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BOOLEAN DYNAMICAL SYSTEMS

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Abstract The asynchronous circuits from the digital electrical engineering are modeled by the so-called asynchronous systems. An autonomous asynchronous system consists in a set X of 'nice' $\mathbf{R} \to \{0,1\}^n$ functions $n \geq 1$, called signals, representing non-deterministically the models of the tensions that describe the behavior of an asynchronous circuit without inputs. A special case of such a system X is the one when a function $\Phi : \{0,1\}^n \to \{0,1\}^n$ is given such that any $x \in X$ fulfills a 'differential' equation involving Φ. In such conditions, by analogy with the 'real' semi-dynamical systems, the Boolean dynamical systems may be defined.

Our paper defines the nullclins, the motions and their speed, the invariant subsets of $\{0,1\}^n$, the attraction, the limit cycles, the evolutions of $x \in X$, the stable manifolds, the Huffman systems and several properties of invariance which are very important in electrical engineering (such as the delay insensitivity). The relation with the discrete time systems is also suggested.

1. PRELIMINARIES

Notation 1.1. *Let M be an arbitrary set. The following notation will be useful*

$$P^*(M) = \{M' | M' \subset M, M' \neq \emptyset\}.$$

Notation 1.2. *The set of the subsequences of the natural numbers set $\mathbf{N}$ is denoted by $Sub(\mathbf{N})$*

$$Sub(\mathbf{N}) = \{(j_k) | j_k \in \mathbf{N}, k \in \mathbf{N}, j_0 < ... < j_k < ...\}.$$

Definition 7. *The set* $\mathbf{B}=\{0,1\}$ *is endowed with the order* $0\leq 1$ *and with the usual laws* $^{-},\cdot,\cup,\oplus$. *It is called the* ***binary Boole algebra****.*

Remark 3. *The set* $\mathbf{B}$ *is a Boole algebra indeed relative to* $^{-},\cdot,\cup$ *and it is also a field relative to* $\oplus,\cdot$. $\mathbf{B}^n$, *together with the sum* $\oplus$ *made coordinatewise and with the scalar product* $\cdot$, *is a linear space over* $\mathbf{B}$.

Notation 1.3. *We denote by*

$$\varepsilon^i=(0,...,\underset{i}{1},...,0),i=\overline{1,n}$$

the vectors of the canonical base of $\mathbf{B}^n$.

Notation 1.4. *The modulo 2 summation of the vectors* $v^j\in\mathbf{B}^n,j\in J$ *(J must be a finite set) is denoted by* $\underset{j\in J}{\Xi}v^j$. *By definition* $\underset{j\in\emptyset}{\Xi}v^j=0$.

Definition 8. *If* $\alpha:\mathbf{N}\rightarrow\mathbf{B}^n$ *is a sequence,* $\alpha(k)\overset{not}{=}\alpha^k,k\in\mathbf{N}$, *then its* ***limit*** $\lim\limits_{k\rightarrow\infty}\alpha^k\in\mathbf{B}^n$ *is defined by the property*

$$\exists k'\in\mathbf{N},\forall k''\geq k',\alpha^{k''}=\lim_{k\rightarrow\infty}\alpha^k.$$

Definition 9. *Consider the function* $x:\mathbf{R}\rightarrow\mathbf{B}^n$. *Its* ***initial value*** $\lim\limits_{t\rightarrow-\infty}x(t)\in\mathbf{B}^n$ *(also denoted by* $x(-\infty+0)$*) and its* ***final value*** $\lim\limits_{t\rightarrow\infty}x(t)\in\mathbf{B}^n$ *(denoted sometimes by* $x(\infty-0)$*) are defined by*

$$\exists t'\in\mathbf{R},\forall t''\leq t',x(t'')=\lim_{t\rightarrow-\infty}x(t),$$

$$\exists t'\in\mathbf{R},\forall t''\geq t',x(t'')=\lim_{t\rightarrow\infty}x(t).$$

Notation 1.5. *We denote by* $\tau^d:\mathbf{R}\rightarrow\mathbf{R}$, $d\in\mathbf{R}$ *the* ***translation***

$$\forall t\in\mathbf{R},\tau^d(t)=t-d.$$

Thus for any $x:\mathbf{R}\rightarrow\mathbf{B}^n$, *we denote by* $x\circ\tau^d:\mathbf{R}\rightarrow\mathbf{B}^n$ *the function*

$$\forall t\in\mathbf{R},(x\circ\tau^d)(t)=x(t-d).$$

Definition 10. *The* ***characteristic function*** $\chi_A : \mathbf{R} \to \mathbf{B}$ *of the set* $A \subset \mathbf{R}$ *is given by*

$$\forall t \in \mathbf{R}, \chi_A(t) = \begin{cases} 1, & t \in A; \\ 0, & otherwise. \end{cases}$$

Notation 1.6. *We use the following notation*

$$Seq = \{(t_k) | t_k \in \mathbf{R}, k \in \mathbf{N},\ t_0 < ... < t_k < ... \text{ is unbounded from above}\}.$$

Definition 11. *The* ***cyclic values*** $\nu \in \mathbf{B}^n$ *and the* ***co-cyclic values*** $\mu \in \mathbf{B}^n$ *of* $x : \mathbf{R} \to \mathbf{B}^n$ *are defined by*

$$\exists (t_k) \in Seq, \forall k \in \mathbf{N}, x(t_k) = \nu,$$

$$\exists (t_k) \in Seq, \forall k \in \mathbf{N}, x(-t_k) = \mu.$$

Remark 4. *The initial and the final values of* $x : \mathbf{R} \to \mathbf{B}^n$ *may not exist and if any of them exists, then it is unique. The cyclic and the co-cyclic values of* x *always exist and they are not unique in general. If* x *has a unique cyclic (co-cyclic) value* ν (μ)*, then* $\nu = x(\infty - 0)$ $(\mu = x(-\infty + 0))$*.*

Definition 12. *A function* $x : \mathbf{R} \to \mathbf{B}^n$ *is called* $n-$***signal****, shortly* ***signal*** *if* $\mu \in B^n$ *and* $(t_k) \in Seq$ *exist so that*

$$x(t) = \mu \cdot \chi_{(-\infty, t_0)}(t) \oplus x(t_0) \cdot \chi_{[t_0, t_1)}(t) \oplus ... \oplus x(t_k) \cdot \chi_{[t_k, t_{k+1})}(t) \oplus ... \quad (1)$$

where in (1) we have abusively used the same symbols $\cdot, \oplus$ *for the laws that are induced by those of* $\mathbf{B}$*. The set of the* $n-$*signals is denoted by* $S^{(n)}$ *and instead of* $S^{(1)}$ *we usually write* S*.*

Definition 13. *Let* $x \in S^{(n)}$ *be given by (1). Its* ***left limit*** $x(t-0)$ *is the* $\mathbf{R} \to \mathbf{B}^n$ *function defined as*

$$x(t-0) = \mu \cdot \chi_{(-\infty, t_0]}(t) \oplus x(t_0) \cdot \chi_{(t_0, t_1]}(t) \oplus ... \oplus x(t_k) \cdot \chi_{(t_k, t_{k+1}]}(t) \oplus ... \quad (2)$$

Definition 14. *By definition, the* ***left derivative*** *of* $x \in S^{(n)}$ *is the function* $Dx : \mathbf{R} \to \mathbf{B}^n$,

$$Dx(t) = x(t-0) \oplus x(t).$$

Remark 5. *From (1) and (2) we infer*

$$Dx(t) = (\mu \oplus x(t_0)) \cdot \chi_{\{t_0\}}(t) \oplus (x(t_0) \oplus x(t_1)) \cdot \chi_{\{t_1\}}(t) \oplus ...$$
$$... \oplus (x(t_{k-1}) \oplus x(t_k)) \cdot \chi_{\{t_k\}}(t) \oplus ...$$

On the other hand the right limit and the right derivative of $x \in S^{(n)}$*:*

$$x(t+0) = x(t),$$
$$D^*x(t) = x(t+0) \oplus x(t) = 0$$

will not be important in our work.

Definition 15. *Let be* $U \in P^*(S^{(m)})$*. A multi-valued function* $f : U \to P^*(S^{(n)})$ *is called* ***asynchronous system****, shortly* ***system****. Any* $u \in U$ *is called* ***an(admissible) input*** *and the functions* $x \in f(u)$ *are called (****the possible) states****.*

Definition 16. *Any of*

a) a system f *with* $\exists X \in P^*(S^{(n)}), \forall u \in U, f(u) = X$,

b) a system f *having the property that* U *has a single element,*

c) a set $X \in P^*(S^{(n)})$

is called ***an autonomous (asynchronous) system****.*

Remark 6. *The concept of system originates in the modeling of the asynchronous circuits. The multi-valued character of the cause-effect association is due to the statistical fluctuations in the fabrication process, the variations in the ambiental temperature, the power supply etc. Sometimes the systems are given by equations and/or inequalities.*

An autonomous system is interpreted as a system without input and we prefer to use for this the concept from Definition 16 c).

2. SEMI-DYNAMICAL SYSTEMS. AN EXAMPLE

Definition 17. *Let* M *be a set. A family of mappings* $\Upsilon_t : M \to M, t \in [0, \infty)$ *that fulfills the conditions*

$$\Upsilon_0 = 1_M, \tag{3}$$

$$\forall t \in [0,\infty), \forall t' \in [0,\infty), \Upsilon_{t+t'} = \Upsilon_t \circ \Upsilon_{t'} \tag{4}$$

is called ***a semi-group of maps*** *of M with one parameter.*

Definition 18. *A couple $\gamma = (M, \Upsilon = (\Upsilon_t)_{t\in[0,\infty)})$, where M is some set and $\Upsilon : [0,\infty) \to M^M, \Upsilon(t) \overset{not}{=} \Upsilon_t, t \in [0,\infty)$ defines a semi-group of maps of M with one parameter, is called a* ***semi-dynamical system*** *and the parameter t is called* ***time***. *The set M is called the* ***phase space*** *or the* ***state space*** *and the points $x \in M$ are called* ***phases*** *or* ***states***.

Remark 7. *A process is called deterministic in [1] if its whole future evolution is uniquely determined by the present evolution. The semi-dynamical systems are mathematical models of the deterministic processes.*

Our purpose is that of adapting Definition 17 and Definition 18 to the study of the asynchronous circuits that are real time, binary, non-deterministic processes and we consider the next

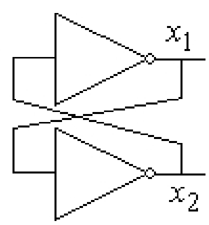

Fig. 1.

Example 4. *In Figure 1 the logical gates compute the Boolean function $\Phi : \mathbf{B}^2 \to \mathbf{B}^2$,*

$$\forall(\mu_1,\mu_2) \in \mathbf{B}^2, \Phi(\mu_1,\mu_2) = (\overline{\mu_2}, \overline{\mu_1}).$$

Let be $(t_k) \in Seq$ and the function $\rho : \mathbf{R} \to \mathbf{B}^2$,

$$\rho(t) = (1,0) \cdot \chi_{\{t_0\}}(t) \oplus (1,1) \cdot \chi_{\{t_1\}}(t) \oplus (0,1) \cdot \chi_{\{t_2\}}(t) \oplus$$

$$\oplus (1,0) \cdot \chi_{\{t_3\}}(t) \oplus (1,1) \cdot \chi_{\{t_4\}}(t) \oplus (0,1) \cdot \chi_{\{t_5\}}(t) \oplus ...$$

showing how Φ_1 *and* Φ_2 *are computed:* Φ_1 *at* t_0, Φ_1 *and* Φ_2 *at* t_1, Φ_2 *at* t_2, Φ_1 *at* t_3, Φ_1 *and* Φ_2 *at* t_4, Φ_2 *at* t_5,... *Denote by* $(\mu_1, \mu_2) \in \mathbf{B}^2$ *the initial state. We can immediately prove that*

$$\begin{array}{ll}
at\ t_0: & (\Phi_1(\mu_1,\mu_2),\mu_2) = (\overline{\mu_2},\mu_2) \\
at\ t_1: & (\Phi_1(\overline{\mu_2},\mu_2),\Phi_2(\overline{\mu_2},\mu_2)) = (\overline{\mu_2},\mu_2) \\
at\ t_2: & (\overline{\mu_2},\Phi_2(\overline{\mu_2},\mu_2)) = (\overline{\mu_2},\mu_2) \\
at\ t_3: & (\Phi_1(\overline{\mu_2},\mu_2),\mu_2) = (\overline{\mu_2},\mu_2) \\
at\ t_4: & (\Phi_1(\overline{\mu_2},\mu_2),\Phi_2(\overline{\mu_2},\mu_2)) = (\overline{\mu_2},\mu_2) \\
at\ t_5: & (\overline{\mu_2},\Phi_2(\overline{\mu_2},\mu_2)) = (\overline{\mu_2},\mu_2)
\end{array}$$

...

thus the behavior of the circuit is modeled by the function

$$x(t) = (\mu_1, \mu_2) \cdot \chi_{(-\infty,t_0)}(t) \oplus (\overline{\mu_2}, \mu_2) \cdot \chi_{[t_0,\infty)}(t).$$

3. PROGRESSIVE SEQUENCES AND PROGRESSIVE FUNCTIONS

Definition 19. *The sequence* $\alpha : \mathbf{N} \to \mathbf{B}^n, \alpha(k) \overset{not}{=} \alpha^k, k \in \mathbf{N}$ *is* ***progressive*** *if* $\forall i \in \{1, ..., n\}$, *the set*

$$\{k | k \in \mathbf{N}, \alpha_i^k = 1\}$$

is infinite. The set of the $\mathbf{N} \to \mathbf{B}^n$ *progressive sequences is denoted by* Π_n.

Definition 20. *The function* $\rho : \mathbf{R} \to \mathbf{B}^n$ *is called* ***progressive****, if* $\alpha \in \Pi_n$ *and* $(t_k) \in Seq$ *exist so that*

$$\rho(t) = \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus ... \oplus \alpha^k \cdot \chi_{\{t_k\}}(t) \oplus ... \tag{5}$$

The set of the progressive functions is denoted by P_n.

Remark 8. *In the previous two definitions, the condition that all the sets* $\{k | k \in \mathbf{N}, \alpha_i^k = 1\}$ *are infinite,* $i = \overline{1, n}$ *expresses the idea that the coordinates* $\Phi_i, i = \overline{1, n}$ *of some function* $\Phi : \mathbf{B}^n \to \mathbf{B}^n$ *are computed countably many times, thus their computation time is arbitrary, finite, possibly variable* *(depending on manufacturing fluctuations in delay related parameters, on the temperature, on the tension of the mains etc.) This was anticipated at Example 4.*

Some properties of invariance must be found in this context, of the form $..., \forall \rho \in P_n, ...$ *meaning that for each manufactured instance of a design, for each admissible temperature and for each admissible tension of the mains, we restrict our attention to the information that is common to all of them. Such properties will be presented in Sections 22,...,28.*

4. BOOLEAN DYNAMICAL SYSTEMS

Definition 21. *Let* $\Phi : \mathbf{B}^n \to \mathbf{B}^n, \Phi = (\Phi_1, ..., \Phi_n)$ *be a function (called sometimes vector field). For* $\lambda \in \mathbf{B}^n, \lambda = (\lambda_1, ..., \lambda_n)$ *we define the function* $\Phi^\lambda : \mathbf{B}^n \to \mathbf{B}^n$,

$$\forall \mu \in \mathbf{B}^n, \Phi^\lambda(\mu) = (\overline{\lambda_1} \cdot \mu_1 \oplus \lambda_1 \cdot \Phi_1(\mu), ..., \overline{\lambda_n} \cdot \mu_n \oplus \lambda_n \cdot \Phi_n(\mu))$$

called Φ ***at the power*** λ, *or the* $\lambda-$***iterate*** *of* Φ.

