ANALELE UNIVERSITATII DIN ORADEA,
FASCICOLA MATEMATICA, TOM V, 1995-1996

The Convolution Product of the $\boldsymbol{R} \to \boldsymbol{B}_2$ Functions

Serban E. Vlad
str. Zimbrului, Nr.3, Bl.PB68, Et.2, Ap.11, 3700, Oradea, Romania
E-mail: serban_e_vlad@yahoo.com

1. Introduction

The study of the $\boldsymbol{R} \to \boldsymbol{B}_2$ functions has its origin in modeling the electrical signals that characterize the switching circuits. The models are called asynchronous automata.

Independently on these interests, there have been defined for the $\boldsymbol{R} \to \boldsymbol{B}_2$ functions, by analogy with the real analysis, some basic notions (derivatives, integrals) and such results have been published in the previous issues of the Annals.

Continuing these ideas, our present purpose is to define and characterize the convolution product.

2. Preliminaries

2.1 $\boldsymbol{B}_2 = \{0,1\}$ is the binary Boole algebra with the discrete topology. We shall take in consideration its field structure, relative to '$\oplus$', the modulo 2 sum and '$\cdot$', the intersection.

2.2 The *support* set of $x : \boldsymbol{R} \to \boldsymbol{B}_2$ is defined by:

$$supp\ x = \{t \mid t \in \boldsymbol{R},\ x(t) = 1\}$$

For some $y : \boldsymbol{R} \to \boldsymbol{B}_2$ we have

$$supp\ (x \oplus y) = supp\ x\ \Delta\ supp\ y$$
$$supp\ (x \cdot y) = supp\ x \wedge supp\ y$$

2.3 If for $A \subset \boldsymbol{R}$, the set $A \wedge supp\ x$ is finite, there is defined the modulo 2 summation:

$$\underset{\xi \in A}{\Xi}\ x(\xi) = \begin{cases} 1, if\ \mid A \wedge supp\ x \mid is\ odd \\ 0, if\ \mid A \wedge supp\ x \mid is\ even \end{cases}$$

Here, $A \wedge supp\ x = \varnothing$ is considered to be a special case of finite set and $0 = \mid \varnothing \mid$ is considered an even number.

2.4 For $B \subset \boldsymbol{R}$ a finite set, there is defined the number:

$$\mu_f(B) = \begin{cases} 1, if\ \mid B \mid is\ odd \\ 0, if\ \mid B \mid is\ even \end{cases}$$

The binary function μ_f is called the *finite Boolean measure*. Its importance is given by the fact that

$$\underset{\xi \in A}{\Xi}\ x(\xi) = \mu_f(A \wedge supp\ x),\ A \wedge supp\ x \text{ is finite}$$

and let us remark also the additivity:

$$\mu_f(B\ \Delta\ C) = \mu_f(B) \oplus \mu_f(C),\ C \subset \boldsymbol{R}, C \text{ is finite}$$

2.5 The *symmetrical intervals* are defined by

$$[[a,b))=[a,b)\vee[b,a)$$
$$((a,b]]=(a,b]\vee(b,a],\ a,b\in \boldsymbol{R}$$

2.6 a) In the above definition

- if $a\neq b$, then exactly one of $[a,b),[b,a)$, respectively one of $(a,b],(b,a]$ is non-empty

- if $a=b$, then both $[a,b),[b,a)$, respectively $(a,b],(b,a]$ are empty

b) 2.5 gives the first example of duality in this theory.

2.7 We define the *integrals* (=binary numbers), for $a,b\in \boldsymbol{R}$ and $x:\boldsymbol{R}\to \boldsymbol{B}_2$

$$\int_a^{b} {}^{-} x=\underset{\xi\in[[a,b))}{\Xi} x(\xi) \qquad \int_a^{b} {}^{+} x=\underset{\xi\in((a,b]]}{\Xi} x(\xi)$$

called the *left definite integral*, respectively the *right definite integral* of x from a to b.

