigcup$ (i.e. inferior and superior, $\underline{(\ \ )},\overline{(\ \ )}$) and the other coming from $\boldsymbol{R},<,>$ (i.e. left and right, $(\ \ ),(\ \ )^*$). For economy, we shall not define all the notions when we shall consider that they result from the context. For example, in this moment we shall only indicate the notations $\underline{D}^*, \overline{D}^*, D^*$ for the right derivatives.

2.3 **Proposition** a) δ, η, η^*, c are left derivable and right derivable, where c is the constant function and

$$D\eta(t) = D\delta(t) = \delta(t)$$
$$D^*\eta^*(t) = D^*\delta(t) = \delta(t)$$
$$D\eta^*(t) = D^*\eta(t) = Dc(t) = D^*c(t) = 0$$

b) $$\underline{D}cx(t) = c\underline{D}x(t),\ \overline{D}cx(t) = c\overline{D}x(t)$$

for any $x : \boldsymbol{R} \to \boldsymbol{B}_2$ and if x, y are left derivable, then $x \oplus y$ and xy are left derivable and

$$D(x \oplus y)(t) = Dx(t) \oplus Dy(t)$$
$$D(xy)(t) = x(t)Dy(t) \oplus y(t)Dx(t) \oplus Dx(t)Dy(t)$$

c) If x is left derivable, then Dx is left derivable and

$$DDx(t) = Dx(t)$$

If x is left derivable and Dx is right derivable, then

$$D^*Dx(t) = Dx(t)$$

3. Integrals Relative to Boolean Measures

3.1 The function $x : \boldsymbol{R} \to \boldsymbol{B}_2$ is called R-*integrable*, or R-*measurable* (on $A \subset \boldsymbol{R}$), if $supp\ x \in R$ (if $A \wedge supp\ x \in R$), where R is a real set ring.

3.2 A *Boolean measure* on the real set ring R is a function $\mu : R \to \boldsymbol{B}_2$ satisfying the condition: for any sequence of sets $A_n \in R, n \in \boldsymbol{N}$, if

$$n \neq m \Rightarrow A_n \wedge A_m = \emptyset$$
and
$$\bigvee_{n \in \boldsymbol{N}} A_n \in R$$

then

$$\{n \mid n \in \boldsymbol{N}, \mu(A_n) = 1\} \text{ is finite}$$
and
$$\mu(\bigvee_{n \in \boldsymbol{N}} A_n) = \underset{n \in \boldsymbol{N}}{\Xi} \mu(A_n)$$

For $R = R_f, \mu_f$ is an example of Boolean measure.

3.3 Let $x : \boldsymbol{R} \to \boldsymbol{B}_2$ be R-integrable on A and $\mu : R \to \boldsymbol{B}_2$ a Boolean measure. The number

$$\int_A x D\mu = \mu(A \wedge supp\ x)$$

is called the *integral of* x *on* A *relative to* μ.

3.4 **Theorem** Let R be a real set ring , $\mu : R \to \boldsymbol{B}_2$ a Boolean measure, $A \subset \boldsymbol{R}$ and $x, y : \boldsymbol{R} \to \boldsymbol{B}_2$.

a) If $A = \emptyset$ or $x(t) = 0, t \in \boldsymbol{R}$, then x is R-integrable on A and

$$\int_A x D\mu = \mu(\emptyset) = 0$$

b) If x, y are R-integrable on A, then $x \oplus y$ is R-integrable on A and

$$\int_A (x \oplus y) D\mu = \int_A x D\mu \oplus \int_A y D\mu$$

As a special case, if $x = y \quad \mu - a.e.$ that is if

$$\mu(\{t \mid x(t) \neq y(t)\}) = 0$$

then

$$\int_A x D\mu = \int_A y D\mu$$

c) Let $B \subset \boldsymbol{R}$. If x is R-integrable on A and B, then it is R-integrable on $A \Delta B$ and