Remark 9. *The role of* λ *in the previous definition is that of showing which coordinate* Φ_i *of* Φ *is computed: if* $\lambda_i = 0$, *then* $\Phi_i^\lambda(\mu) = \mu_i$ *and* Φ_i *is not computed, while if* $\lambda_i = 1$, *then* $\Phi_i^\lambda(\mu) = \Phi_i(\mu)$ *and* Φ_i *is computed,* $i = \overline{1, n}$.

Definition 22. *Let be* $\alpha \in \Pi_n$. *We define the functions* $\Phi^{\alpha^0 ... \alpha^k} : \mathbf{B}^n \to \mathbf{B}^n, k \in \mathbf{N}$ *by*

$$\forall k \in \mathbf{N}, \forall \mu \in \mathbf{B}^n, \Phi^{\alpha^0 ... \alpha^k \alpha^{k+1}}(\mu) = \Phi^{\alpha^{k+1}}(\Phi^{\alpha^0 ... \alpha^k}(\mu)). \tag{6}$$

Definition 23. *Let be* $\rho \in P_n$,

$$\rho(t) = \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus ... \oplus \alpha^k \cdot \chi_{\{t_k\}}(t) \oplus ... \tag{7}$$

where $\alpha \in \Pi_n$ *and* $(t_k) \in Seq$. *We define* Φ ***at the power*** ρ, $\Phi^\rho : \mathbf{R} \to (\mathbf{B}^n)^{\mathbf{B}^n}$ *by* $\forall t \in \mathbf{R}, \forall \mu \in \mathbf{B}^n$,

$$\Phi^\rho(t)(\mu) = \mu \cdot \chi_{(-\infty, t_0)}(t) \oplus \Phi^{\alpha^0}(\mu) \cdot \chi_{[t_0, t_1)}(t) \oplus ... \tag{8}$$

$$... \oplus \Phi^{\alpha^0 ... \alpha^k}(\mu) \cdot \chi_{[t_k, t_{k+1})}(t) \oplus ...$$

Remark 10. *Suppose that (7) is true. A distinction should be made between* $\Phi^\rho(t)(\mu)$ *given by (8) and*

$$\Phi^{\rho(t)}(\mu) = \begin{cases} \Phi^{\alpha^k}(\mu), t = t_k \\ \mu, t \notin \{t_0, ..., t_k, ...\}. \end{cases}$$

Definition 24. *The couple* $\phi = (\mathbf{B}^n, (\Phi^\rho)_{\rho \in P_n})$ *is called* ***a Boolean dynamical system****.* $\mathbf{B}^n$ *is called the* ***phase space****, or the* ***state space*** *and its points* $\mu \in \mathbf{B}^n$ *are called* ***phases****, or* ***states***[1]*. The function* Φ *is called the* ***generator function*** *of* ϕ *and* $\Phi^\rho, \rho \in P_n$ *are called the* ***computations*** *of* Φ*. The domain* $\mathbf{R}$ *of the computations* Φ^ρ *is the* ***time set*** *and* $t \in \mathbf{R}$ *is the* ***time parameter****.*

Remark 11. *We have given up the prefix 'semi' in the terminology of 'Boolean semi-dynamical system' because, in the theory of the asynchronous systems, the time set is always bounded from below (in the sense of the existence of the initial values of the states* $x(-\infty + 0)$ *and of the initial time* t_0*); the possibility of allowing a time set which is unbounded from below corresponds to the prefix 'pseudo' in the terminology of 'Boolean dynamical pseudo-system' and we are not interested in the study of this concept at the moment.*

The major difference between ϕ *(Definition 24) and the usual semi-dynamical systems* γ *(Definition 18) comes from the fact that here the modelled processes are not deterministic, meaning that different ways that* Φ *may be computed exist and they are indicated by the power* ρ *of* Φ^ρ*. We conclude that the deterministic process in* γ *has been replaced by a family of deterministic processes in* ϕ*. The property (3) of no advance in time is replaced by*

$$\Phi^{(0,...,0)} = 1_{\mathbf{B}^n}$$

and the request (4) of advancing time is replaced by (6), (7), (8).

Definition 25. *For* $\mu \in \mathbf{B}^n$ *and* $\rho \in P_n$*, the function* $\Phi^\rho(\cdot)(\mu) : \mathbf{R} \to \mathbf{B}^n$ *is called the* $\rho-$***motion*** *of the point* μ*.*

Definition 26. *We define the* $\rho-$***orbit*** *(or the* $\rho-$***phase trajectory)*** ***of*** μ *in the following way:*

$$Orb_\rho(\mu) = \{\Phi^\rho(t)(\mu) | t \in \mathbf{R}\}.$$

Definition 27. *By definition, we call* ***integral curve*** *the graph*

$$G = \{(t, \Phi^\rho(t)(\mu)) | t \in \mathbf{R}\}.$$

5. REGULAR AUTONOMOUS ASYNCHRONOUS SYSTEMS

Definition 28. *Let be* $\Phi : \mathbf{B}^n \to \mathbf{B}^n$*. The set* $X_\Phi \subset S^{(n)}$,

$$X_\Phi = \{\mu \cdot \chi_{(-\infty,t_0)}(t) \oplus \Phi^{\alpha^0}(\mu) \cdot \chi_{[t_0,t_1)}(t) \oplus ... \oplus \Phi^{\alpha^0 ... \alpha^k}(\mu) \cdot \chi_{[t_k,t_{k+1})}(t) \oplus ...$$

$$| \mu \in \mathbf{B}^n, \alpha \in \Pi_n, (t_k) \in Seq\}$$

is called the ***universal regular autonomous asynchronous system*** *that is* ***generated*** *by* Φ*. The function* Φ *is called the* ***generator function*** *of* X_Φ*.*

Definition 29. *A non-empty set* $X \in P^*(S^{(n)})$ *is called* ***regular autonomous asynchronous system*** *if* $\Phi : \mathbf{B}^n \to \mathbf{B}^n$ *exists so that* $X \subset X_\Phi$*. If this inclusion holds, then the function* Φ *is called the* ***generator function*** *of* X*.*

Remark 12. *In Definitions 28, 29 the terminology is the following: 'universal' means maximal relative to the inclusion and 'regular' means the existence of a generator function* Φ*.*

While the system X_Φ *is associated with the Boolean dynamical system* $\phi = (\mathbf{B}^n, (\Phi^\rho)_{\rho \in P_n})$*, its subsystems* $X \subset X_\Phi$ *may result by requesting that the initial states* μ *run over a subset of* $\mathbf{B}^n$ *only (initial conditions), or perhaps* Φ^ρ *run over a subset of all the computations of* Φ *only (restrictions imposed to the computation time of the coordinate functions* $\Phi_i, i = \overline{1,n}$*).*

Notation 5.1. *Consider a set* $\Theta_0 \in P^*(\mathbf{B}^n)$*. Denote by* $X_\Phi^{\Theta_0} \subset X_\Phi$ *the system*

$$X_\Phi^{\Theta_0} = \{\mu \cdot \chi_{(-\infty,t_0)}(t) \oplus \Phi^{\alpha^0}(\mu) \cdot \chi_{[t_0,t_1)}(t) \oplus ... \oplus \Phi^{\alpha^0 ... \alpha^k}(\mu) \cdot \chi_{[t_k,t_{k+1})}(t) \oplus ...$$

$$|\mu \in \Theta_0, \alpha \in \Pi_n, (t_k) \in Seq\}.$$

Remark 13. *In general the regular autonomous systems do not have a unique generator function*

$$\exists \Phi, \Phi' : \mathbf{B}^n \to \mathbf{B}^n, \Phi \neq \Phi' \text{ and } X_\Phi \cap X_{\Phi'} \neq \emptyset.$$

Such a situation results in the intersection of the systems from Example 6 and Example 7 to follow.

6. EXAMPLES OF REGULAR AUTONOMOUS ASYNCHRONOUS SYSTEMS

Example 5. *Suppose that in Figure 1 the gates are identical, the computation time of the coordinate functions is equal to* 1*, the initial state is* $(0,0)$ *and the initial time instant is* $t_0 = 0$*. Then the model of this circuit is given by the autonomous system*

$$X = \{(1,1) \cdot \chi_{[1,2)\cup[3,4)\cup[5,6)\cup\ldots}\}.$$

Example 6. *Suppose that* $\Phi : \mathbf{B} \to \mathbf{B}$ *is the constant function*

$$\exists \mu^0 \in \mathbf{B}, \forall \mu \in \mathbf{B}, \Phi(\mu) = \mu^0.$$

We identify the constant function $x \in S$ *with the constant* $\mu^0 \in \mathbf{B}$ *and we remark that*

$$X_\Phi = \{\mu^0\} \cup \{\overline{\mu^0} \cdot \chi_{(-\infty,t)} \oplus \mu^0 \cdot \chi_{[t,\infty)} | t \in \mathbf{R}\}.$$

Example 7. *For* $\Phi : \mathbf{B} \to \mathbf{B}$ *the identity function*

$$\forall \mu \in \mathbf{B}, \Phi(\mu) = \mu$$

we have

$$X_\Phi = \mathbf{B}$$

($\mathbf{B}$ *has been identified with the set of the two constant 1-signals). The circuit was drawn in Fig. 2.*

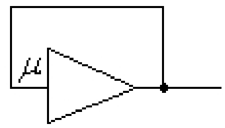

Fig. 2.

Example 8. *We define* $\Phi : \mathbf{B} \to \mathbf{B}$ *by*

$$\forall \mu \in \mathbf{B}, \Phi(\mu) = \overline{\mu}.$$

We infer that

$$X_\Phi = \{\mu \cdot \chi_{(-\infty,t_0)} \oplus \overline{\mu} \cdot \chi_{[t_0,t_1)} \oplus \mu \cdot \chi_{[t_1,t_2)} \oplus \overline{\mu} \cdot \chi_{[t_2,t_3)} \oplus ... | \mu \in \mathbf{B}, (t_k) \in Seq\}$$

and the circuit is the one from Fig. 3.

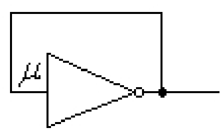

Fig. 3.

Example 9. *We associate the function* $\Phi_\lambda : \mathbf{B}^2 \to \mathbf{B}^2$,

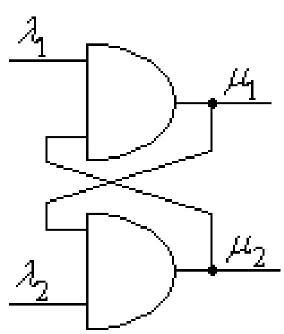

Fig. 4.

$$\forall \mu \in \mathbf{B}^2, \Phi_\lambda(\mu) = (\lambda_1\mu_2, \lambda_2\mu_1)$$

-where $\lambda \in \mathbf{B}^2$ is a parameter- with the universal regular autonomous system X_{Φ_λ}. The states $x \in X_{\Phi_\lambda}$ model the behavior of the circuit from Fig. 4 under the constant input λ. For example if $\lambda = (0,0)$, then

$$X_{\Phi_{(0,0)}} = \{(\mu_1 \cdot \chi_{(-\infty,t_1)}, \mu_2 \cdot \chi_{(-\infty,t_2)}) | \mu \in \mathbf{B}^2, t_1, t_2 \in \mathbf{R}\}.$$

7. FIXED POINTS VS. FINAL VALUES

Remark 14. *The study of the fixed points is important because the final values of the states of the stable $X \subset X_\Phi$ systems are fixed points of Φ and the accessible fixed points of Φ are final values of the states of those systems.*

Theorem 7.1. *The following fixed point property holds*

$$\forall \mu \in \mathbf{B}^n, \forall \mu' \in \mathbf{B}^n, \forall \rho \in P_n, \lim_{t\to\infty} \Phi^\rho(t)(\mu) = \mu' \Longrightarrow \Phi(\mu') = \mu'.$$

Proof. We take $\mu \in \mathbf{B}^n, \mu' \in \mathbf{B}^n$ and $\rho \in P_n$ arbitrarily with the property that $t' \in \mathbf{R}$ exists so that

$$\forall t \geq t', \Phi^\rho(t)(\mu) = \mu'. \tag{9}$$

We have the sequences $\alpha \in \Pi_n$ and $(t_k) \in Seq$ so that

$$\rho(t) = \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus ... \oplus \alpha^k \cdot \chi_{\{t_k\}}(t) \oplus ...$$

and we can suppose that $t' = t_{k'}$ for some $k' \in \mathbf{N}$, thus from (9) we can write

$$\Phi^\rho(t_{k'})(\mu) = \Phi^{\alpha^0...\alpha^{k'}}(\mu) = \mu',$$

$$\forall k \geq 1, \Phi^\rho(t_{k'+k})(\mu) = \Phi^{\alpha^0...\alpha^{k'}\alpha^{k'+1}...\alpha^{k'+k}}(\mu) = \tag{10}$$

$$= \Phi^{\alpha^{k'+1}...\alpha^{k'+k}}(\Phi^{\alpha^0...\alpha^{k'}}(\mu)) = \Phi^{\alpha^{k'+1}...\alpha^{k'+k}}(\mu') = \mu'.$$

We define the sets $\Psi_{k'+1}, ..., \Psi_{k'+p} \subset \{1, ..., n\}$ in the following way

$$\Psi_{k'+1} = \{i | i \in \{1, ..., n\}, \alpha_i^{k'+1} = 1\},$$

...

$$\Psi_{k'+p} = \{i | i \in \{1,...,n\}, \alpha_i^{k'+p} = 1\},$$

$$\Psi_{k'+1} \cup ... \cup \Psi_{k'+p} = \{1,...,n\}.$$

This is always possible and we have

$t \in [t_{k'+1}, t_{k'+2}) : \forall i \in \{1,...,n\},$

$$x_i(t) = \begin{cases} \Phi_i(\mu'), i \in \Psi_{k'+1} \\ \mu'_i, i \in \{1,...,n\} \setminus \Psi_{k'+1} \end{cases} \overset{(10)}{=} \mu'_i,$$

$t \in [t_{k'+2}, t_{k'+3}) : \forall i \in \{1,...,n\},$

$$x_i(t) = \begin{cases} \Phi_i(\mu'), i \in \Psi_{k'+1} \cup \Psi_{k'+2} \\ \mu'_i, i \in \{1,...,n\} \setminus (\Psi_{k'+1} \cup \Psi_{k'+2}) \end{cases} \overset{(10)}{=} \mu'_i,$$

$$...$$

$t \in [t_{k'+p}, \infty) : \forall i \in \{1,...,n\},$

$$x_i(t) = \begin{cases} \Phi_i(\mu'), i \in \Psi_{k'+1} \cup ... \cup \Psi_{k'+p} \\ \mu'_i, i \in \{1,...,n\} \setminus (\Psi_{k'+1} \cup ... \cup \Psi_{k'+p}) \end{cases} \overset{(10)}{=} \mu'_i.$$

It has followed that

$$\forall i \in \{1,...,n\}, \Phi_i(\mu') = \mu'_i.$$

∎

Theorem 7.2. *We have* $\forall \mu \in \mathbf{B}^n, \forall \mu' \in \mathbf{B}^n, \forall \rho \in P_n,$

$$(\Phi(\mu') = \mu' \text{ and } \exists t' \in \mathbf{R}, \Phi^\rho(t')(\mu) = \mu') \Longrightarrow \lim_{t \to \infty} \Phi^\rho(t)(\mu) = \mu'.$$

Proof. We fix $\mu, \mu' \in \mathbf{B}^n$, $\alpha \in \Pi_n$ and $(t_k) \in Seq$ arbitrarily so that

$$\rho(t) = \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus ... \oplus \alpha^k \cdot \chi_{\{t_k\}}(t) \oplus ...$$

$$\Phi(\mu') = \mu'. \tag{11}$$

If $t' < t_0$, we obtain

$$\mu = \Phi^\rho(t')(\mu) \overset{hypothesis}{=} \mu'.$$

On the other hand, from (11) we get that

$$\Phi^{\alpha^0}(\mu) = ... = \Phi^{\alpha^0...\alpha^k}(\mu) = ... = \mu'$$

and the conclusion is fulfilled under the form

$$\forall t \in \mathbf{R}, \Phi^\rho(t)(\mu) = \mu',$$

rest position.