2.8 It is of interest the case when in 2.7 a is a fixed parameter and $b=t$ is the variable. We get the *left* and the *right primitives* (= $\boldsymbol{R}\to \boldsymbol{B}_2$ functions):

$$\int^{-} x(t)=\int_a^{t} {}^{-} x \qquad \int^{+} x(t)=\int_a^{t} {}^{+} x$$

defined up to an additive constant, since it may be easily proved that:

$$\int_a^{t} {}^{-} x=\int_{a'}^{t} {}^{-} x\oplus\int_a^{a'} {}^{-} x \qquad \int_a^{t} {}^{+} x=\int_{a'}^{t} {}^{+} x\oplus\int_a^{a'} {}^{+} x$$

2.9 There are obvious the conditions of existence (see 2.3) of:

a) $\int_a^{b} {}^{-} x,\int_a^{b} {}^{+} x:\{t\,|\,t\in(\min(a,b),\max(a,b)),x(t)=1\}$ is finite

b) $\int^{-} x,\int^{+} x:\forall a,b\in \boldsymbol{R},(a,b)\wedge \mathit{supp}\ x$ is finite

c) $\int_{-\infty}^{\infty} {}^{-} x=\int_{-\infty}^{\infty} {}^{+} x=\underset{\xi\in \boldsymbol{R}}{\Xi} x(\xi):\mathit{supp}\ x$ is finite

2.10 The *limits*: *left* $x(t-0)$, *right* $x(t+0)$ and the *derivatives*: *left* $D^{-}x(t)$, *right* $D^{+}x(t)$ of $x:\boldsymbol{R}\to \boldsymbol{B}_2$ are the $\boldsymbol{R}\to \boldsymbol{B}_2$ functions defined by:

$$\forall t\in \boldsymbol{R},\exists x(t-0)\in \boldsymbol{B}_2,\exists\varepsilon>0,\forall\xi\in(t-\varepsilon,t),x(\xi)=x(t-0)$$
$$\forall t\in \boldsymbol{R},\exists x(t+0)\in \boldsymbol{B}_2,\exists\varepsilon>0,\forall\xi\in(t,t+\varepsilon),x(\xi)=x(t+0)$$
$$D^{-}x(t)=x(t-0)\oplus x(t) \qquad D^{+}x(t)=x(t+0)\oplus x(t)$$

2.11 We shall use the following notations:

$$\chi_A(t)=\begin{cases}1, \mathit{if}\ t\in A\\ 0, \mathit{if}\ t\notin A\end{cases}$$ the *characteristic function* of the set $A\subset \boldsymbol{R}$

$$\delta(t)=\chi_{\{0\}}(t)$$

$$\eta^{-}(t)=\chi_{[0,\infty)}(t),\ \eta^{+}(t)=\chi_{(-\infty,0]}(t),\ t\in \boldsymbol{R}$$

3. The Definition of the Convolution Product

3.1 Let $x, y : \boldsymbol{R} \to \boldsymbol{B}_2$ be two functions having the property that for any $t \in \boldsymbol{R}$, the functions in ξ:

$$h_t(\xi) = x(\xi) \cdot y(t - \xi)$$

are integrable, in the sense that $supp\ h_t$ is finite, see 2.9 c). We shall call the *convolution product* (or the *product of convolution*) of x and y, in this order, the function in t:

$$(x * y)(t) \overset{def}{=} \overline{\int_{-\infty}^{\infty}} x(\xi) \cdot y(t - \xi) = \underset{\xi \in \boldsymbol{R}}{\Xi}\, x(\xi) \cdot y(t - \xi)$$

where the integral is taken relative to ξ.