$$\int_A x D\mu \oplus \int_B x D\mu = \int_{A \Delta B} x D\mu$$

4. Boole-Stieltjes Integrals

4.1 Let $a \neq b$. A *division* of $[[a,b))$ or $((a,b]]$ is a set

$$d_n : \min(a,b) = t_0 < t_1 < ... < t_n = \max(a,b), n \geq 1$$

and the *norm of the division* is the real number $\nu(d_n)$ given by

$$\nu(d_n) = \max_{i=0,n-1} (t_{i+1} - t_i)$$

A *sequence of divisions* $(d_n)_{n \geq 1}$ is a family of divisions that satisfy

$$d_n \subset d_{n+1}, n \geq 1$$

We say about $(d_n)_{n \geq 1}$ that $\nu(d_n) \to 0$ when $n \to \infty$ if

$$\forall \varepsilon > 0, \exists N \in \boldsymbol{N}, n \geq N \Rightarrow \nu(d_n) < \varepsilon$$

4.2 **Theorem** Let $[[a,b))$ and $x, y : \boldsymbol{R} \to \boldsymbol{B}_2$. The next statements are equivalent:

a) The integral sum

$$\underline{\sigma}_{d_n}(x,y) = \mathop{\Xi}_{i=0}^{n-1} \bigcup_{\xi_i \in [t_i, t_{i+1})} x(\xi_i) \cdot (y(\xi_i) \oplus \bigcap_{\xi_i' \in (\xi_i, t_{i+1})} y(\xi_i'))$$

satisfies the property that for any sequence $(d_n)_{n \geq 1}$ of divisions of $[[a,b))$ with $\nu(d_n) \to 0$ when $n \to \infty$, it converges (as $\boldsymbol{N} \to \boldsymbol{B}_2$ sequence) to a limit that does not depend on $(d_n)_{n \geq 1}$

b) $$[[a,b)) \wedge supp\ x \wedge supp\ \underline{D}^* y \in R_f$$

4.3 In any of the above situations, x is called *inferiorly left Boole-Stieltjes integrable on* $[[a,b))$ *relative to* y and the number

$$\int_a^b x \underline{D}^* y = \lim_{\substack{n \to \infty \\ \nu(d_n) \to 0}} \underline{\sigma}_{d_n}(x,y) = \mu_f([[a,b)) \wedge supp\ x \wedge supp\ \underline{D}^* y) = \int_{[[a,b))} x \underline{D}^* y D\mu_f$$

is the *inferior left Boole-Stieltjes integral of* x *on* $[[a,b))$ *relative to* y. By definition

$$\int_a^a x \underline{D}^* y = \mu_f(\emptyset) = 0$$

Similarly, there are defined:

$$\overline{\sigma}_{d_n}(x,y)=\underset{i=0}{\overset{n-1}{\Xi}}\bigcup_{\xi_i\in[t_i,t_{i+1})}x(\xi_i)\cdot(y(\xi_i)\oplus\bigcup_{\xi_i'\in(\xi_i,t_{i+1})}y(\xi_i'))$$

$$\int_a^b x\overline{D}*y=\lim_{\substack{n\to\infty\\ \nu(d_n)\to 0}}\overline{\sigma}_{d_n}(x,y)=\mu_f([[a,b))\wedge supp\ x\wedge supp\ \overline{D}*y)=\int_{[[a,b))}x\overline{D}*yD\mu_f$$

$$\underline{\sigma}^*_{d_n}(x,y)=\underset{i=0}{\overset{n-1}{\Xi}}\bigcup_{\xi_i\in(t_i,t_{i+1}]}x(\xi_i)\cdot(y(\xi_i)\oplus\bigcap_{\xi_i'\in(t_i,\xi_i)}y(\xi_i'))$$

$$\int_a^{b\ *} x\underline{D}y=\lim_{\substack{n\to\infty\\ \nu(d_n)\to 0}}\underline{\sigma}^*_{d_n}(x,y)=\mu_f(((a,b]]\wedge supp\ x\wedge supp\ \underline{D}y)=\int_{((a,b]]}x\underline{D}yD\mu_f$$