We suppose now that $t' \geq t_0$, thus $k' \in \mathbf{N}$ exists with $t' \in [t_{k'}, t_{k'+1})$ and

$$\Phi^\rho(t')(\mu) = \Phi^{\alpha^0...\alpha^{k'}}(\mu) = \mu'.$$

But

$$\Phi^{\alpha^0...\alpha^{k'}\alpha^{k'+1}}(\mu) = \Phi^{\alpha^{k'+1}}(\Phi^{\alpha^0...\alpha^{k'}}(\mu)) = \Phi^{\alpha^{k'+1}}(\mu') \overset{(11)}{=} \mu',$$

$$...$$

$$\Phi^{\alpha^0...\alpha^{k'}\alpha^{k'+1}...\alpha^{k'+k}}(\mu) = \Phi^{\alpha^{k'+1}...\alpha^{k'+k}}(\Phi^{\alpha^0...\alpha^{k'}}(\mu)) = \Phi^{\alpha^{k'+1}...\alpha^{k'+k}}(\mu') \overset{(11)}{=} \mu',$$

$$...$$

and the conclusion is $\forall t \geq t', \Phi^\rho(t)(\mu) = \mu'$. ∎

Remark 15. *Here are two other results that are inferred from Theorems 7.1 and 7.2*

$$\forall x \in X_\Phi, \exists \lim_{t\to\infty} x(t) \Longrightarrow \Phi(\lim_{t\to\infty} x(t)) = \lim_{t\to\infty} x(t),$$

$$\forall x \in X_\Phi, \forall \mu' \in \mathbf{B}^n, (\Phi(\mu') = \mu' \text{ and } \exists t' \in \mathbf{R}, x(t') = \mu') \Longrightarrow \lim_{t\to\infty} x(t) = \mu'.$$

8. NULLCLINS AND FIXED POINTS

Definition 30. *Let be the function $\Phi : \mathbf{B}^n \to \mathbf{B}^n$. For any $i \in \{1, ..., n\}$, the **nullclins** of Φ are the sets*

$$NC_i = \{\mu | \mu \in \mathbf{B}^n, \Phi_i(\mu) = \mu_i\}.$$

*If $\mu \in NC_i$, then the coordinate i is said to be **not excited**, or **not enabled**, or **stable** and otherwise it is called **excited**, or **enabled**, or **unstable**.*

Theorem 8.1. *The following statements are equivalent for* $\mu \in \mathbf{B}^n$*:*

a) $\exists \rho \in P_n, Orb_\rho(\mu) = \{\mu\}$;

b) $\exists \rho \in P_n, \forall t \in \mathbf{R}, \Phi^\rho(t)(\mu) = \mu$;

c) $\Phi(\mu) = \mu$;

d) $\mu \in NC_1 \cap ... \cap NC_n$.

Proof. a)$\Longleftrightarrow$b) and c)$\Longleftrightarrow$d) are obvious from the way that $Orb_\rho(\mu)$ and $NC_1, ..., NC_n$ were defined.

b)$\Longrightarrow$c) Let be $\alpha \in \Pi_n$ and $(t_k) \in Seq$ with the property that

$$\rho(t) = \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus ... \oplus \alpha^k \cdot \chi_{\{t_k\}}(t) \oplus ... \tag{12}$$

and b) is fulfilled. We obtain

$$\Phi^\rho(t)(\mu) = \tag{13}$$

$$= \mu \cdot \chi_{(-\infty,t_0)}(t) \oplus \Phi^{\alpha^0}(\mu) \cdot \chi_{[t_0,t_1)}(t) \oplus ... \oplus \Phi^{\alpha^0 ... \alpha^k}(\mu) \cdot \chi_{[t_k,t_{k+1})}(t) \oplus ... = \mu$$

$$\Longleftrightarrow \Phi^{\alpha^0}(\mu) = ... = \Phi^{\alpha^0 ... \alpha^k}(\mu) = ... = \mu.$$

Denote

$$\Psi_0 = \{i | i \in \{1, ..., n\}, \alpha_i^0 = 1\}, ..., \Psi_k = \{i | i \in \{1, ..., n\}, \alpha_i^k = 1\}, ...$$

and we infer that

$$\forall k \in \mathbf{N}, \forall i \in \Psi_0 \cup ... \cup \Psi_k, \Phi_i(\mu) = \mu_i.$$

The existence of $k \in \mathbf{N}$ such that $\Psi_0 \cup ... \cup \Psi_k = \{1, ..., n\}$ shows the validity of c).

c)$\Longrightarrow$b) Let $\alpha \in \Pi_n$ and $(t_k) \in Seq$ be arbitrary and define $\rho \in P_n$ by (12). By induction on $k \in \mathbf{N}$ it is shown that

$$\Phi^{\alpha^0}(\mu) = ... = \Phi^{\alpha^0 ... \alpha^k}(\mu) = ... = \mu$$

wherefrom we get (see (13)) the truth of

$$\forall t \in \mathbf{R}, \Phi^\rho(t)(\mu) = \mu$$

for arbitrary $\rho \in P_n$. b) is true. ∎

Notation 8.1. *The set of the fixed points (also called points of equilibrium) of* Φ *is denoted by Eq.*

9. THE SPEED

Definition 31. *For any* $\mu \in \mathbf{B}^n$ *and* $\rho \in P_n$*, we define the* ***speed of the*** $\rho-$***motion of*** μ, $v^\rho(\cdot)(\mu) : \mathbf{R} \to \mathbf{B}^n$ *in the following manner*

$$\forall t \in \mathbf{R}, v^\rho(t)(\mu) = \Phi^\rho(t-0)(\mu) \oplus \Phi^\rho(t)(\mu).$$

Remark 16. *From the previous definition we infer that if*

$$\rho(t) = \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus ... \oplus \alpha^k \cdot \chi_{\{t_k\}}(t) \oplus ...,$$

then

$$v^\rho(t)(\mu) = D\Phi^\rho(t)(\mu) =$$
$$= (\mu \oplus \Phi^{\alpha^0}(\mu)) \cdot \chi_{\{t_0\}}(t) \oplus (\Phi^{\alpha^0}(\mu) \oplus \Phi^{\alpha^0\alpha^1}(\mu)) \cdot \chi_{\{t_1\}}(t) \oplus ...$$
$$... \oplus (\Phi^{\alpha^0 ... \alpha^{k-1}}(\mu) \oplus \Phi^{\alpha^0 ... \alpha^k}(\mu)) \cdot \chi_{\{t_k\}}(t) \oplus ...$$

10. THE RELATION BETWEEN EQUATIONS AND DYNAMICAL SYSTEMS

Theorem 10.1. *We consider the point* $\mu \in \mathbf{B}^n$ *and the function* $\rho \in P_n$.

i) The equations

$$\begin{cases} x(-\infty+0) = \mu \\ Dx(t) = v^\rho(t)(\mu) \end{cases},$$

$$\begin{cases} x(-\infty+0) = \mu \\ x(t) = \Phi^{\rho(t)}(x(t-0)) \end{cases}$$

have both the unique solution

$$x(t) = \Phi^\rho(t)(\mu).$$

ii) μ *is a fixed point of* $\Phi \iff \forall t \in \mathbf{R}, v^\rho(t)(\mu) = 0.$

Proof. Let be $\mu \in \mathbf{B}^n, \alpha \in \Pi_n$ and $(t_k) \in Seq$. Denote

$$\rho(t) = \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus ... \oplus \alpha^k \cdot \chi_{\{t_k\}}(t) \oplus ... \tag{14}$$

We search the solutions of the two equations under the form

$$x(t) = \mu \cdot \chi_{(-\infty,t_0)}(t) \oplus x(t_0) \cdot \chi_{[t_0,t_1)}(t) \oplus ... \oplus x(t_k) \cdot \chi_{[t_k,t_{k+1})}(t) \oplus ... \tag{15}$$

where the unknowns are $x(t_k) \in \mathbf{B}^n, k \in \mathbf{N}$. As we know, we have

$$\Phi^\rho(t)(\mu) = \mu \cdot \chi_{(-\infty,t_0)}(t) \oplus \Phi^{\alpha^0}(\mu) \cdot \chi_{[t_0,t_1)}(t) \oplus ... \tag{16}$$

$$... \oplus \Phi^{\alpha^0 ... \alpha^k}(\mu) \cdot \chi_{[t_k,t_{k+1})}(t) \oplus ...$$

$$v^\rho(t)(\mu) = (\mu \oplus \Phi^{\alpha^0}(\mu)) \cdot \chi_{\{t_0\}}(t) \oplus (\Phi^{\alpha^0}(\mu) \oplus \Phi^{\alpha^0\alpha^1}(\mu)) \cdot \chi_{\{t_1\}}(t) \oplus ... \tag{17}$$

$$... \oplus (\Phi^{\alpha^0 ... \alpha^{k-1}}(\mu) \oplus \Phi^{\alpha^0 ... \alpha^k}(\mu)) \cdot \chi_{\{t_k\}}(t) \oplus ...$$

We solve the first equation, where

$$Dx(t) = (\mu \oplus x(t_0)) \cdot \chi_{\{t_0\}}(t) \oplus (x(t_0) \oplus x(t_1)) \cdot \chi_{\{t_1\}}(t) \oplus ... \tag{18}$$

$$\oplus (x(t_{k-1}) \oplus x(t_k)) \cdot \chi_{\{t_k\}}(t) \oplus ...$$

and we infer from (17) and (18)

$$\mu \oplus x(t_0) = \mu \oplus \Phi^{\alpha^0}(\mu),\ x(t_0) \oplus x(t_1) = \Phi^{\alpha^0}(\mu) \oplus \Phi^{\alpha^0\alpha^1}(\mu),$$

$$...$$

$$x(t_{k-1}) \oplus x(t_k) = \Phi^{\alpha^0 ... \alpha^{k-1}}(\mu) \oplus \Phi^{\alpha^0 ... \alpha^k}(\mu),$$

$$...$$

In other words

$$x(t_0) = \Phi^{\alpha^0}(\mu),\ x(t_1) = \Phi^{\alpha^0\alpha^1}(\mu),$$

$$...$$

$$x(t_k) = \Phi^{\alpha^0 ... \alpha^k}(\mu),$$

$$...$$

thus $\Phi^{\rho}(t)(\mu)$ is a solution of the first equation.

We solve the second equation and we take into account the fact that

$$x(t-0) = \mu \cdot \chi_{(-\infty,t_0]}(t) \oplus x(t_0) \cdot \chi_{(t_0,t_1]}(t) \oplus ... \oplus x(t_k) \cdot \chi_{(t_k,t_{k+1}]}(t) \oplus ...$$

$$\Phi^{\rho(t)}(x(t-0)) = \begin{cases} \Phi^{\alpha^k}(x(t-0)), t = t_k \\ x(t-0), t \notin \{t_0, ..., t_k, ...\} \end{cases} .$$

We have

$t < t_0 :$ $\quad x(t) = x(t-0) = x(-\infty+0) = \mu,$

$t = t_0 :$ $\quad x(t_0) = \Phi^{\alpha^0}(x(t_0-0)) = \Phi^{\alpha^0}(\mu),$

$t \in (t_0, t_1) :$ $\quad x(t) = x(t-0) = \Phi^{\alpha^0}(\mu),$

$t = t_1 :$ $\quad x(t_1) = \Phi^{\alpha^1}(x(t_1-0)) = \Phi^{\alpha^1}(\Phi^{\alpha^0}(\mu)) = \Phi^{\alpha^0\alpha^1}(\mu),$

$t \in (t_1, t_2) :$ $\quad x(t) = x(t-0) = \Phi^{\alpha^0\alpha^1}(\mu),$

...

$t = t_k :$ $\quad x(t_k) = \Phi^{\alpha^k}(x(t_k-0)) = \Phi^{\alpha^k}(\Phi^{\alpha^0...\alpha^{k-1}}(\mu)) = \Phi^{\alpha^0...\alpha^k}(\mu),$

$t \in (t_k, t_{k+1}) : x(t) = x(t-0) = \Phi^{\alpha^0...\alpha^k}(\mu),$

...

The fact that $\Phi^{\rho}(t)(\mu)$ satisfies the second equation was proved.

The uniqueness of the solution is proved for both equations like this. We suppose against all reason that two distinct solutions x, x' exist. Then we observe that $\forall t < t_0$, $x(t) = x'(t) = \mu$, thus $t_1 \geq t_0$ should exist such that $\forall t < t_1, x(t) = x'(t)$ and $x(t_1) \neq x'(t_1)$. This supposition gives a contradiction in both cases, meaning that the solution is unique.

The statement ii) is a consequence of the fact that $\Phi(\mu) = \mu \iff \mu = \Phi^{\alpha^0}(\mu) = \Phi^{\alpha^0\alpha^1}(\mu) = ...$ ∎

Remark 17. *With any of the equations from Theorem 10.1, when μ runs in $\mathbf{B}^n$ and ρ runs in P_n, we can associate the dynamical system $\phi = (\mathbf{B}^n, (\Phi^{\rho})_{\rho \in P_n})$.*

11. INVARIANT SETS IN THE PHASE SPACE

Definition 32. *The set $A \in P^*(\mathbf{B}^n)$ is $\rho-$**invariant**, $\rho \in P_n$ if*

$$\forall \mu \in A, \forall t \in \mathbf{R}, \Phi^{\rho}(t)(\mu) \in A. \tag{19}$$

Other definitions of invariance of A are the following:

$$\forall \mu \in A, \exists \rho' \in P_n, \forall t \in \mathbf{R}, \Phi^{\rho'}(t)(\mu) \in A; \tag{20}$$

$$\forall \mu \in A, \forall \rho' \in P_n, \forall t \in \mathbf{R}, \Phi^{\rho'}(t)(\mu) \in A; \tag{21}$$

$$\forall \mu \in A, \forall \lambda \in \mathbf{B}^n, \Phi^{\lambda}(\mu) \in A. \tag{22}$$

Theorem 11.1. *The following implications hold:*

$$(22) \Longleftrightarrow (21) \Longrightarrow (19) \Longrightarrow (20).$$

Proof. We prove (21)$\Longleftrightarrow$(22), because the other implications are obvious.