3.2 We are interested now to show some possibilities when the convolution product exists and we define for this purpose the next spaces of functions $x : \boldsymbol{R} \to \boldsymbol{B}_2$:

$\boldsymbol{I}_\infty = \{x \mid supp\ x\ is\ \ finite\}$, the *integrable* functions

$\boldsymbol{I}_{Loc} = \{x \mid \forall a, b \in \boldsymbol{R}, (a,b) \wedge supp\ x\ is\ \ finite\}$, the *locally integrable* functions (also called the functions with *locally finite support*)

$\boldsymbol{B} = \{x \mid \exists \alpha, \beta \in \boldsymbol{R}, supp\ x \subset [\alpha, \beta]\}$, the functions with *bounded support*

$\boldsymbol{B}_i = \{x \mid \exists \alpha \in \boldsymbol{R}, supp\ x \subset [\alpha, \infty)\}$, the functions with *inferiorly bounded support*

$\boldsymbol{B}_s = \{x \mid \exists \beta \in \boldsymbol{R}, supp\ x \subset (-\infty, \beta]\}$, the functions with *superiorly bounded support*

$\boldsymbol{B}_{i,Loc} = \{x \mid \forall \alpha \in \boldsymbol{R}, (-\infty, \alpha] \wedge supp\ x\ is\ \ finite\}$, the functions with *locally finite inferiorly bounded support*

$\boldsymbol{B}_{s,Loc} = \{x \mid \forall \alpha \in \boldsymbol{R}, [\alpha, \infty) \wedge supp\ x\ is\ \ finite\}$, the functions with *locally finite superiorly bounded support*

3.3 There are true:

$$\boldsymbol{I}_\infty = \boldsymbol{I}_{Loc} \wedge \boldsymbol{B},\ \boldsymbol{B} = \boldsymbol{B}_i \wedge \boldsymbol{B}_s$$

$$\boldsymbol{B}_{i,Loc} = \boldsymbol{I}_{Loc} \wedge \boldsymbol{B}_i,\ \boldsymbol{B}_{s,Loc} = \boldsymbol{I}_{Loc} \wedge \boldsymbol{B}_s$$

All these spaces of functions are algebras relative to the sum $'\oplus'$ of functions, the product $'\cdot'$ of functions and the product $'\cdot'$ of functions with scalars from $\boldsymbol{B}_2$.

3.4 **Theorem** In any of the following situations:

a) $x : \boldsymbol{R} \to \boldsymbol{B}_2, y \in \boldsymbol{I}_\infty$

b) $x \in \boldsymbol{I}_{Loc}, y \in \boldsymbol{B}$

c) $x \in \boldsymbol{B}_i, y \in \boldsymbol{B}_{i,Loc}$

c)* $x \in \boldsymbol{B}_s, y \in \boldsymbol{B}_{s,Loc}$

the convolution products $x * y, y * x$ exist.

Proof We must show that the functions in ξ:

$$h_t(\xi) = x(\xi) \cdot y(t - \xi),\ g_t(\xi) = x(t - \xi) \cdot y(\xi)$$

have a finite support for any $t \in \boldsymbol{R}$.

a) If $y \in \boldsymbol{I}_\infty$, the function in $\xi : y(t - \xi)$ belongs to $\boldsymbol{I}_\infty$ for any t and the set

$$supp\ h_t = supp\ x \wedge \{\xi \mid y(t - \xi) = 1\} \subset \{\xi \mid y(t - \xi) = 1\}$$

is finite. $supp\ g_t$ is obviously finite.

b) If $y \in \boldsymbol{B}$, the function in $\xi : y(t-\xi)$ has a bounded support and from the definition of the locally integrable functions, we get the existence of some $\alpha, \beta \in \boldsymbol{R}$ so that

$$supp\ h_t = supp\ x \wedge \{\xi \mid y(t-\xi) = 1\} \subset supp\ x \wedge [\alpha, \beta]$$

is finite. On the other hand, if $x \in \boldsymbol{I}_{Loc}$, the function in $\xi : x(t-\xi) \in \boldsymbol{I}_{Loc}$ etc.