$$\overline{\sigma}^*_{d_n}(x,y)=\underset{i=0}{\overset{n-1}{\Xi}}\bigcup_{\xi_i\in(t_i,t_{i+1}]}x(\xi_i)\cdot(y(\xi_i)\oplus\bigcup_{\xi_i'\in(t_i,\xi_i)}y(\xi_i'))$$

$$\int_a^{b\ *} x\overline{D}y=\lim_{\substack{n\to\infty\\ \nu(d_n)\to 0}}\overline{\sigma}^*_{d_n}(x,y)=\mu_f(((a,b]]\wedge supp\ x\wedge supp\ \overline{D}y)=\int_{((a,b]]}x\overline{D}yD\mu_f$$

4.4 If

$$[[a,b))\wedge supp\ x\wedge supp\ \underline{D}*y=[[a,b))\wedge supp\ x\wedge supp\ \overline{D}*y\in R_f$$

then their common value is written $[[a,b))\wedge supp\ x\wedge supp\ D*y$ and we note $\int_a^b xD*y$ for that integral where the attributes inferior and superior are missing.

4.5 The integral sums

$$\underline{\sigma}_{d_n}(1,x)=\underset{i=0}{\overset{n-1}{\Xi}}\bigcup_{\xi_i\in[t_i,t_{i+1})}(x(\xi_i)\oplus\bigcap_{\xi_i'\in(\xi_i,t_{i+1})}x(\xi_i'))$$

$$\varphi_{d_n}(x)=\underset{i=0}{\overset{n-1}{\Xi}}\bigcup_{\xi_i\in[t_i,t_{i+1})}x(\xi_i)$$

give the *inferior left Boole-differential integrals* and the *left Boole-Riemann integrals*:

$$\int_a^b\underline{D}*x=\mu_f([[a,b))\wedge supp\ \underline{D}*x)=\int_{[[a,b))}\underline{D}*xD\mu_f$$

$$\int_a^b x=\mu_f([[a,b))\wedge supp\ x)=\int_{[[a,b))}xD\mu_f$$

4.6 If the $\boldsymbol{N}\to\boldsymbol{B}_2$ sequence

$$a_n=\int_a^n x\underline{D}*y$$

has a limit when $n \to \infty$, then the limit will be noted $\int_a^{\infty} x\underline{D}*y$ etc.

4.7 **Theorem** The next statements are true:

a) If x is inferiorly left integrable relative to y on $[[a,b))$ then it is also inferiorly left integrable relative to y on $[[b,a))$ and

$$\int_a^b x\underline{D}*y = \int_b^a x\underline{D}*y$$

b) Let the function x inferiorly left integrable relative to y on $[[a,b))$. Then it is inferiorly left integrable relative to y on any $[[a',b')) \subset [[a,b))$.

c) If x_1 and x_2 are inferiorly left integrable relative to y on $[[a,b))$, then $x_1 \oplus x_2$ is inferiorly left integrable relative to y on $[[a,b))$ and the integral is additive:

$$\int_a^b (x_1 \oplus x_2)\underline{D}*y = \int_a^b x_1\underline{D}*y \oplus \int_a^b x_2\underline{D}*y$$

d) If x is inferiorly left integrable relative to y on $[[a,b))$ and $[[b,c))$, then it is inferiorly left integrable relative to y on $[[a,c))$ and

$$\int_a^b x\underline{D}*y \oplus \int_b^c x\underline{D}*y = \int_a^c x\underline{D}*y$$

e) Let the function x be inferiorly left integrable relative to y on $[a,b), b > a$. The functions

$$\varphi(\varepsilon) = \int_a^{a+\varepsilon} x\underline{D}*y, \psi(\varepsilon) = \int_{b-\varepsilon}^b x\underline{D}*y$$

exist for $\varepsilon \in (0,b-a)$, they have a right limit in the origin and

$$\int_a^{a+0} x\underline{D}*y = x(a)\underline{D}*y(a)$$

$$\int_{b-0}^b x\underline{D}*y = 0$$

f) In a similar way to e), let us suppose that the function x is inferiorly left integrable relative to y on $[a,b)$ and $a < b$. The functions