(21)$\Longrightarrow$(22) Let $\mu \in A, \lambda \in \mathbf{B}^n$ and the sequence $\rho'' \in P_n$ be arbitrary,

$$\rho''(t) = \alpha''^0 \cdot \chi_{\{t''_0\}}(t) \oplus ... \oplus \alpha''^k \cdot \chi_{\{t''_k\}}(t) \oplus ...$$

with $\alpha'' \in \Pi_n$ and $(t''_k) \in Seq$. Define

$$\rho'(t) = \lambda \cdot \chi_{\{t_0\}}(t) \oplus \alpha''^0 \cdot \chi_{\{t_0+t''_0\}}(t) \oplus ... \oplus \alpha''^k \cdot \chi_{\{t_0+t''_k\}}(t) \oplus ...$$

where $t_0 \in \mathbf{R}$ is arbitrary and we can see that $\rho' \in P_n$. (21) implies $\Phi^{\lambda}(\mu) = \Phi^{\rho'}(t_0)(\mu) \in A$.

(22)$\Longrightarrow$(21) Let $\mu \in A$ and $\rho' \in P_n$,

$$\rho'(t) = \alpha'^0 \cdot \chi_{\{t'_0\}}(t) \oplus ... \oplus \alpha'^k \cdot \chi_{\{t'_k\}}(t) \oplus ...$$

be arbitrary, with $\alpha' \in \Pi_n, (t'_k) \in Seq$. (22) implies that

$t < t'_0$: $\quad \Phi^{\rho'}(t)(\mu) = \mu \in A$,

$t \in [t'_0, t'_1)$: $\quad \Phi^{\rho'}(t)(\mu) = \Phi^{\alpha'^0}(\mu) \in A$,

...

$t \in [t'_k, t'_{k+1})$: $\quad \Phi^{\alpha'^0 ... \alpha'^{k-1}}(\mu) \in A$

due to the hypothesis of the induction and we have

$\Phi^{\rho'}(t)(\mu) = \Phi^{\alpha'^k}(\Phi^{\alpha'^0 ... \alpha'^{k-1}}(\mu)) \in A, ...$ ∎

12. EXAMPLES OF INVARIANT SETS

Example 10. *Consider a* $\Phi : \mathbf{B}^2 \to \mathbf{B}^2, \forall \mu \in \mathbf{B}^2, \Phi(\mu_1, \mu_2) = (\overline{\mu_1}, \overline{\mu_2})$ *and* $\rho(t) = (1,1) \cdot \chi_{\{0,1,2,...\}}(t)$. *The set* $A = \{(0,1),(1,0)\}$ *is* $\rho-$*invariant i.e. it satisfies* (19)

$$\Phi^\rho(t)(0,1) = (0,1) \cdot \chi_{(-\infty,0)}(t) \oplus (1,0) \cdot \chi_{[0,1)}(t) \oplus$$

$$\oplus (0,1) \cdot \chi_{[1,2)}(t) \oplus (1,0) \cdot \chi_{[2,3)}(t) \oplus ...$$

$$\Phi^\rho(t)(1,0) = (1,0) \cdot \chi_{(-\infty,0)}(t) \oplus (0,1) \cdot \chi_{[0,1)}(t) \oplus$$

$$\oplus (1,0) \cdot \chi_{[1,2)}(t) \oplus (0,1) \cdot \chi_{[2,3)}(t) \oplus ...$$

Similarly, $A = \{(0,0),(1,1)\}$ *satisfies the same invariance property.*

Example 11. *Consider a* $\mu \in \mathbf{B}^n$. *The set* $\bigcup\limits_{\rho \in P_n} Orb_\rho(\mu)$ *is invariant in the sense of the satisfaction of (21). In order to see this we take an arbitrary vector* $\mu' \in \bigcup\limits_{\rho \in P_n} Orb_\rho(\mu)$, *for which we have two possibilities:*

a) $\mu' = \mu$,

in this situation (21) is obviously fulfilled;

b) $\mu' \neq \mu$,

in this case $\alpha \in \Pi_n$, $(t_k) \in Seq$ *and* $k' \in \mathbf{N}$ *exist such that*

$$\rho(t) = \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus ... \oplus \alpha^k \cdot \chi_{\{t_k\}}(t) \oplus ...$$

$$\mu' = \Phi^\rho(t_{k'})(\mu) = \Phi^{\alpha^0 ... \alpha^{k'}}(\mu).$$

If $\rho' \in P_n$ *is arbitrary,*

$$\rho'(t) = \alpha'^0 \cdot \chi_{\{t'_0\}}(t) \oplus ... \oplus \alpha'^k \cdot \chi_{\{t'_k\}}(t) \oplus ...$$

$\alpha' \in \Pi_n, (t'_k) \in Seq$, *then the sequence*

$$\gamma(t) = \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus ... \oplus \alpha^{k'} \cdot \chi_{\{t_{k'}\}}(t) \oplus$$

$$\oplus \alpha'^0 \cdot \chi_{\{t_{k'}+t'_0\}}(t) \oplus ... \oplus \alpha'^k \cdot \chi_{\{t_{k'}+t'_k\}}(t) \oplus ...$$

belongs to P_n *and for* $t \in \mathbf{R}$ *we have the following possibilities:*

b.1) $t < t'_0$ *when* $\Phi^{\rho'}(t)(\mu') = \mu' \in \bigcup_{\rho \in P_n} Orb_\rho(\mu)$;

b.2) $t \geq t'_0$ *when* $k'' \in \mathbf{N}$ *exists such that* $t \in [t'_{k''}, t'_{k''+1})$ *and*

$$\Phi^{\rho'}(t)(\mu') = \Phi^{\rho'}(t'_{k''})(\Phi^{\rho}(t_{k'})(\mu)) = \Phi^{\rho'}(t'_{k''})(\Phi^{\alpha^0 ... \alpha^{k'}}(\mu)) =$$

$$= \Phi^{\alpha'^0 ... \alpha'^{k''}}(\Phi^{\alpha^0 ... \alpha^{k'}}(\mu)) = \Phi^{\alpha^0 ... \alpha^{k'} \alpha'^0 ... \alpha'^{k''}}(\mu) = \Phi^{\gamma}(t_{k'} + t'_{k''})(\mu)$$

thus $\Phi^{\rho'}(t)(\mu') \in Orb_\gamma(\mu) \subset \bigcup_{\rho \in P_n} Orb_\rho(\mu)$.

Example 12. *We show that Eq is invariant in the sense of satisfaction of (22). Indeed, let be* $\mu \in Eq$ *meaning that*

$$\Phi(\mu) = \mu \tag{23}$$

is true and we take arbitrarily some $\lambda \in \mathbf{B}^n$. *Because*

$$\forall i \in \{1, ..., n\}, \Phi_i^{\lambda}(\mu) = \begin{cases} \Phi_i(\mu), if\ \lambda_i = 1 \\ \mu_i, if\ \lambda_i = 0 \end{cases} = \mu_i, \tag{24}$$

we conclude that

$$\Phi(\Phi^{\lambda}(\mu)) \overset{(24)}{=} \Phi(\mu) \overset{(23)}{=} \mu \overset{(24)}{=} \Phi^{\lambda}(\mu),$$

thus $\Phi^{\lambda}(\mu) \in Eq$.

13. ATTRACTION

Definition 33. *Let be* $A, B \in P^*(\mathbf{B}^n)$ *and the function* $\rho \in P_n$. *We say that* A *is* $\rho-$***attractive for*** B *and that the points of* B *are* $\rho-$***attracted by the set*** A *if*

$$\forall \mu \in B, \exists t \in \mathbf{R}, \forall t' \geq t, \Phi^{\rho}(t')(\mu) \in A, \tag{25}$$

i.e. A *contains all the cyclic values of* $\Phi^{\rho}(t)(\mu), \mu \in B$. *Similarly,* A *is* ***attractive for*** B *and the points of* B *are* ***attracted by*** A *if one of the non-equivalent properties*

$$\forall \mu \in B, \exists \rho' \in P_n, \exists t \in \mathbf{R}, \forall t' \geq t, \Phi^{\rho'}(t')(\mu) \in A, \tag{26}$$

$$\forall \mu \in B, \forall \rho' \in P_n, \exists t \in \mathbf{R}, \forall t' \geq t, \Phi^{\rho'}(t')(\mu) \in A \tag{27}$$

holds.

Remark 18. *The implications*

$$(27) \Longrightarrow (25) \Longrightarrow (26)$$

hold. On the other hand if $A \in P^*(\mathbf{B}^n)$ is $\rho-$invariant ((19) is fulfilled) then it is $\rho-$attractive for itself ((25) is true with $B = A$).

Here is the special case of Definition 33 when $B = \{\mu\}$:

$$\exists t \in \mathbf{R}, \forall t' \geq t, \Phi^{\rho}(t')(\mu) \in A;$$

$$\exists \rho' \in P_n, \exists t \in \mathbf{R}, \forall t' \geq t, \Phi^{\rho'}(t')(\mu) \in A;$$

$$\forall \rho' \in P_n, \exists t \in \mathbf{R}, \forall t' \geq t, \Phi^{\rho'}(t')(\mu) \in A.$$

14. LIMIT CYCLES

Definition 34. *Given the set B and the function ρ, the $\rho-$**limit cycle** ($\rho-$**limit set**) **of** B is given by*

$$LC_B^{\rho} = \{\mu' | \mu' \in \mathbf{B}^n, \exists \mu \in B, \forall t \in \mathbf{R}, \exists t' > t, \Phi^{\rho}(t')(\mu) = \mu'\}.$$

*Each $\mu' \in LC_B^{\rho}$ is called $\rho-$**cyclic**, or $\rho-$**limit point of** B. Similarly, the **superior** and the **inferior limit cycle (limit set)** of B are defined by*

$$\overline{LC}_B = \{\mu' | \mu' \in \mathbf{B}^n, \exists \mu \in B, \exists \rho' \in P_n, \forall t \in \mathbf{R}, \exists t' > t, \Phi^{\rho'}(t')(\mu) = \mu'\},$$

$$\underline{LC}_B = \{\mu' | \mu' \in \mathbf{B}^n, \exists \mu \in B, \forall \rho' \in P_n, \forall t \in \mathbf{R}, \exists t' > t, \Phi^{\rho'}(t')(\mu) = \mu'\}.$$

Remark 19. *The limit cycles are the sets of the cyclic values of the motions $\Phi^{\rho}(t)(\mu)$ and the following inclusions*

$$\underline{LC}_B \subset LC_B^{\rho} \subset \overline{LC}_B$$

are true.

We have the special case at Definition 34, when $B = \{\mu\}$:

$$LC^{\rho}_{\mu} = \{\mu' | \mu' \in \mathbf{B}^n, \forall t \in \mathbf{R}, \exists t' > t, \Phi^{\rho}(t')(\mu) = \mu'\};$$

$$\overline{LC}_{\mu} = \{\mu' | \mu' \in \mathbf{B}^n, \exists \rho' \in P_n, \forall t \in \mathbf{R}, \exists t' > t, \Phi^{\rho'}(t')(\mu) = \mu'\};$$

$$\underline{LC}_{\mu} = \{\mu' | \mu' \in \mathbf{B}^n, \forall \rho' \in P_n, \forall t \in \mathbf{R}, \exists t' > t, \Phi^{\rho'}(t')(\mu) = \mu'\}.$$

Theorem 14.1. *Consider a $B \in P^*(\mathbf{B}^n)$, $\rho \in P_n$ and consider the Definition 33, property (25). Then the set LC^{ρ}_{B} is $\rho-$attractive for B and any $A \in P^*(\mathbf{B}^n)$ which is $\rho-$attractive for B fulfills $LC^{\rho}_{B} \subset A$. Similar properties hold for the properties (26), (27) and for the sets $\overline{LC}_B$, $\underline{LC}_B$.*

Example 13. *The function $\Phi : \mathbf{B}^2 \to \mathbf{B}^2$ defined by the following table:*

(μ_1, μ_2)	Φ
$(0,0)$	$(0,1)$
$(0,1)$	$(1,1)$
$(1,0)$	$(1,1)$
$(1,1)$	$(0,1)$

has the interesting property that $\forall \mu \in \mathbf{B}^2, \forall \rho \in P_n$,

$$LC^{\rho}_{\mu} = \{(0,1), (1,1)\}.$$

Theorem 14.2. *Let be $\mu \in \mathbf{B}^n$ and $\rho \in P_n$. We have*

a) $LC^{\rho}_{\mu} \subset Orb_{\rho}(\mu)$,

b) $LC^{\rho}_{\mu} \neq \emptyset$,

c) $LC^{\rho}_{\mu} = \{\mu'\} \Longrightarrow \lim\limits_{t\to\infty} \Phi^{\rho}(t)(\mu) = \mu'$,

d) $\exists t_0 \in \mathbf{R}, \forall t \geq t_0, \Phi^{\rho}(t)(\mu) \in LC^{\rho}_{\mu}$,

e) the set LC^{ρ}_{μ} is invariant in the sense of satisfying (20).

Proof. d) This is true from the definition of LC^{ρ}_{μ}.

e) Let $t_0 \in \mathbf{R}$ be the number that makes d) be true and we take some arbitrary $\mu' \in LC^{\rho}_{\mu}$. As all the points of LC^{ρ}_{μ} are $\rho-$cyclic values of $\Phi^{\rho}(\cdot)(\mu)$, we have

$$\exists t' > t_0, \Phi^{\rho}(t')(\mu) = \mu'.$$

We can see that the function

$$\rho'(t) = \rho(t) \cdot \chi_{(t',\infty)}(t)$$

is progressive and we infer that

$$\Phi^{\rho'}(t)(\mu') = \begin{cases} \mu', t \leq t' \\ \Phi^{\rho}(t)(\mu), t > t' \end{cases} \in LC^{\rho}_{\mu}.$$

∎

15. THE EVOLUTIONS OF A SYSTEM

Definition 35. *We have the following terminology, given by Moisil and Gavrilov [3], [4]:*

a) $Orb_{\rho}(\mu) \setminus LC^{\rho}_{\mu} \neq \emptyset,\ |LC^{\rho}_{\mu}| > 1$

Moisil: ***finally cyclic evolution,***

Gavrilov: ***successively repeated evolution;***

b) $Orb_{\rho}(\mu) \setminus LC^{\rho}_{\mu} \neq \emptyset,\ |LC^{\rho}_{\mu}| = 1$

Moisil: ***finally stabilized evolution,***

Gavrilov: ***successive evolution;***

c) $Orb_{\rho}(\mu) = LC^{\rho}_{\mu},\ |LC^{\rho}_{\mu}| > 1$

Moisil: ***cyclic evolution,***

Gavrilov: ***repeated evolution;***

d) $Orb_{\rho}(\mu) = LC^{\rho}_{\mu},\ |LC^{\rho}_{\mu}| = 1$

Moisil: ***rest position*** *(or* ***stable position****).*

Remark 20. *We see that both Moisil and Gavrilov avoid referring to periodicity or maybe to pseudo-periodicity; for them the evolution of a system is just cyclic or repeated. On the other hand, in cases b), d), when* $|LC^{\rho}_{\mu}| = 1$*, the* $\rho-$*limit cycle* LC^{ρ}_{μ} *consists of one point* μ' *that is a final value of* $\Phi^{\rho}(t)(\mu)$*, thus a fixed point of* Φ *(see Theorem 7.1).*

16. STABLE MANIFOLD

Definition 36. *Let be $A \in P^*(\mathbf{B}^n)$ and $\rho \in P_n$. The $\rho-$**stable manifold**[2] **of** A, or the **kingdom** (the **basin**) **of** $\rho-$**attraction of** A is, by definition, the set*

$$M_s^\rho(A) = \{\mu | \mu \in \mathbf{B}^n, \exists t \in \mathbf{R}, \forall t' \geq t, \Phi^\rho(t')(\mu) \in A\}. \tag{28}$$