c) If $y \in \boldsymbol{B}_{i,Loc}$, then the function in $\xi : y(t-\xi)$ belongs to $\boldsymbol{B}_{s,Loc}$ and if the numbers α, β_t satisfy:

$$x(\xi) = 1 \Rightarrow \xi \geq \alpha$$

$$y(t-\xi) = 1 \Rightarrow \xi \leq \beta_t$$

then $h_t(\xi)$ is locally integrable (because of $y(t-\xi)$) with the support included in $[\alpha, \beta_t]$ i.e. it has a finite support, being integrable.

On the other hand, if $x \in \boldsymbol{B}_i$, then the function in $\xi : x(t-\xi) \in \boldsymbol{B}_s$ etc.

4. The Properties of the Convolution Product

4.1 The convolution product is commutative:

$$(x * y)(t) = \underset{\xi \in \boldsymbol{R}}{\Xi}\, x(\xi) \cdot y(t-\xi) = \underset{\xi' \in \boldsymbol{R}}{\Xi}\, x(t-\xi') \cdot y(\xi') = (y * x)(t)\text{, where } t-\xi = \xi'$$

4.2 Let $x, y : \boldsymbol{R} \to \boldsymbol{B}_2$ with the property that

- $x * y$ exists
- $\exists \alpha, \beta$ so that $supp\ x \subset [\alpha, \infty),\ supp\ y \subset [\beta, \infty)$

see 3.4 c). Then:

$$supp\ (x * y) \subset [\alpha + \beta, \infty)$$

Proof We define $y_t^* : \boldsymbol{R} \to \boldsymbol{B}_2,\ y_t^*(\xi) = y(t-\xi)$ and we get

$$supp\ y_t^* \subset (-\infty, t-\beta]$$

$$\{\xi \mid x(\xi) \cdot y(t-\xi) = 1\} = supp\ x \wedge supp\ y_t^* \subset [\alpha, \infty) \wedge (-\infty, t-\beta] = [\alpha, t-\beta]$$

$$\alpha \leq t - \beta \Leftrightarrow t \geq \alpha + \beta$$

4.3 For $x, y : \boldsymbol{R} \to \boldsymbol{B}_2$, we suppose that $x * y$ exists. The next formula is true:

$$(x * y)(t) = \mu_f\,(supp\ x \times \boldsymbol{R} \wedge \boldsymbol{R} \times supp\ y \wedge \{(\xi, \xi') \mid \xi + \xi' = t\})$$

Proof $(x * y)(t) = \underset{\xi \in \boldsymbol{R}}{\Xi}\, x(\xi) \cdot y(t-\xi) = \underset{\xi + \xi' \in \boldsymbol{R}}{\Xi}\, x(\xi) \cdot y(\xi') =$

$$= \mu_f\,(\{(\xi, \xi') \mid x(\xi) = 1\ \&\ y(\xi') = 1\ \&\ \xi + \xi' = t\}) =$$

$$= \mu_f\,(supp\ x \times \boldsymbol{R} \wedge \boldsymbol{R} \times supp\ y \wedge \{(\xi, \xi') \mid \xi + \xi' = t\})$$

4.4 a) For $x : \boldsymbol{R} \to \boldsymbol{B}_2$ and $\delta_\tau(t) = \delta(t-\tau)$, we have that $\delta_\tau * x,\ x * \delta_\tau$ exist and

$$(\delta_\tau * x)(t) = \underset{\xi \in \boldsymbol{R}}{\Xi}\, \delta_\tau(\xi) \cdot x(t-\xi) = \underset{\xi \in \{\tau\}}{\Xi}\, \delta(\xi) \cdot x(t-\xi) = x(t-\tau)$$

b) $\delta * x,\ x * \delta$ always exist and δ is the neuter element of $* : \delta * x = x * \delta = x$.