$$\varphi(\varepsilon) = \int_{a+\varepsilon}^b x\underline{D}*y, \psi(\varepsilon) = \int_a^{b-\varepsilon} x\underline{D}*y$$

are defined for $\varepsilon \in (0,b-a)$ and they have a right limit in the origin. There hold

$$\int_{a+0}^b x\underline{D}*y = \int_a^b x\underline{D}*y \oplus x(a)\underline{D}*y(a)$$

$$\int_a^{b-0} x\underline{D}*y = \int_a^b x\underline{D}*y$$

g) If x is left integrable on $[[a,b))$ relative to y_1 and y_2, then x is left integrable on $[[a,b))$ relative to $y_1 \oplus y_2$ and it is true

$$\int_a^b xD^* y_1 \oplus \int_a^b xD^* y_2 = \int_a^b xD^*(y_1 \oplus y_2)$$

h) Let us suppose that x, y have right limits on $[a,b)$ and that the sets

$$[a,b) \wedge supp\ D^*(xy)$$
$$[a,b) \wedge supp\ D^* x \wedge supp\ D^* y$$
$$[a,b) \wedge supp\ x \wedge supp\ D^* y$$
$$[a,b) \wedge supp\ D^* x \wedge supp\ y$$

are finite. In this situation, the next formula is true:

$$\int_a^b D^*(xy) \oplus \int_a^b D^* xD^* y = \int_a^b xD^* y \oplus \int_a^b yD^* x$$

4.8 A function $h:[[a',b')) \to [[a,b))$ is called *right continuous* if for any $t = h(t')$ and any $\varepsilon > 0$, there exists $\varepsilon' > 0$ with the property that

$$h((t',t'+\varepsilon')) \subset (t, t+\varepsilon)$$

If h is a bijection and h, h^{-1} are right continuous, then it is called *right homeomorphism*.

4.9 **Theorem** Let x be left integrable on $[[a,b))$ relative to y and the right homeomorphism $h:[[a',b')) \to [[a,b))$. Then the integral $\int_{a'}^{b'} (x \circ h)D^*(y \circ h)$ exists and

$$\int_a^b xD^* y = \int_{a'}^{b'} (x \circ h)D^*(y \circ h)$$

4.10 If we note with

$$x_\tau(t) = x(t-\tau)$$

then it is true

$$\int_a^b \delta_\tau = \eta(\tau - a) \oplus \eta(\tau - b) = \chi_{[[a,b))}(\tau)$$

4.11 In the integral $\int_a^b x$, let us put $a = \alpha$ a parameter and $b = t$ the real variable. The $\boldsymbol{R} \to \boldsymbol{B}_2$ function

$$\int x(t) = \int_\alpha^t x$$

is called the *left primitive of* x. We say that x *has a left primitive.*

4.12 The function x has a left primitive if it is of the form

$$x(t) = \underset{i \in I}{\Xi}\ \delta(t-i)$$

where for any $a < b$, the set $(a,b) \wedge I$ is finite. As a special case, for $I = \emptyset$, the null function has a left primitive.

Any two left primitives of x differ by a constant.

4.13 The formula of Leibniz-Newton is true: if x has a left primitive then for any $a, b \in \boldsymbol{R}$, x is left Boole-Riemann integrable on $[[a,b))$ and we have

$$\int_a^b x = \int x(a) \oplus \int x(b)$$

4.14 Let us suppose that x has a left primitive; then its left primitive is left derivable and right derivable and we have:

$$D\int x(t) = 0,\ D^*\int x(t) = x(t)$$

4.15 Let x with left limits and right limits. Then the functions Dx and $D^* x$ have left primitives and right primitives and

$$\int D^* x(t) \oplus \int^* Dx(t) = x(t) \oplus c,\ c \in \boldsymbol{B}_2$$

4.16 Let $x, y : \boldsymbol{R} \to \boldsymbol{B}_2$, where x has a finite support and y has left limits and right limits. Then the function

$$h(t) = \int_{-\infty}^{\infty} x_t y$$

is defined and has left limits and right limits.

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