Two versions of this definition are the following:

$$\overline{M}_s(A) = \{\mu | \mu \in \mathbf{B}^n, \exists \rho' \in P_n, \exists t \in \mathbf{R}, \forall t' \geq t, \Phi^{\rho'}(t')(\mu) \in A\}; \tag{29}$$

$$\underline{M}_s(A) = \{\mu | \mu \in \mathbf{B}^n, \forall \rho' \in P_n, \exists t \in \mathbf{R}, \forall t' \geq t, \Phi^{\rho'}(t')(\mu) \in A\}, \tag{30}$$

*giving the **superior** and the **inferior stable manifold of** A.*

Remark 21. *The stable manifolds of A are the sets of points which are attracted by A and the following inclusions hold*

$$\underline{M}_s(A) \subset M_s^\rho(A) \subset \overline{M}_s(A).$$

We have the special case of Definition 36, when $A = \{\mu^0\}$:

$$M_s^\rho(\mu^0) = \{\mu | \mu \in \mathbf{B}^n, \lim_{t\to\infty} \Phi^\rho(t)(\mu) = \mu^0\},$$

$$\overline{M}_s(\mu^0) = \{\mu | \mu \in \mathbf{B}^n, \exists \rho' \in P_n, \lim_{t\to\infty} \Phi^{\rho'}(t)(\mu) = \mu^0\},$$

$$\underline{M}_s(\mu^0) = \{\mu | \mu \in \mathbf{B}^n, \forall \rho' \in P_n, \lim_{t\to\infty} \Phi^{\rho'}(t)(\mu) = \mu^0\}.$$

Theorem 16.1. *Let $A \in P^*(\mathbf{B}^n), \rho \in P_n$ be given and we consider the Definition 36, property (28). We have that the points of $M_s^\rho(A)$ are $\rho-$attracted by A and for any set $B \in P^*(\mathbf{B}^n)$ whose points are $\rho-$attracted by A. We infer $B \subset M_s^\rho(A)$. Similar statements are true for the properties (29), (30) and the sets $\overline{M}_s(A), \underline{M}_s(A)$.*

17. TOTAL AND PARTIAL ATTRACTION

Definition 37. *If $M_s^\rho(A) \neq \emptyset$, then the set A is called $\rho-$**attractive**. In this case we have the possibilities:*

a) $M_s^\rho(A) = \mathbf{B}^n$, *when A is called* ***totally*** $\rho-$***attractive****;*

b) $M_s^\rho(A) \neq \mathbf{B}^n$, *when A is called* ***partially*** $\rho-$***attractive****.*

By replacing $M_s^\rho(A)$ *with* $\overline{M}_s(A)$ *(with* $\underline{M}_s(A)$*), we obtain the* ***superior*** *(the* ***inferior****)* ***attractive****,* ***totally attractive*** *and* ***partially attractive*** *sets.*

Remark 22. *We have the special case of Definition 37, when* $A = \mu^0$*. If* $M_s^\rho(\mu^0) \neq \emptyset$*, then* μ^0 *is a point of* $\rho-$*attraction*

$$\exists \mu \in \mathbf{B}^n, \lim_{t\to\infty} \Phi^\rho(t)(\mu) = \mu^0$$

and we have the possibilities: $M_s^\rho(\mu^0) = \mathbf{B}^n$*, when* μ^0 *is totally* $\rho-$*attractive, respectively* $M_s^\rho(\mu^0) \neq \mathbf{B}^n$*, when* μ^0 *is partially* $\rho-$*attractive.*

The situation is similar by replacing $M_s^\rho(\mu^0)$ *with* $\overline{M}_s(\mu^0)$ *and* $\underline{M}_s(\mu^0)$*.*

18. EQUIVALENCIES

Definition 38. *Let be the generator functions* $\Phi, \Psi : \mathbf{B}^n \to \mathbf{B}^n$*. We say that the Boolean dynamical systems* $\phi = (\mathbf{B}^n, (\Phi^\rho)_{\rho\in P_n})$ *and* $\psi = (\mathbf{B}^n, (\Psi^\rho)_{\rho\in P_n})$ *are* ***equivalent*** *if a bijection* $H : \mathbf{B}^n \to \mathbf{B}^n$ *exists so that the following diagram is commutative*

$$\begin{array}{ccc} \mathbf{B}^n & \overset{\Phi}{\to} & \mathbf{B}^n \\ H\downarrow & & \downarrow H \\ \mathbf{B}^n & \overset{\Psi}{\to} & \mathbf{B}^n \end{array}$$

If this is true, we say that H ***transforms the generator function*** Φ ***in the generator function*** Ψ*.*

Remark 23. *This definition concerning the equivalence of the Boolean dynamical systems refers to a change of the system of coordinates.*

Lemma 1. *If the function* $h : \mathbf{R} \to \mathbf{R}$ *is bijective, continuous and strictly increasing, then* h^{-1} *has the same properties.*

Proof. [3]In [2], page 233, it is shown that if $h : I \to \mathbf{R}$ is continuous and strictly monotonous on the interval $I \subset \mathbf{R}$ (the choice $I = \mathbf{R}$ is possible), then it has a continuous inverse. Obviously, if $h : \mathbf{R} \to \mathbf{R}$ is bijective and strictly monotonous, its inverse is strictly monotonous, with the same monotonicity like h.

The conclusion is that if h is bijective, continuous and strictly increasing, its inverse has the same properties. ∎

Definition 39. *Let us consider a* $\mu \in \mathbf{B}^n$ *and the functions* $\rho, \rho' \in P_n$. *The equations*

$$\begin{cases} x(-\infty + 0) = \mu \\ x(t) = \Phi^{\rho(t)}(x(t-0)), \end{cases} \tag{31}$$

$$\begin{cases} y(-\infty + 0) = \mu \\ y(t) = \Phi^{\rho'(t)}(y(t-0)) \end{cases} \tag{32}$$

and the motions $\Phi^{\rho}(\cdot)(\mu)$ *and* $\Phi^{\rho'}(\cdot)(\mu)$ *are* ***equivalent*** *if the bijective, continuous strictly increasing function* $h : \mathbf{R} \to \mathbf{R}$ *exists so that*

$$\Phi^{\rho'}(t)(\mu) = \Phi^{\rho}(h(t))(\mu). \tag{33}$$

Remark 24. *This definition states that the solutions* $\Phi^{\rho}(\cdot)(\mu)$, $\Phi^{\rho'}(\cdot)(\mu)$ *of (31),(32) are equivalent if they are equal functions regardless the time flow, which is given by* t *for* $\Phi^{\rho'}(\cdot)(\mu)$ *and by* $h(t)$ *for* $\Phi^{\rho}(\cdot)(\mu)$. *Lemma 1 guarantees that Definition 39 is, indeed, that of an equivalence relation. We have the sufficient condition* $\rho' = \rho \circ h$ *in order that (33) is true:*

Theorem 18.1. *Let be* $\mu \in \mathbf{B}^n, \rho \in P_n$ *and* $h : \mathbf{R} \to \mathbf{R}$ *bijective, continuous strictly increasing. Then*

$$\Phi^{\rho \circ h}(t)(\mu) = \Phi^{\rho}(h(t))(\mu).$$

Proof. We take the sequences $\alpha \in \Pi_n$ and $(t_k) \in Seq$ so that

$$\rho(t) = \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus ... \oplus \alpha^k \cdot \chi_{\{t_k\}}(t) \oplus ...$$

and we remark the truth of the following statements

$$h(t) \in (-\infty, t_0) \Longleftrightarrow t \in (-\infty, h^{-1}(t_0)),$$

$$h(t) \in [t_0, t_1) \Longleftrightarrow t \in [h^{-1}(t_0), h^{-1}(t_1)),$$

$$...$$

$$h(t) \in [t_k, t_{k+1}) \Longleftrightarrow t \in [h^{-1}(t_k), h^{-1}(t_{k+1})),$$

$$...$$

Because

$$(\rho \circ h)(t) = \rho(h(t)) = \alpha^0 \cdot \chi_{\{t_0\}}(h(t)) \oplus ... \oplus \alpha^k \cdot \chi_{\{t_k\}}(h(t)) \oplus ...$$

$$= \alpha^0 \cdot \chi_{\{h^{-1}(t_0)\}}(t) \oplus ... \oplus \alpha^k \cdot \chi_{\{h^{-1}(t_k)\}}(t) \oplus ...$$

we infer that for arbitrary $\mu \in \mathbf{B}^n$ we can write

$$\Phi^{\rho \circ h}(t)(\mu) =$$

$$= \mu \cdot \chi_{(-\infty, h^{-1}(t_0))}(t) \oplus \Phi^{\alpha^0}(\mu) \cdot \chi_{[h^{-1}(t_0), h^{-1}(t_1))}(t) \oplus ...$$

$$... \oplus \Phi^{\alpha^0 ... \alpha^k}(\mu) \cdot \chi_{[h^{-1}(t_k), h^{-1}(t_{k+1}))}(t) \oplus ...$$

$$= \mu \cdot \chi_{(-\infty, t_0)}(h(t)) \oplus \Phi^{\alpha^0}(\mu) \cdot \chi_{[t_0, t_1)}(h(t)) \oplus ... \oplus \Phi^{\alpha^0 ... \alpha^k}(\mu) \cdot \chi_{[t_k, t_{k+1})}(h(t)) \oplus ...$$

$$= \Phi^{\rho}(h(t))(\mu). \blacksquare$$

Definition 40. *Let be $\mu \in \mathbf{B}^n$ and $\rho, \rho' \in P_n$. The equations (31), (32) and the motions $\Phi^{\rho}(\cdot)(\mu)$, $\Phi^{\rho'}(\cdot)(\mu)$ are* ***equivalent*** *if*

$$LC_{\mu}^{\rho} = LC_{\mu}^{\rho'}.$$

Remark 25. *The motions $\Phi^{\rho}(\cdot)(\mu)$, $\Phi^{\rho'}(\cdot)(\mu)$ are equivalent conformably to Definition 40 if they start from the same initial value μ and they reach the same limit cycle. The equivalence class of $\Phi^{\rho}(\cdot)(\mu)$ has the property that the unique limit cycle reached by all its elements depends on μ only and it does not depend on ρ.*

Definition 41. *The regular system $X \subset X_{\Phi}$ is called* ***perfect*** *if $\forall \mu \in \mathbf{B}^n, \forall \rho \in P_n, \forall \rho' \in P_n$,*

$$(\Phi^{\rho}(\cdot)(\mu) \in X \text{ and } \Phi^{\rho'}(\cdot)(\mu) \in X) \Longrightarrow LC_{\mu}^{\rho} = LC_{\mu}^{\rho'}.$$

19. THE PROPERTIES OF THE REGULAR AUTONOMOUS SYSTEMS

Remark 26. *If $X \subset X_\Phi$ is regular, then any subsystem $X' \subset X$ is regular and it has the same generator function like X. The intersection and the union of $X \subset X_\Phi$ and $X' \subset X_\Phi$ are regular: $X \cap X' \subset X_\Phi, X \cup X' \subset X_\Phi$.*

Theorem 19.1. *Let be $\Theta_0 \in P^*(\mathbf{B}^n)$. Then $\forall x \in X_\Phi^{\Theta_0}, \forall d \in \mathbf{R}$, we have $x \circ \tau^d \in X_\Phi^{\Theta_0}$ i.e. $X_\Phi^{\Theta_0}$ is invariant to translations.*

Proof. Suppose that $x \in X_\Phi^{\Theta_0}$ is given by

$$x(t) = \mu \cdot \chi_{(-\infty,t_0)}(t) \oplus \Phi^{\alpha^0}(\mu) \cdot \chi_{[t_0,t_1)}(t) \oplus ... \oplus \Phi^{\alpha^0 ... \alpha^k}(\mu) \cdot \chi_{[t_k,t_{k+1})}(t) \oplus ...,$$

where $\mu \in \Theta_0$, $\alpha \in \Pi_n$ and $(t_k) \in Seq$ are arbitrary. Take $d \in \mathbf{R}$ arbitrarily and remark that $(t_k + d) \in Seq$. On the other hand, we get

$$(x \circ \tau^d)(t) = x(t-d) =$$

$$= \mu \cdot \chi_{(-\infty,t_0)}(t-d) \oplus \Phi^{\alpha^0}(\mu) \cdot \chi_{[t_0,t_1)}(t-d) \oplus ...$$

$$... \oplus \Phi^{\alpha^0 ... \alpha^k}(\mu) \cdot \chi_{[t_k,t_{k+1})}(t-d) \oplus ...$$

$$= \mu \cdot \chi_{(-\infty,t_0+d)}(t) \oplus \Phi^{\alpha^0}(\mu) \cdot \chi_{[t_0+d,t_1+d)}(t) \oplus ...$$

$$... \oplus \Phi^{\alpha^0 ... \alpha^k}(\mu) \cdot \chi_{[t_k+d,t_{k+1}+d)}(t) \oplus ...,$$

thus, $x \circ \tau^d \in X_\Phi^{\Theta_0}$. ∎

Remark 27. *The dynamical system $\phi = (\mathbf{B}^n, (\Phi^\rho)_{\rho \in P_n})$ is given. The sets $Orb_\rho(\mu)$ with $\rho \in P_n$ fixed do not represent a partition of $\mathbf{B}^n$ when μ runs over $\mathbf{B}^n$ and we give the counterexample represented by $\Phi : \mathbf{B} \to \mathbf{B}, \Phi = 1$ (the constant function), $\rho = \chi_{\{0,1,2,...\}}$ for which*

$$Orb_\rho(0) = \{0, 1\},$$

$$Orb_\rho(1) = \{1\}$$

are non-disjoint sets. The consequence is that points from $\mathbf{R} \times \mathbf{B}^n$ exist representing the intersection of several integral curves.

Theorem 19.2. *Let be $\mu \in \mathbf{B}^n$ and $\rho \in P_n$ arbitrary. Then the line $\mathbf{R} \times \{\mu\}$ is an integral curve $\Longleftrightarrow \forall t \in \mathbf{R}, \Phi^\rho(t)(\mu) = \mu$.*

Proof. We have

$\{(t, \Phi^\rho(t)(\mu)) | t \in \mathbf{R}\} = \mathbf{R} \times \{\mu\} \Longleftrightarrow \forall t \in \mathbf{R}, \Phi^\rho(t)(\mu) = \mu.$ ∎

20. ACCESSIBLE STATES

Definition 42. *We say that the state $\mu' \in \mathbf{B}^n$ is* ***directly accessible from*** *$\mu \in \mathbf{B}^n$ if $\mu \neq \mu'$* [4] *and some $\lambda \in \mathbf{B}^n$ exists with*

$$\mu' = \Phi^\lambda(\mu).$$

Theorem 20.1. *Take $\Theta_0 \in P^*(\mathbf{B}^n), \mu', \mu'' \in \mathbf{B}^n$ and suppose that μ'' is directly accessible from μ'*

$$\exists \lambda \in \mathbf{B}^n, \mu'' = \Phi^\lambda(\mu'). \tag{34}$$

If μ' is accessible from any initial state under the form

$$\forall \mu \in \Theta_0, \exists \rho \in P_n, \exists t \in \mathbf{R}, \Phi^\rho(t)(\mu) = \mu', \tag{35}$$

then μ'' is accessible from any initial state

$$\forall \mu \in \Theta_0, \exists \rho' \in P_n, \exists t' > t, \Phi^{\rho'}(t')(\mu) = \mu''. \tag{36}$$

Proof. Let $\mu \in \Theta_0$ be arbitrary. The truth of (35) for $\rho \in P_n$,

$$\rho = \alpha^0 \cdot \chi_{\{t_0\}} \oplus ... \oplus \alpha^k \cdot \chi_{\{t_k\}} \oplus ...$$

$\alpha \in \Pi_n$,$(t_k) \in Seq$ and t shows the existence of two possibilities.