4.5 Let $x, y, z : \boldsymbol{R} \to \boldsymbol{B}_2$ with the property that $x * z$ and $y * z$ exist. Then $(x \oplus y) * z$ exists and the property of distributivity is true:

$$(x \oplus y) * z = x * z \oplus y * z$$

Proof $(x * z)(t) \oplus (y * z)(t) =$

$$= \mu_f\,(supp\ x \times \boldsymbol{R} \wedge \boldsymbol{R} \times supp\ z \wedge \{(\xi, \xi') \mid \xi + \xi' = t\}) \oplus$$

$\oplus \mu_f(supp\ y \times \boldsymbol{R} \wedge \boldsymbol{R} \times supp\ z \wedge \{(\xi,\xi') \mid \xi+\xi'=t\}) =$

$= \mu_f(supp\ x \times \boldsymbol{R} \wedge \boldsymbol{R} \times supp\ z \wedge \{(\xi,\xi') \mid \xi+\xi'=t\}\ \Delta$

$\wedge\ supp\ y \times \boldsymbol{R} \wedge \boldsymbol{R} \times supp\ z \wedge \{(\xi,\xi') \mid \xi+\xi'=t\}) =$

$= \mu_f((supp\ x\ \Delta\ supp\ y) \times \boldsymbol{R} \wedge \boldsymbol{R} \times supp\ z \wedge \{(\xi,\xi') \mid \xi+\xi'=t\}) =$

$= \mu_f(supp\ (x \oplus y) \times \boldsymbol{R} \wedge \boldsymbol{R} \times supp\ z \wedge \{(\xi,\xi') \mid \xi+\xi'=t\}) =$

$= ((x \oplus y) * z)(t)$

4.6 We suppose that $x, y : \boldsymbol{R} \to \boldsymbol{B}_2$ satisfy the property that $x * y$, $y * z$, $(x * y) * z$, $x * (y * z)$ are defined. Then the associativity is true:

$$(x * y) * z = x * (y * z)$$

Proof $((x*y)*z)(t) = \underset{\xi\in\boldsymbol{R}}{\Xi}(x*y)(\xi)\cdot z(t-\xi) = \underset{\xi\in\boldsymbol{R}}{\Xi}\ \underset{\omega\in\boldsymbol{R}}{\Xi} x(\omega)\cdot y(\xi-\omega)\cdot z(t-\xi) =$

$= \underset{\omega\in\boldsymbol{R}}{\Xi}\ \underset{\xi\in\boldsymbol{R}}{\Xi} x(\omega)\cdot y(\xi-\omega)\cdot z(t-\xi) = \underset{\omega\in\boldsymbol{R}}{\Xi}\ \underset{\xi'\in\boldsymbol{R}}{\Xi} x(\omega)\cdot y(\xi')\cdot z(t-\xi'-\omega) =$

$= \underset{\omega\in\boldsymbol{R}}{\Xi} x(\omega)\cdot(y*z)(t-\omega) = (x*(y*z))(t)$

the sum being always defined, accordingly to the suppositions from the hypothesis.

4.7 **Remark** In the hypothesis from 4.6 (see also 4.3):

$$(x*y*z)(t) = \mu_f(supp\ x \times \boldsymbol{R} \times \boldsymbol{R} \wedge \boldsymbol{R} \times supp\ y \times \boldsymbol{R} \wedge$$
$$\wedge\ \boldsymbol{R} \times \boldsymbol{R} \times supp\ z \wedge \{(\xi,\xi',\xi'') \mid \xi+\xi'+\xi''=t\})$$