Case $t < t_0$. We choose $\rho' \in P_n$,

$$\rho' = \alpha'^0 \cdot \chi_{\{t'_0\}} \oplus ... \oplus \alpha'^k \cdot \chi_{\{t'_k\}} \oplus ...$$

$\alpha \in \Pi_n$,$(t_k) \in Seq$ such that $\alpha'^0 =$ the λ that makes (34) true and $t' = t'_0 > t$. We have

$$\Phi^\rho(t)(\mu) = \mu = \mu',$$

$$\Phi^{\rho'}(t')(\mu) = \Phi^{\rho'}(t'_0)(\mu') = \Phi^{\alpha'^0}(\mu') = \Phi^{\lambda}(\mu') = \mu''.$$

Case $t \geq t_0$ and we can suppose that $k \in \mathbf{N}$ exists such that $t = t_k$. We choose ρ' such that $\alpha'^0 = \alpha^0, ..., \alpha'^k = \alpha^k, \alpha'^{k+1} =$ the λ that makes (34) true, $(t'_k) = (t_k)$ and $t' = t_{k+1}$. We infer

$$\Phi^{\rho'}(t')(\mu) = \Phi^{\rho'}(t_{k+1})(\mu) = \Phi^{\alpha^0 ... \alpha^k \lambda}(\mu) = \Phi^{\lambda}(\Phi^{\alpha^0 ... \alpha^k}(\mu)) =$$

$$= \Phi^{\lambda}(\Phi^{\rho}(t_k)(\mu)) = \Phi^{\lambda}(\Phi^{\rho}(t)(\mu)) = \Phi^{\lambda}(\mu') = \mu''.$$

∎

21. HUFFMAN REGULAR AUTONOMOUS SYSTEMS

Definition 43. *The autonomous system X is called **Huffman** if it fulfills one of the next two conditions a), b):*

a) $X \in P^*(S^{(n)})$; the function $\Phi : \mathbf{B}^n \to \mathbf{B}^n$ and the systems $f, g : S^{(n)} \to P^*(S^{(n)})$ exist so that

$$\forall y \in S^{(n)}, \exists \lim_{t\to\infty} \Phi(y(t)) \Longrightarrow \forall x \in f(y), \lim_{t\to\infty} x(t) = \lim_{t\to\infty} \Phi(y(t)), \tag{37}$$

$$\forall x \in S^{(n)}, \exists \lim_{t\to\infty} x(t) \Longrightarrow \forall y \in g(x), \lim_{t\to\infty} y(t) = \lim_{t\to\infty} x(t), \tag{38}$$

$$X = \{x | \exists y \in S^{(n)}, x \in f(y) \text{ and } y \in g(x)\}; \tag{39}$$

b) $X \in P^*(S^{(n+n')})$; the function $\Phi : \mathbf{B}^n \to \mathbf{B}^n$ and the systems $f, g : S^{(n)} \to P^*(S^{(n)})$, $f' : S^{(n)} \to P^*(S^{(n')})$ exist so that (37), (38) are true as well as

$$X = \{(x, x') | \exists y \in S^{(n)}, x \in f(y), x' \in f'(y) \text{ and } y \in g(x)\}. \tag{40}$$

The two conditions a), b) have been drawn in Fig. 5.

Remark 28. *A system f having the property that Φ exists with (37) true is called combinational (or race-free stable relative to the function Φ). Property (37) shows that f is a combinational system that computes the function Φ*

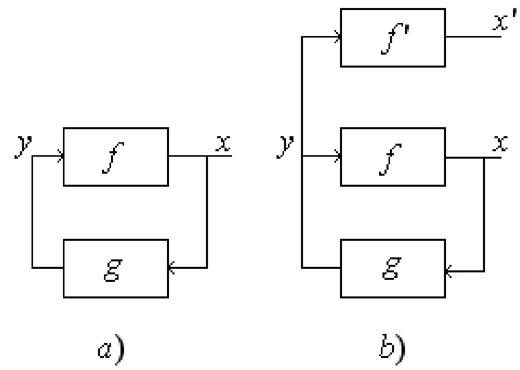

Fig. 5.

and (38) shows that g is a combinational system that computes the identity function $1_{\mathbf{B}^n}$; (39) links f and g and similarly (40) links f, g, f'. As g from (38) models the delay elements, we conclude that the Huffman autonomous systems consist in combinational systems f having feedback loops with delay elements.

Theorem 21.1. *Consider a $d > 0$ and the systems $f, g : S^{(n)} \to P^*(S^{(n)}), X_\Phi^d \in P^*(S^{(n)})$ defined as*

$$\forall y \in S^{(n)}, f(y) = \{x | x(t) = \begin{cases} \Phi^{\alpha^k}(y(t_k)), t = t_k \\ x(t-0), t \notin \{t_0, ..., t_k, ...\}, \end{cases}$$

$$x(-\infty + 0) \in \mathbf{B}^n, \alpha \in \Pi_n, (t_k) \in Seq, \forall k \in \mathbf{N}, t_{k+1} - t_k > d\},$$

$$g(x) = \{x \circ \tau^d\},$$

$$X_\Phi^d = \{x | \exists y \in S^{(n)}, x \in f(y) \text{ and } y \in g(x)\}.$$

Then:

a) f satisfies (37);

b) g satisfies (38);

c) X_Φ^d satisfies (39);

d) $\forall x \in X_\Phi^d$, *we have the existence of* $\mu \in \mathbf{B}^n, \alpha \in \Pi_n, (t_k) \in Seq$ *with* $\forall k \in \mathbf{N}, t_{k+1} - t_k > d$ *and*

$$x(t) = \Phi^{\alpha^0 \cdot \chi_{\{t_0\}} \oplus \ldots \oplus \alpha^k \cdot \chi_{\{t_k\}} \oplus \ldots}(t)(\mu),$$

i.e. $X_\Phi^d \subset X_\Phi$.

Proof. a) Let be $\alpha \in \Pi_n$ and $(t_k) \in Seq$ arbitrary so that $\forall k \in \mathbf{N}, t_{k+1} - t_k > d$. From the hypothesis we can suppose that for an arbitrary $y \in S^{(n)}$, $k' \in \mathbf{N}$ exists so that

$$\forall t \geq t_{k'}, \Phi(y(t)) = \Phi(y(t_{k'})).$$

Define the sets $\Psi_{k'}, ..., \Psi_{k'+p} \subset \{1, ..., n\}, p \in \mathbf{N}$ in the following way

$$\Psi_{k'} = \{i | i \in \{1, ..., n\}, \alpha_i^{k'} = 1\}, ..., \Psi_{k'+p} = \{i | i \in \{1, ..., n\}, \alpha_i^{k'+p} = 1\},$$

$$\Psi_{k'} \cup ... \cup \Psi_{k'+p} = \{1, ..., n\}.$$

Because α is progressive, the definition of these sets is always possible. We can also suppose that $p \geq 1$. We infer for any $x \in f(y)$ that

$t \in [t_{k'}, t_{k'+1}) : \ \forall i \in \Psi_{k'}$,

$$x_i(t) = \Phi_i(y(t_{k'})),$$

$t \in [t_{k'+1}, t_{k'+2}) : \ \forall i \in \Psi_{k'} \cup \Psi_{k'+1}$,

$$\begin{aligned} x_i(t) &= \begin{cases} \Phi_i(y(t_{k'+1})), i \in \Psi_{k'+1} \\ x_i(t_{k'+1} - 0), i \in \Psi_{k'} \setminus \Psi_{k'+1} \end{cases} \\ &= \begin{cases} \Phi_i(y(t_{k'})), i \in \Psi_{k'+1} \\ \Phi_i(y(t_{k'})), i \in \Psi_{k'} \setminus \Psi_{k'+1} \end{cases} = \Phi_i(y(t_{k'})), \\ &\ldots \end{aligned}$$

$t \in [t_{k'+p}, \infty) : \ \forall i \in \Psi_{k'} \cup ... \cup \Psi_{k'+p}$,

$$\begin{aligned} x_i(t) &= \begin{cases} \Phi_i(y(t_{k'+p})), i \in \Psi_{k'+p} \\ x_i(t_{k'+p} - 0), i \in (\Psi_{k'} \cup ... \cup \Psi_{k'+p-1}) \setminus \Psi_{k'+p} \end{cases} \\ &= \begin{cases} \Phi_i(y(t_{k'})), i \in \Psi_{k'+p} \\ \Phi_i(y(t_{k'})), i \in (\Psi_{k'} \cup ... \cup \Psi_{k'+p-1}) \setminus \Psi_{k'+p} \end{cases} = \Phi_i(y(t_{k'})). \end{aligned}$$

b) Some $t' \in \mathbf{R}$ exists so that $\forall t \geq t', x(t) = x(t')$, wherefrom

$$\forall t \geq t' + d, y(t) = g(x)(t) = x(t-d) = x(t').$$

c) Obvious.

d) Let be $x \in X_{\Phi}^{d}$ arbitrary, in other words $\mu \in \mathbf{B}^n, \alpha \in \Pi_n$ and $(t_k) \in Seq$ exist so that $\forall k \in \mathbf{N}, t_{k+1} - t_k > d$ and

$$\begin{cases} x(-\infty + 0) = \mu \\ x(t) = \begin{cases} \Phi^{\alpha^k}(x(t_k - d)), t = t_k \\ x(t-0), t \notin \{t_0, ..., t_k, ...\}. \end{cases} \end{cases} \tag{41}$$

Denote

$$\rho(t) = \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus ... \oplus \alpha^k \cdot \chi_{\{t_k\}}(t) \oplus ...$$

such that (41) becomes

$$\begin{cases} x(-\infty + 0) = \mu, \\ x(t) = \begin{cases} \Phi^{\rho(t_k)}(x(t_k - d)), t = t_k, \\ x(t-0), t \notin \{t_0, ..., t_k, ...\}. \end{cases} \end{cases} \tag{42}$$

Because $x(t) = x(t-0), t \notin \{t_0, ..., t_k, ...\}$, the only discontinuity points of x (i.e. for which $x(t) \neq x(t-0)$) are found in the set (t_k). Thus, taking into account the fact that $t_k - d > t_{k-1}, k \geq 1$, we conclude that

$$\forall k \in \mathbf{N}, x(t_k - d) = x(t_k - 0).$$

Equation (42) is equivalent to (i.e. it has the same solution like)

$$\begin{cases} x(-\infty + 0) = \mu \\ x(t) = \Phi^{\rho(t)}(x(t-0)), \end{cases}$$

see also Theorem 10.1. ∎

22. DELAY-INSENSITIVITY

Definition 44. *The autonomous system $X \in P^*(S^{(n)})$ is, by definition, **delay-insensitive** if the following property of stability is true*

$$\exists \mu' \in \mathbf{B}^n, \forall x \in X, \lim_{t \to \infty} x(t) = \mu'.$$

Theorem 22.1. *Let be the set* $\Theta_0 \in P^*(\mathbf{B}^n)$*. The next statements concerning the delay-insensitivity of* $X_\Phi^{\Theta_0}$ *are equivalent:*

$$\exists \mu' \in \mathbf{B}^n, \forall x \in X_\Phi^{\Theta_0}, \lim_{t\to\infty} x(t) = \mu'; \tag{43}$$

$$\exists \mu' \in \mathbf{B}^n, \forall \mu \in \Theta_0, \forall \rho \in P_n, \lim_{t\to\infty} \Phi^\rho(t)(\mu) = \mu'; \tag{44}$$

$$\exists \mu' \in \mathbf{B}^n, \forall \mu \in \Theta_0, \forall \alpha \in \Pi_n, \lim_{k\to\infty} \Phi^{\alpha^0 \ldots \alpha^k}(\mu) = \mu'. \tag{45}$$

Proof. (43)$\Longleftrightarrow$(44) Because $X_\Phi^{\Theta_0} = \{\Phi^\rho(\cdot)(\mu) | \mu \in \Theta_0, \rho \in P_n\}$, the equivalence is obvious.

(44)$\Longrightarrow$(45) Let be $\mu \in \Theta_0, \alpha \in \Pi_n$, $(t_k) \in Seq$ arbitrary and use the notation

$$\rho(t) = \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus \ldots \oplus \alpha^k \cdot \chi_{\{t_k\}}(t) \oplus \ldots \tag{46}$$

The hypothesis states the existence of $\mu' \in \mathbf{B}^n$ with

$$\exists t' \in \mathbf{R}, \forall t \geq t', \Phi^\rho(t)(\mu) = \mu' \tag{47}$$

and we can suppose the existence of some $k' \in \mathbf{N}$ with $t' = t_{k'}$. Because $\forall t \geq t', \exists k \in \mathbf{N}$,

$$t \in [t_{k'+k}, t_{k'+k+1}) \text{ } and \text{ } \Phi^\rho(t)(\mu) = \Phi^{\alpha^0 \ldots \alpha^{k'} \alpha^{k'+1} \ldots \alpha^{k'+k}}(\mu), \tag{48}$$

we have that (47) implies

$$\exists k' \in \mathbf{N}, \forall k \in \mathbf{N}, \Phi^{\alpha^0 \ldots \alpha^{k'} \alpha^{k'+1} \ldots \alpha^{k'+k}}(\mu) = \mu'. \tag{49}$$

(45)$\Longrightarrow$(44) Let be $\mu \in \Theta_0, \alpha \in \Pi_n$ and $(t_k) \in Seq$ arbitrary, for which we define ρ like in (46), thus $\rho \in P_n$ is an arbitrary element. The hypothesis states the existence of $\mu' \in \mathbf{B}^n$ such that (49) holds. For $t' = t_{k'}$, because (48) is true, it follows the truth of (47). ∎

23. EXAMPLES OF DELAY-INSENSITIVE SYSTEMS

Example 14. *The constant function* $\Phi : \mathbf{B}^n \to \mathbf{B}^n$,

$$\exists \mu' \in \mathbf{B}^n, \forall \mu \in \mathbf{B}^n, \Phi(\mu) = \mu'$$

fulfills (45) with $\Theta_0 = \mathbf{B}^n$. *Indeed, let* $\mu \in \mathbf{B}^n$ *and* $\alpha \in \Pi_n$ *be arbitrary and we have the following possibilities.*

Case $\mu = \mu'$.

Then $\Phi(\mu) = \mu$ *and (45) is true under the form*

$$\forall k \in \mathbf{N}, \Phi^{\alpha^0 \ldots \alpha^k}(\mu) = \mu'.$$

Case $\exists p \in \{1, ..., n\}, \exists i_1 \in \{1, ..., n\}, ..., \exists i_p \in \{1, ..., n\}$ *such that* $\mu' = \mu \oplus \varepsilon^{i_1} \oplus ... \oplus \varepsilon^{i_p}$.

We define $q \in \mathbf{N}$ *as the least number that satisfies*

$$\Psi_0 = \{i | i \in \{1, ..., n\}, \alpha_i^0 = 1\}, ..., \Psi_q = \{i | i \in \{1, ..., n\}, \alpha_i^q = 1\},$$

$$\{i_1, ..., i_q\} \subset \Psi_0 \cup ... \cup \Psi_q.$$

We have

$$\Phi^{\alpha^0}(\mu) = \mu \oplus \underset{j \in \Psi_0 \cap \{i_1, ..., i_p\}}{\Xi} \varepsilon^j,$$

$$...$$

$$\Phi^{\alpha^0 \ldots \alpha^q}(\mu) = \mu \oplus \underset{j \in (\Psi_0 \cup ... \cup \Psi_q) \cap \{i_1, ..., i_p\}}{\Xi} \varepsilon^j = \mu'.$$

From this moment, for any $q' \geq q$, *we have*

$$\Phi^{\alpha^0 \ldots \alpha^{q'}}(\mu) = \mu'.$$

Notation 23.1. *We use to underline sometimes the excited coordinates (see Definition 30), for example* $(..., \underline{\mu_i}, ..., \mu_j, ...)$ *shows the fact that*

$$..., \mu_i \neq \Phi_i(\mu), ..., \mu_j = \Phi_j(\mu), ...$$

Notation 23.2. *If* μ' *is directly accessible from* μ *(see Definition 42), we use to denote this fact by an arrow* $\mu \to \mu'$.

Example 15. *In Fig. 6 we have drawn an RS flip-flop where* λ_1, λ_2, μ_1, $\mu_2 \in \mathbf{B}$.

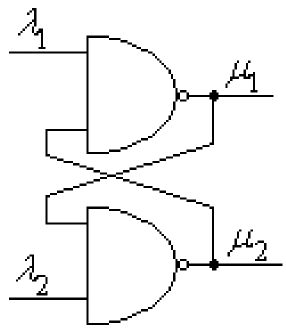

Fig. 6.

Corresponding to $\lambda_1 = 0, \lambda_2 = 1$, *respectively to* $\lambda_1 = 1, \lambda_2 = 0$ *we have two functions* $\Phi, \Phi' : \mathbf{B}^2 \to \mathbf{B}^2$,

$$\forall(\mu_1, \mu_2) \in \mathbf{B}^2, \Phi(\mu_1, \mu_2) = (1, \overline{\mu_1}),$$

$$\forall(\mu_1, \mu_2) \in \mathbf{B}^2, \Phi'(\mu_1, \mu_2) = (\overline{\mu_2}, 1)$$

for which (43),...,(45) are fulfilled when $\Theta_0 = \mathbf{B}^2$. *We show the fulfillment of (45) by* Φ *and* $\mu' = (1, 0)$.

The behavior of the circuit from Fig. 6 with $\lambda_1 = 0, \lambda_2 = 1$ *is the one from Fig. 7.*

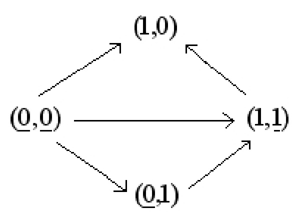

Fig. 7.

Conformably to the Notations 23.1, 23.2, the excited coordinates were underlined and the directly accessible states were outlined with arrows. The fact that in the state $(0, 0)$ *any coordinate may be computed (both coordinates are exited) implies that from there we can get in any of* $(0, 1), (1, 1), (1, 0)$; *the* *states* $(0,1)$ *and* $(1,1)$ *have one excited coordinate only, thus one arrow towards some directly accessible state. The only state that has no excited coordinates is* $(1,0)$, *the system arrives and remains there from any initial value* $\mu \in \mathbf{B}^n$.

The satisfaction of (45) is proved.

24. HAZARD-FREEDOM, THE FIRST DEFINITION

Definition 45. *The system* $X \in P^*(S^{(n)})$ *is, by definition,* ***hazard-free*** *if the property*

$$\exists \mu' \in \mathbf{B}^n, \forall x \in X, x(t) \underset{monotonous}{\rightarrow} \mu'$$

is fulfilled. We have denoted by $\underset{monotonous}{\rightarrow}$ *the coordinatewise monotonous convergence, i.e. each coordinate function* x_i *is allowed to change value at most once.*

Remark 29. *The coordinatewise monotonous* $x \in S^{(n)}$ *functions have a final value. The hazard-freedom of* X *states that all the elements* $x \in X$ *are coordinatewise monotonous and that they have the same final value. Note that, like in the case of delay-insensitivity, this definition of hazard-freedom does not ask that* X *be regular.*

Theorem 24.1. *Let be the set* $\Theta_0 \in P^*(\mathbf{B}^n)$. *The next statements concerning the hazard-freedom of* $X_\Phi^{\Theta_0}$ *are equivalent:*

$$\exists \mu' \in \mathbf{B}^n, \forall x \in X_\Phi^{\Theta_0}, x(t) \underset{monotonous}{\rightarrow} \mu'; \tag{50}$$

$$\exists \mu' \in \mathbf{B}^n, \forall \mu \in \Theta_0, \forall \rho \in P_n, \Phi^\rho(t)(\mu) \underset{monotonous}{\rightarrow} \mu'; \tag{51}$$

$$\exists \mu' \in \mathbf{B}^n, \forall \mu \in \Theta_0, \forall \alpha \in \Pi_n, \Phi^{\alpha^0 \ldots \alpha^k}(\mu) \underset{monotonous}{\rightarrow} \mu'. \tag{52}$$

Proof. Similar with the proof of Theorem 22.1. ∎

Example 16. *For* $\Theta_0 = \{(0,0,0),(1,1,1)\}$, *we have*

$$(0,0,\underline{0}) \rightarrow (0,\underline{0},1) \rightarrow (0,1,1),$$

$$(\underline{1},1,1) \to (0,1,1)$$

and because the state $(0,1,1)$ *has no underlined coordinates, it is a fixed point of* Φ *and a final state of the system.*

25. HAZARD FREEDOM, THE SECOND DEFINITION

Remark 30. *In Definition 45 of hazard-freedom, the vector* $\mu' \in \mathbf{B}^n$ *towards which all* $x \in X$ *converge is independent of* $x(-\infty+0)$*. We give now a possibility of defining a hazard-freedom property for the regular systems where the states* $x \in X$ *converge towards the limit* $\Phi(x(-\infty+0))$*.*

Definition 46. *The regular system* $X \subset X_\Phi$ *is called* ***hazard-free*** *if the following property is true*

$$\forall x \in X, x(t) \underset{monotonous}{\to} \Phi(x(-\infty+0)). \tag{53}$$

Theorem 25.1. *Let be* $\Theta_0 \in P^*(\mathbf{B}^n)$*. The next statements of hazard-freedom of* $X_\Phi^{\Theta_0}$ *are equivalent:*

$$\forall x \in X_\Phi^{\Theta_0}, x(t) \underset{monotonous}{\to} \Phi(x(-\infty+0)); \tag{54}$$

$$\forall \mu \in \Theta_0, \forall \rho \in P_n, \Phi^\rho(t)(\mu) \underset{monotonous}{\to} \Phi(\mu); \tag{55}$$

$$\forall \mu \in \Theta_0, \forall \alpha \in \Pi_n, \Phi^{\alpha^0 \ldots \alpha^k}(\mu) \underset{monotonous}{\to} \Phi(\mu). \tag{56}$$

Proof. Similar with the proof of Theorem 22.1. ∎

Remark 31. *The theorem which follows gives a property that is equivalent to the hazard-freedom of* $X_\Phi^{\Theta_0}$*. However, at the moment, we do not know the proof of its necessity part.*

Theorem 25.2. *Let be* $\Theta_0 \in P^*(\mathbf{B}^n)$*. If*

$$\forall \mu \in \Theta_0, \forall \lambda \in \mathbf{B}^n, \Phi(\Phi^\lambda(\mu)) = \Phi(\mu),$$

then $X_\Phi^{\Theta_0}$ *is hazard-free.*

Proof. We show that property (56) is implied and let for this $\mu \in \Theta_0, \alpha \in \Pi_n$ be arbitrary. If $\Phi(\mu) = \mu$ holds, then the conclusion is true, thus we suppose the existence of $p \geq 1$ and $\{i_1, ..., i_p\} \subset \{1, ..., n\}$ such that

$$\mu \oplus \Phi(\mu) = \varepsilon^{i_1} \oplus ... \oplus \varepsilon^{i_p}.$$

With

$$\Psi_k = \{i | i \in \{1, ..., n\}, \alpha_i^k = 1\}, k \in \mathbf{N},$$

we infer

$$\Phi^{\alpha^0}(\mu) = \mu \oplus \underset{j \in \Psi_0 \cap \{i_1,...,i_p\}}{\Xi} \varepsilon^j,$$

$$\Phi^{\alpha^0\alpha^1}(\mu) = \Phi^{\alpha^1}(\Phi^{\alpha^0}(\mu)) = \Phi^{\alpha^1}(\mu \oplus \underset{j \in \Psi_0 \cap \{i_1,...,i_p\}}{\Xi} \varepsilon^j) =$$

$$= \begin{cases} \Phi_i(\mu \oplus \underset{j \in \Psi_0 \cap \{i_1,...,i_p\}}{\Xi} \varepsilon^j), i \in \Psi_1 \\ \mu_i \oplus \underset{j \in \Psi_0 \cap \{i_1,...,i_p\}}{\Xi} \varepsilon_i^j, i \in \{1, ..., n\} \setminus \Psi_1 \end{cases}$$

$$= \begin{cases} \Phi_i(\mu), i \in \Psi_1 \\ \mu_i \oplus \underset{j \in \Psi_0 \cap \{i_1,...,i_p\}}{\Xi} \varepsilon_i^j, i \in \{1, ..., n\} \setminus \Psi_1 \end{cases}$$

$$= \begin{cases} \mu_i \oplus 1, i \in \Psi_1 \cap \{i_1, ..., i_p\} \\ \mu_i, i \in \Psi_1 \setminus \{i_1, ..., i_p\} \\ \mu_i \oplus 1, i \in (\{1, ..., n\} \setminus \Psi_1) \cap \Psi_0 \cap \{i_1, ..., i_p\} \\ \mu_i, i \in (\{1, ..., n\} \setminus \Psi_1) \setminus (\Psi_0 \cap \{i_1, ..., i_p\}) \end{cases}$$

$$= \begin{cases} \mu_i \oplus 1, i \in (\Psi_1 \cap \{i_1, ..., i_p\}) \cup ((\{1, ..., n\} \setminus \Psi_1) \cap \Psi_0 \cap \{i_1, ..., i_p\}) \\ \mu_i, otherwise \end{cases}$$

We denote by $\overline{\Psi_1}$ the set $\{1, ..., n\} \setminus \Psi_1$. We compute

$$(\Psi_1 \cap \{i_1, ..., i_p\}) \cup ((\{1, ..., n\} \setminus \Psi_1) \cap \Psi_0 \cap \{i_1, ..., i_p\}) =$$

$$= ((\Psi_1 \cap \{i_1, ..., i_p\}) \cup \overline{\Psi_1}) \cap ((\Psi_1 \cap \{i_1, ..., i_p\}) \cup (\Psi_0 \cap \{i_1, ..., i_p\}))$$

$$= (\{i_1, ..., i_p\} \cup \overline{\Psi_1}) \cap (\Psi_1 \cup \Psi_0) \cap \{i_1, ..., i_p\} = (\Psi_1 \cup \Psi_0) \cap \{i_1, ..., i_p\}.$$

By induction on k we can prove that the general term is

$$\Phi^{\alpha^0 \ldots \alpha^k}(\mu) = \mu \oplus \underset{j \in (\Psi_0 \cup \ldots \cup \Psi_k) \cap \{i_1, \ldots, i_p\}}{\Xi} \varepsilon^j$$

and it converges monotonously to

$$\mu \oplus \underset{j \in \{i_1, \ldots, i_p\}}{\Xi} \varepsilon^j = \Phi(\mu).$$

∎

Example 17. *Define* $\Phi : \mathbf{B}^3 \to \mathbf{B}^3$ *by the following table:*

(μ_1, μ_2, μ_3)	Φ
$(0, 0, 0)$	$(1, 1, 0)$
$(1, 0, 0)$	$(1, 1, 0)$
$(0, 1, 0)$	$(1, 1, 0)$
$(1, 1, 0)$	$(1, 1, 0)$
$(0, 0, 1)$	$(1, 1, 1)$
$(1, 0, 1)$	$(1, 1, 1)$
$(0, 1, 1)$	$(1, 1, 1)$
$(1, 1, 1)$	$(1, 1, 1)$

and we see that for $\Theta_0 = \mathbf{B}^3$ *the hazard-freedom property (53) is fulfilled.*

Remark 32. *The first definition of hazard-freedom implies delay-insensitivity, but the second does not.*

26. SYNCHRONOUS-LIKENESS

Definition 47. *We define for* $k \in \mathbf{N}$ *the function* $\Phi^{(k)} : \mathbf{B}^n \to \mathbf{B}^n$ *in the following manner:* $\forall \mu \in \mathbf{B}^n$,

$$\Phi^{(0)}(\mu) = \mu,$$

$$\Phi^{(k+1)}(\mu) = \Phi(\Phi^{(k)}(\mu)).$$

$\Phi^{(k)}$ *is called the* ***iterate of order*** k *(or the* $k-$***th iterate****)* ***of*** Φ*.*

Definition 48. *The autonomous system $X \subset X_\Phi$ is **synchronous-like** if*

$$\forall x \in X, \exists (t_k) \in Seq, \forall k \in \mathbf{N}, x(t_k) = \Phi^{(k+1)}(x(-\infty + 0)). \tag{57}$$

Remark 33. *In formula (57) during some interval $[t_k, t_{k+1})$ x may change value. The point is here that the values $x(-\infty+0)$, $\Phi(x(-\infty+0))$, $\Phi(\Phi(x(-\infty+0)))$, $\Phi(\Phi(\Phi(x(-\infty+0))))$, ... are reached, in this order.*

Theorem 26.1. *Consider the set $\Theta_0 \in P^*(\mathbf{B}^n)$. The next statements concerning the synchronous-likeness of $X_\Phi^{\Theta_0}$ are equivalent:*

$$\forall x \in X_\Phi^{\Theta_0}, \exists (t_k) \in Seq, \forall k \in \mathbf{N}, x(t_k) = \Phi^{(k+1)}(x(-\infty + 0)); \tag{58}$$

$$\forall \mu \in \Theta_0, \forall \rho \in P_n, \exists (t_k) \in Seq, \forall k \in \mathbf{N}, \Phi^\rho(t_k)(\mu) = \Phi^{(k+1)}(\mu); \tag{59}$$

$$\forall \mu \in \Theta_0, \forall \alpha \in \Pi_n, \exists (j_k) \in Sub(\mathbf{N}), \forall k \in \mathbf{N}; \tag{60}$$

$$\Phi^{\alpha^0 \ldots \alpha^{j_k}}(\mu) = \Phi^{(k+1)}(\mu).$$

Proof. The line of the proof is similar with the proofs of Theorems 22.1, 24.1 and 25.1. ∎

Theorem 26.2. *Suppose that the system X satisfies the hazard-freedom property (53)*

$$\forall x \in X, x(t) \underset{monotonous}{\rightarrow} \Phi(x(-\infty + 0)).$$

Then it is synchronous-like.