4.8 **Example** Let $x, y \in \boldsymbol{I}_\infty, z : \boldsymbol{R} \to \boldsymbol{B}_2$. We have:

$$x(t) = \underset{i=1}{\overset{n}{\Xi}} \delta(t-\tau_i)$$

$$y(t) = \underset{j=1}{\overset{m}{\Xi}} \delta(t-\varphi_j)$$

$$(x*y)(t) = \underset{\xi\in\boldsymbol{R}}{\Xi}(\underset{i=1}{\overset{n}{\Xi}}\delta(\xi-\tau_i)\cdot\underset{j=1}{\overset{m}{\Xi}}\delta(t-\varphi_j-\xi)) = \underset{i=1}{\overset{n}{\Xi}}\ \underset{j=1}{\overset{m}{\Xi}}\delta(t-\tau_i-\varphi_j)$$

$$(x*y*z)(t) = \underset{\xi\in\boldsymbol{R}}{\Xi}(\underset{i=1}{\overset{n}{\Xi}}\ \underset{j=1}{\overset{m}{\Xi}}\delta(\xi-\tau_i-\varphi_j)\cdot z(t-\xi)) = \underset{i=1}{\overset{n}{\Xi}}\ \underset{j=1}{\overset{m}{\Xi}} z(t-\tau_i-\varphi_j)$$

5. The Convolution Product of the Periodical Functions

5.1 The function $x : \boldsymbol{R} \to \boldsymbol{B}_2$ is *periodical* if

$$\exists\tau > 0, \forall t \in \boldsymbol{R}, x(t) = x(t-\tau)$$

Any such τ is called *period* of x and the smallest τ like above is called the *principal period* of x.

5.2 a) The two $\boldsymbol{R} \to \boldsymbol{B}_2$ constant functions do not have a principal period.

b) If τ is a period of x, then $n\tau, n \geq 1$ is a period of x.

c) The functions:

$$x(t) = \underset{z\in\boldsymbol{Z}}{\Xi}\delta(t - z\tau)$$

$$y(t) = \underset{z\in\boldsymbol{Z}}{\Xi}\chi_{[2z\frac{\tau}{2},(2z+1)\frac{\tau}{2})}(t)$$

are periodical, of period τ.

5.3 If $x : \boldsymbol{R} \to \boldsymbol{B}_2, x \neq 0$ is periodical of period $\tau > 0$, then $supp\ x$ is superiorly and inferiorly unbounded.

Proof Let α so that $supp\ x \subset [\alpha, \infty)$, in contradiction with the conclusion. There exists $t \geq \alpha$ and $n \in \boldsymbol{N}$ so that

$$t - n\tau < \alpha$$
$$1 = x(t) = x(t - n\tau) = 0$$

contradiction.

5.4 In $\boldsymbol{I}_\infty, \boldsymbol{I}_{Loc}, \boldsymbol{B}, \boldsymbol{B}_i, \boldsymbol{B}_s, \boldsymbol{B}_{i,Loc}, \boldsymbol{B}_{s,Loc}$, the only periodical function is $x = 0$.

5.5 There exist the non-null functions x, y with the property that the product $x * y$ is defined and

$$x * y = x$$

Proof If x is periodical of period τ and $y = \delta_\tau$, then (see also 4.4 a)):

$$(x * \delta_\tau)(t) = x(t - \tau) = x(t)$$

5.6 There exist the non-null functions x, y with the property that $x * y$ is defined and

$$x * y = 0$$

Proof Let x be periodical of period τ. We have:

$$(x * (\delta_{n\tau} \oplus \delta_{m\tau}))(t) = (x * \delta_{n\tau})(t) \oplus (x * \delta_{m\tau})(t) = x(t - n\tau) \oplus x(t - m\tau) = x(t) \oplus x(t) = 0$$

5.7 Let x be periodical with the period τ and y so that $x * y$ exists. Then $x * y$ is periodical, of period τ.

Proof $(x * y)(t - \tau) = (\delta_\tau * (x * y))(t) = ((\delta_\tau * x) * y)(t) = (x * y)(t)$

5.8 As we have seen before, the convolution product has a neuter element, which is δ. The search of an inverse for x relative to the convolution product means looking for some x^{-1} with the property that

$$x * x^{-1} = x^{-1} * x = \delta$$

We have the following

Theorem The periodical functions $x \neq 0$ do not have an inverse relative to $*$.