Proof. For arbitrary $x \in X$, (57) is fulfilled with

$$\forall k \geq 2, \Phi^{(k)}(x(-\infty + 0)) = \Phi(x(-\infty + 0)).$$

∎

27. SYNCHRONICITY (THE TECHNICAL CONDITION OF PROPER OPERATION)

Definition 49. *The autonomous system $X \subset X_\Phi$ is called **synchronous** and we also say that it fulfills the **technical condition of proper operation** if* *the following property is satisfied:* $\forall x \in X$, $\exists (t_k) \in Seq$,

$$x(t) = x(-\infty+0) \cdot \chi_{(-\infty,t_0)}(t) \oplus \Phi(x(-\infty+0)) \cdot \chi_{[t_0,t_1)}(t) \oplus ... \quad (61)$$

$$... \oplus \Phi^{(k+1)}(x(-\infty+0)) \cdot \chi_{[t_k,t_{k+1})}(t) \oplus ...$$

Theorem 27.1. *Given the set* $\Theta_0 \in P^*(\mathbf{B}^n)$*, the following statements concerning the synchronicity of* $X_\Phi^{\Theta_0}$ *are equivalent:*

$$\forall x \in X_\Phi^{\Theta_0}, \exists (t_k) \in Seq, x(t) = x(-\infty+0) \cdot \chi_{(-\infty,t_0)} \oplus \quad (62)$$

$$\oplus \Phi(x(-\infty+0)) \cdot \chi_{[t_0,t_1)} \oplus ... \oplus \Phi^{(k)}(\mu) \cdot \chi_{[t_k,t_{k+1})} \oplus ...$$

$$\forall \mu \in \Theta_0, \forall \alpha \in \Pi_n, \exists j_0 \in \mathbf{N}, \exists j_1 \in \mathbf{N}^*, ..., \exists j_k \in \mathbf{N}^*, ... \quad (63)$$

$$(\Phi^{\alpha^0...\alpha^k}(\mu))_{k\in\mathbf{N}} = (\underbrace{\mu, ..., \mu}_{j_0}, \underbrace{\Phi(\mu), ..., \Phi(\mu)}_{j_1}, ..., \underbrace{\Phi^{(k)}(\mu), ..., \Phi^{(k)}(\mu)}_{j_k}, ...),$$

where $\underbrace{\mu, ..., \mu}_{0}$ *means non-existing values,*

$$\forall \mu \in \Theta_0, \forall k \in \mathbf{N}, \Phi^{(k+1)}(\mu) = \Phi^{(k)}(\mu) \text{ or} \quad (64)$$

$$\text{or } \exists i \in \{1, ..., n\}, \Phi^{(k+1)}(\mu) = \Phi^{(k)}(\mu) \oplus \varepsilon^i.$$

Proof. (62)$\Longrightarrow$(63) Let $\mu \in \Theta_0$, $\alpha \in \Pi_n$ and $(t_k) \in Seq$ be arbitrary, thus

$$x(t) = \mu \cdot \chi_{(-\infty,t_0)}(t) \oplus \Phi^{\alpha^0}(\mu) \cdot \chi_{[t_0,t_1)}(t) \oplus ... \quad (65)$$

$$... \oplus \Phi^{\alpha^0...\alpha^k}(\mu) \cdot \chi_{[t_k,t_{k+1})}(t) \oplus ...$$

is an arbitrary element from $X_\Phi^{\Theta_0}$. The hypothesis of synchronicity of $X_\Phi^{\Theta_0}$ shows that $\exists (t'_k) \in Seq$ such that

$$x(t) = \mu \cdot \chi_{(-\infty,t'_0)}(t) \oplus \Phi(\mu) \cdot \chi_{[t'_0,t'_1)}(t) \oplus ... \oplus \Phi^{(k)}(\mu) \cdot \chi_{[t'_k,t'_{k+1})}(t) \oplus ... \quad (66)$$

By comparing (65) with (66) the validity of (63) follows.

(63)$\Longrightarrow$(62) Take some arbitrary $\mu \in \Theta_0, \alpha \in \Pi_n, (t_k) \in Seq$. In this situation (65) represents an arbitrary $x \in X_\Phi^{\Theta_0}$. The hypothesis states the existence of $j_0 \in \mathbf{N}$, $j_1 \in \mathbf{N}^*$, ..., $j_k \in \mathbf{N}^*$, ... such that

$$(\Phi^{\alpha^0...\alpha^k}(\mu))_{k\in\mathbf{N}} = (\underbrace{\mu, ..., \mu}_{j_0}, \underbrace{\Phi(\mu), ..., \Phi(\mu)}_{j_1}, ..., \underbrace{\Phi^{(k)}(\mu), ..., \Phi^{(k)}(\mu)}_{j_k}, ...),$$

and this, looking at (65), means the existence of a subsequence $(t'_k) \in Seq$ of (t_k) such that (66) be true. $X_{\Phi}^{\Theta_0}$ is synchronous.

(63)$\Longrightarrow$(64) Suppose, against all reason, that (64) is not true. In this situation $\mu \in \Theta_0$, $k \in \mathbf{N}$, $p \in \{2, ..., n\}$ and $i_1, ..., i_p \in \{1, ..., n\}$ distinct exist such that

$$\Phi^{(k+1)}(\mu) = \Phi^{(k)}(\mu) \oplus \varepsilon^{i_1} \oplus ... \oplus \varepsilon^{i_p}.$$

Then $\alpha \in \Pi_n$ and $k' \in \mathbf{N}$ exist such that

$$\Phi^{(k)}(\mu) = \Phi^{\alpha^0 ... \alpha^{k'}}(\mu),$$

$$\Phi^{\alpha^0 ... \alpha^{k'} \alpha^{k'+1}}(\mu) \notin \{\Phi^{(k)}(\mu), \Phi^{(k+1)}(\mu)\}$$

and this happens if we take $\alpha^{k'+1} \in \mathbf{B}^n$ with

$$\exists j \in \{1, ..., p\}, \exists j' \in \{1, ..., p\}, \alpha_j^{k'+1} = 1 \text{ and } \alpha_{j'}^{k'+1} = 0.$$

We have obtained a contradiction with (63).

(64)$\Longrightarrow$(63) Let be $\mu \in \Theta_0$ and $\alpha \in \Pi_n$ arbitrary and prove (63) by induction on $k \in \mathbf{N}$.

$k = 0 : \Phi^{(0)}(\mu) = \mu$ and we have the possibilities:

- $\Phi^{(1)}(\mu) = \Phi^{(0)}(\mu)$, then $(\Phi^{\alpha^0 ... \alpha^k}(\mu))_{k \in \mathbf{N}}$ has all the terms equal to μ and $j_0 \in \mathbf{N}, j_1 \in \mathbf{N}^*, ..., j_k \in \mathbf{N}^*, ...$ may be taken arbitrarily;
- $\exists i \in \{1, ..., n\}, \Phi^{(1)}(\mu) = \Phi^{(0)}(\mu) \oplus \varepsilon^i$, then

$$\Phi^{\alpha^0}(\mu) = \begin{cases} \mu, \ if \ \alpha_i^0 = 0 \ (\Longrightarrow j_0 \geq 1) \\ \Phi(\mu), \ if \ \alpha_i^0 = 1 \ (\Longrightarrow j_0 = 0) \end{cases}$$

k : from the hypothesis of the induction, some $k' \in \mathbf{N}$ exists such that $\Phi^{\alpha^0 ... \alpha^{k'}}(\mu) = \Phi^{(k)}(\mu)$ and we have the possibilities:

- $\Phi^{(k+1)}(\mu) = \Phi^{(k)}(\mu)$, then $\Phi^{\alpha^0 ... \alpha^{k'}}(\mu) = \Phi^{\alpha^0 ... \alpha^{k'} \alpha^{k'+1}}(\mu) = ... = \Phi^{(k)}(\mu)$ and $j_k \in \mathbf{N}^*, j_{k+1} \in \mathbf{N}^*, ...$ may be chosen arbitrarily;
- $\exists i \in \{1, ..., n\}, \Phi^{(k+1)}(\mu) = \Phi^{(k)}(\mu) \oplus \varepsilon^i$, then

$$\Phi^{\alpha^0 ... \alpha^{k'} \alpha^{k'+1}}(\mu) = \begin{cases} \Phi^{(k)}(\mu), \ if \ \alpha_i^{k'+1} = 0 \\ \Phi^{(k+1)}(\mu), \ if \ \alpha_i^{k'+1} = 1 \end{cases}.$$

∎

Remark 34. *The synchronous systems are also synchronous-like. The point is that in (57), during the intervals* $[t_k, t_{k+1})$, x *may change value, while in (61), during the intervals* $[t_k, t_{k+1})$, x *is constant.*

28. THE GENERALIZED TECHNICAL CONDITION OF PROPER OPERATION

Definition 50. *The regular system* $X \subset X_\Phi$ *is said to satisfy the* ***generalized technical condition of proper operation*** *if*

$$\forall x \in X, \forall \nu \in \{\Phi^{(k)}(x(-\infty + 0)) | k \in \mathbf{N}\}, \forall \lambda \in \mathbf{B}^n,$$

$$\Phi^\lambda(\nu) \neq \Phi(\nu) \Longrightarrow \Phi(\Phi^\lambda(\nu)) = \Phi(\nu).$$

Theorem 28.1. *If the system* X *having the generator function* Φ *satisfies the technical condition of proper operation (synchronicity), then it fulfills also the generalized condition of proper operation.*

Proof. Let be $\lambda \in \mathbf{B}^n$, $\mu \in \mathbf{B}^n$ and $k \in \mathbf{N}$ arbitrary (see (64)), for which $\nu = \Phi^{(k)}(\mu)$.

Case $\Phi(\nu) = \nu$

Then $\Phi^\lambda(\nu) = \nu = \Phi(\nu)$ and the conclusion follows.

Case $\exists i \in \{1, ..., n\}, \Phi(\nu) = \nu \oplus \varepsilon^i$

There are two possibilities, $\Phi^\lambda(\nu) = \nu$ ($\Longrightarrow \Phi(\Phi^\lambda(\mu)) = \Phi(\nu)$), $\Phi^\lambda(\nu) = \Phi(\nu)$ and the conclusion is true in both of them. ∎

Theorem 28.2. *If* $X \subset X_\Phi$ *satisfies the generalized technical condition of proper operation then it is synchronous-like.*

Proof. Let be $x \in X$ arbitrary,

$$x(t) = \mu \cdot \chi_{(-\infty,t_0)}(t) \oplus \Phi^{\alpha^0}(\mu) \cdot \chi_{[t_0,t_1)}(t) \oplus ... \oplus \Phi^{\alpha^0...\alpha^k}(\mu) \cdot \chi_{[t_k,t_{k+1})}(t) \oplus ...$$

with $\mu \in \mathbf{B}^n$, $\alpha \in \Pi_n$ and $(t_k) \in Seq$. If $\Phi(\mu) = \mu$, then $x(t) = \mu$ is the constant function and (57) is fulfilled, thus we can suppose that

$$\exists i_1 \in \{1, ..., n\}, ..., \exists i_p \in \{1, ..., n\}, \Phi(\mu) = \mu \oplus \varepsilon^{i_1} \oplus ... \oplus \varepsilon^{i_p}.$$

We define $q \in \mathbf{N}$ as the least integer such that

$$\Psi_0 = \{i | i \in \{1, ..., n\}, \alpha_i^0 = 1\},$$

$$...$$

$$\Psi_q = \{i | i \in \{1, ..., n\}, \alpha_i^q = 1\},$$

$$\{i_1, ..., i_p\} \subset \Psi_0 \cup ... \cup \Psi_q$$

and we have, see also the proof of Theorem 25.2,

$$\Phi^{\alpha^0 ... \alpha^q}(\mu) = \mu \oplus \underset{j \in (\Psi_0 \cup ... \cup \Psi_q) \cap \{i_1, ..., i_p\}}{\Xi} \varepsilon^j = \Phi(\mu).$$

We can define $t'_0 = t_q$, for which (57) is true under the form $x(t'_0) = \Phi(\mu)$.

The hypothesis of the induction shows the existence of the rank $k_1 \in \mathbf{N}^*$ and of the numbers $t'_0 < t'_1 < ... < t'_{k-1}$ such that $t'_{k-1} = t_{k_1 - 1}, x(t'_1) = \Phi^{(2)}(\mu), ..., x(t'_{k-1}) = \Phi^{(k)}(\mu)$. If $\Phi^{(k+1)}(\mu) = \Phi^{(k)}(\mu)$, then $\forall t \geq t'_{k-1}, x(t) = x(t'_{k-1})$ and (57) is fulfilled with arbitrary $t'_k > t'_{k-1}$ and $x(t'_k) = \Phi^{(k+1)}(\mu)$, thus we can suppose that

$$\exists j_1 \in \{1, ..., n\}, ..., \exists j_{p'} \in \{1, ..., n\}, \Phi^{(k+1)}(\mu) = \Phi^{(k)}(\mu) \oplus \varepsilon^{j_1} \oplus ... \oplus \varepsilon^{j_{p'}}.$$

We define $q' \in \mathbf{N}$ as the least integer such that

$$\Psi_{k_1} = \{i | i \in \{1, ..., n\}, \alpha_i^{k_1} = 1\},$$

$$...$$

$$\Psi_{k_1 + q'} = \{i | i \in \{1, ..., n\}, \alpha_i^{k_1 + q'} = 1\},$$

$$\{j_1, ..., j_{p'}\} \subset \Psi_{k_1} \cup ... \cup \Psi_{k_1 + q'}$$

for which we get

$$\Phi^{\alpha^0 ... \alpha^{k_1 + q'}}(\mu) = \Phi^{(k)}(\mu) \oplus \underset{j \in (\Psi_{k_1} \cup ... \cup \Psi_{k_1 + q'}) \cap \{j_1, ..., j_{p'}\}}{\Xi} \varepsilon^j = \Phi^{(k+1)}(\mu).$$

We define $t'_k = t_{k_1 + q'}$ and (57) is fulfilled under the form $x(t'_k) = \Phi^{(k+1)}(\mu)$. ∎

29. DISCRETE TIME

Notation 29.1. *Denote*

$$\mathbf{N}_{\_} = \{-1, 0, 1, 2, ...\}.$$

Definition 51. *Let be* $\alpha \in \Pi_n$. *Define* Φ ***at the power*** $\alpha, \Phi^\alpha : \mathbf{N}_{\_} \to (\mathbf{B}^n)^{\mathbf{B}^n}$ *by*

$$\forall k \in \mathbf{N}_{\_}, \forall \mu \in \mathbf{B}^n, \Phi^\alpha(k)(\mu) = \begin{cases} \mu, k = -1 \\ \Phi^{\alpha^0 \ldots \alpha^k}(\mu), k \geq 0. \end{cases}$$

Definition 52. *The couple* $\phi' = (\mathbf{B}^n, (\Phi^\alpha)_{\alpha \in \Pi_n})$ *is called* ***a discrete Boolean dynamical system****.* $\mathbf{B}^n$ *is the* ***phase space****, or the* ***state space*** *and* $\mu \in \mathbf{B}^n$ *is called* ***phase****, or* ***state****. The function* Φ *is the* ***generator function*** *of* ϕ' *and* $\Phi^\alpha, \alpha \in \Pi_n$ *are called the* ***computations*** *of* Φ*. The domain* $\mathbf{N}_{\_}$ *of the computations* Φ^α *is the* ***time set*** *and* $k \in \mathbf{N}_{\_}$ *is the* ***time parameter****.*

Remark 35. *To be compared Definition 51 with Definition 23 and Definition 52 with Definition 24. Many definitions and also the reasoning from this paper may be formulated in discrete time and, as a matter of fact, such discrete time reasoning has been used.*

Notes

1. We abusively identify a function $x \in S^{(n)}$ normally called state with its values $\mu = x(t)$.
2. The word 'manifold' is traditional and it was borroughed from the literature by analogy; in our case it has not a precise meaning, since the manifolds were not defined in the binary context.
3. The proof of this Lemma was suggested to us by Professor Sorin Gal.
4. this request is that of a strict accessibility

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