Proof Accordingly to 5.7, $x * x^{-1} = \delta$ should be a periodical function, but this is not the case.

6. On the Inverse Relative to the Convolution Product

6.1 **Example 1** We shall construct the inverse of

$$x(t) = \chi_{\{1,2,4,6,8,...\}}(t)$$

helped by the following table:

$*$	1	2	4	6	8
-1	0	~~1~~	~~3~~	~~5~~	~~7~~
0	~~1~~	~~2~~	~~4~~	~~6~~	8
1	~~2~~	~~3~~	~~5~~	~~7~~	9
3	~~4~~	5	7	9	11
			...		

table 6.1

The first row contains the support of x and the first column contains the support of x^{-1} to be computed:

$$x^{-1}(t) = \chi_{\{-1,0,1,3,...\}}(t)$$

The rest of the elements of the table are obtained by the summation of the corresponding points of the first row and the first column.

We have also indicated that the alike terms are reduced. We shall show now the way that the first column was obtained.

From

$$(x * x^{-1})(t) = \delta(t),\ supp\ (x * x^{-1}) = \{0\}$$

the element -1 results from the fact that the sequence $\{1,2,4,6,8,...\}$ summed term by term with -1 (translation with -1) contains 0. More exactly, the second row of the table is given by:

$$\{1,2,4,6,8,...\} - 1 = \{0,1,3,5,7,...\}$$

$$\chi_{\{1,2,4,6,8,...\}} * \chi_{\{-1\}} = \chi_{\{0,1,3,5,7,...\}}$$

Now, the term 1 is the first to be reduced and it is 0 to sum $\{1,2,4,6,8,\ldots\}$ with in order to reduce 1, giving the third row of the table:

$$\{1,2,4,6,8,\ldots\}+0=\{1,2,4,6,8,\ldots\}$$

$$\chi_{\{1,2,4,6,8,...\}} * \chi_{\{-1,0\}} = \chi_{\{1,2,4,6,8,...\}} * (\chi_{\{-1\}} \oplus \chi_{\{0\}}) =$$

$$= \chi_{\{1,2,4,6,8,...\}} * \chi_{\{-1\}} \oplus \chi_{\{1,2,4,6,8,...\}} * \chi_{\{0\}} = \chi_{\{0,1,3,5,7,...\}} \oplus \chi_{\{1,2,4,6,8,...\}} =$$

$$= \chi_{\{0,2,3,4,5,6,7,8,...\}}$$

We have after reducing 1, that the first term to be reduced is 2. Because

$$\{1,2,4,6,8,\ldots\}+1=\{2,3,5,7,9,\ldots\}$$

giving the fourth row of the table, there results:

$$\chi_{\{1,2,4,6,8,...\}} * \chi_{\{-1,0,1\}} = \chi_{\{1,2,4,6,8,...\}} * (\chi_{\{-1,0\}} \oplus \chi_{\{1\}}) =$$

$$= \chi_{\{1,2,4,6,8,...\}} * \chi_{\{-1,0\}} \oplus \chi_{\{1,2,4,6,8,...\}} * \chi_{\{1\}} =$$

$$= \chi_{\{0,2,3,4,5,6,7,8,...\}} \oplus \chi_{\{2,3,5,7,9,...\}} = \chi_{\{0,4,6,8,...\}}$$

6.2 **Example 2** The function

$$x(t) = \chi_{\{0,1\}}(t)$$

has two inverses, accordingly to the tables:

*	0	1
0	0	~~1~~
1	~~1~~	~~2~~
2	~~2~~	3
3	3	4
…		

table 6.2 a

*	0	1
−1	−~~1~~	0
−2	−~~2~~	−~~1~~
−3	−3	−~~2~~
−4	−4	−3
…		

table 6.2 b

$$x_1^{-1}(t) = \chi_{\{0,1,2,3,...\}}(t)$$

$$x_2^{-1}(t) = \chi_{\{-1,-2,-3,-4,...\}}(t)$$

6.3 The way that the inverses were constructed in the previous example gives the following conclusions:

a) $x \in \boldsymbol{B}_{i,Loc}$ has an inverse. If $supp\ x$ is superiorly unbounded, then this inverse is unique and it belongs to $\boldsymbol{B}_{i,Loc}$

a)* $x \in \boldsymbol{B}_{s,Loc}$ has an inverse. If $supp\ x$ is inferiorly unbounded, then this inverse is unique and it belongs to $\boldsymbol{B}_{s,Loc}$.

b) $x \in \boldsymbol{I}_\infty = \boldsymbol{B}_{i,Loc} \wedge \boldsymbol{B}_{s,Loc}$, $x \neq \delta$ may have two inverses: $x_1^{-1} \in \boldsymbol{B}_{i,Loc}$ inverts x as a function from $\boldsymbol{B}_{i,Loc}$ and it has a superiorly unbounded support, whilst $x_2^{-1} \in \boldsymbol{B}_{s,Loc}$ inverts x as a function from $\boldsymbol{B}_{s,Loc}$ and it has an inferiorly unbounded support.

$x = \delta$ has exactly one inverse, which is δ itself.

6.4 A function x with the property that $(a,b) \subset supp\ x$ does not have an inverse relative to the convolution product. We state the problem of proving that if there exists a convergent real sequence $\tau_n \to \tau, \tau_n \in supp\ x$ then x does not have an inverse relative to the convolution product.

7. The Behavior of the Convolution Product Relative to the Derivation Operator

7.1 Let us consider the (finite or infinite, strictly increasing locally finite) families $(t_n), (t_j^{'})$ satisfying

$$t_0 < t_1 < ... < t_n < ..., \forall a < b, (a,b) \wedge (t_n) \text{ is finite}$$
$$t_0^{'} < t_1^{'} < ... < t_j^{'} < ..., \forall a < b, (a,b) \wedge (t_j^{'}) \text{ is finite}$$

and the functions (see the notations 2.11)

$$x(t) = \underset{n=0,1,2,...}{\Xi} \eta^{-}(t - t_n) \in \boldsymbol{B}_i$$
$$y(t) = \underset{j=0,1,2,...}{\Xi} \delta(t - t_j^{'}) \in \boldsymbol{B}_{i,Loc}$$

We have that

$$(x * y)(t) = \underset{j=0,1,2,...}{\Xi} \underset{n=0,1,2,...}{\Xi} \eta^{-}(t - t_n - t_j^{'})$$

has left limit (see 2.10, 4.8) and

$$D^{-}(x * y)(t) = \underset{j=0,1,2,...}{\Xi} \underset{n=0,1,2,...}{\Xi} \delta(t - t_n - t_j^{'}) = (D^{-}x * y)(t) \in \boldsymbol{B}_{i,Loc}$$
$$(x * y)(t) = (x * D^{-}y)(t)$$

7.2 Under the previous assumptions on (t_n) and $(t_j^{'})$, for

$$x(t) = \underset{n=0,1,2,...}{\Xi} \eta^{-}(t - t_n)$$
$$y(t) = \underset{j=0,1,2,...}{\Xi} \eta^{-}(t - t_j^{'})$$

it is true ($x * y$ does not exist):

$$(D^{-}x * y)(t) = (x * D^{-}y)(t) = \underset{j=0,1,2,...}{\Xi} \underset{n=0,1,2,...}{\Xi} \eta^{-}(t - t_n - t_j^{'})$$

8. Conclusions

Our purpose was to define and characterize the pseudoboolean convolution product. The most interesting open problems that were stated are probably related to the existence of the convolution product and to the existence and construction of the inverse.