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Dedicated to the memory of Academician Mitrofan M. Cioban (1942-2021)

Boolean asynchronous systems vs. Daizhan Cheng's theory

Serban E. Vlad

Abstract. The theory of Daizhan Cheng [1] replaces $\mathbf{B} = \{0, 1\}$ with $\mathbf{D} = \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$, and Boolean functions with logical matrices. Interesting and very important algebraical opportunities result, which can be used in systems theory. Our purpose is to give a hint on the theory of Cheng and its application to asynchronicity.
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Keywords: Boolean function, Boolean asynchronous system, structure matrix, semi-tensor product, theory of Daizhan Cheng.

Sisteme booleene asincrone din perspectiva teoriei lui Daizhan Cheng

Rezumat. Teoria lui Daizhan Cheng [1] înlocuieşte $\mathbf{B} = \{0, 1\}$ cu $\mathbf{D} = \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$, şi funcţiile booleene cu matrici logice. Rezultă de aici oportunităţi algebrice importante, care pot fi folosite în teoria sistemelor. Scopul nostru este acela de a schiţa teoria lui Cheng şi aplicaţiile sale în asincronism.
Cuvinte cheie: funcţie booleană, sistem asincron boolean, matrice de structură, produs semi-tensorial, teoria lui Daizhan Cheng.

1. Preliminaries

Notation 1.1. We denote with $\mathbf{B} = \{0, 1\}$ the binary Boolean algebra.

Definition 1.1. The $\lambda-$*iterate* of $\Phi : \mathbf{B}^n \to \mathbf{B}^n, \lambda \in \mathbf{B}^n$ is the function $\Phi^\lambda : \mathbf{B}^n \to \mathbf{B}^n$ defined like this: $\forall \mu \in \mathbf{B}^n, \forall i \in \{1, ..., n\}$,

$$(\Phi^\lambda)_i(\mu) = \begin{cases} \Phi_i(\mu), \ if \ \lambda_i = 1, \\ \mu_i, \ if \ \lambda_i = 0. \end{cases}$$

Dedicated to the memory of Academician Mitrofan Cioban, who was the core of the intellectual and spiritual life of the Moldovan mathematicians for so many years, and also a steady bridge connecting the mathematicians from Moldova and Romania. We shall keep in our hearts his common sense and support. May he rest in peace!

Definition 1.2. Given Φ, the function $\widetilde{\Phi} : \mathbf{B}^n \times \mathbf{B}^n \longrightarrow \mathbf{B}^n$ is defined by $\forall \mu \in \mathbf{B}^n, \forall \lambda \in \mathbf{B}^n$,

$$\widetilde{\Phi}(\mu, \lambda) = \Phi^{\lambda}(\mu). \tag{1}$$

Definition 1.3. The function $\alpha : \mathbf{N} \longrightarrow \mathbf{B}^n$, $\mathbf{N} \ni k \longmapsto \alpha^k \in \mathbf{B}^n$, with

$$\forall i \in \{1, ..., n\}, \text{ the set } \{k | k \in \mathbf{N}, \alpha_i^k = 1\} \text{ is infinite}$$

is called *progressive computation function,* and we denote with Π_n the set of these functions.

Remark 1.1. Two ways of making the discrete time iterations of the function $\Phi : \mathbf{B}^n \longrightarrow \mathbf{B}^n$ exist: synchronously $1_{\mathbf{B}^n}, \Phi, \Phi \circ \Phi, ...$ when $\Phi_1, ..., \Phi_n$ are computed always, all of them, and asynchronously, when the coordinates of Φ are computed sometimes, independently on each other. The functions $\alpha \in \Pi_n$ indicate how Φ is computed: $\forall k \in \mathbf{N}, \forall i \in \{1, ..., n\}$,

$$\begin{cases} \alpha_i^k = 1, \text{ at time instant } k, \ \Phi_i \text{ is computed}, \\ \alpha_i^k = 0, \text{ at time instant } k, \ \Phi_i \text{ is not computed}. \end{cases}$$

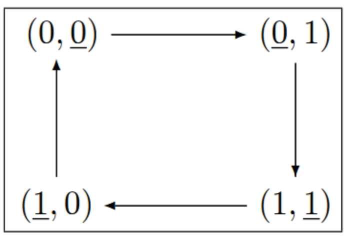
Synchronicity is that special case of asynchronicity when $\forall k \in \mathbf{N}, \alpha^k = (1, ..., 1)$.

Definition 1.4. The *unbounded delay model of computation of* Φ consists in the equation

$$x(k+1) = \Phi^{\alpha^k}(x(k)), \tag{2}$$

where $\Phi : \mathbf{B}^n \longrightarrow \mathbf{B}^n$, $x : \mathbf{N} \longrightarrow \mathbf{B}^n$, $\alpha \in \Pi_n$ and $k \in \mathbf{N}$. In (2) the function x, called *state*, is unknown, and $x(0)$, together with α, are parameters.

Example 1.1. We consider the function $\Phi : \mathbf{B}^2 \longrightarrow \mathbf{B}^2, \forall \mu \in \mathbf{B}^2, \Phi(\mu_1, \mu_2) = (\mu_2, \overline{\mu_1})$, with the following state portrait



In the drawing, the underlined coordinates $\underline{\mu_i}$, $i \in \{1, 2\}$ show that $\Phi_i(\mu) \neq \mu_i$ and, by their computation, the system moves to a distinct state, while the arrows indicate the evolution of the system. The equation (2) is $\forall k \in \mathbf{N}$,

$$\begin{cases} x_1(k+1) = x_2(k)\alpha_1^k \cup x_1(k)\overline{\alpha_1^k}, \\ x_2(k+1) = \overline{x_1(k)}\alpha_2^k \cup x_2(k)\overline{\alpha_2^k}, \end{cases} \tag{3}$$

where $x : \mathbf{N} \longrightarrow \mathbf{B}^2$ fulfils $x(0) = (0, 0)$ and $\alpha \in \Pi_2$ is defined as

$$\alpha = (1, 0), (0, 1), (1, 1), (0, 1), (1, 0), ...$$

We get

$$x(1) = \Phi^{\alpha^0}(x(0)) = \Phi^{(1,0)}(0,0) = (0,0), \tag{4}$$

$$x(2) = \Phi^{\alpha^1}(x(1)) = \Phi^{(0,1)}(0,0) = (0,1), \tag{5}$$

$$x(3) = \Phi^{\alpha^2}(x(2)) = \Phi^{(1,1)}(0,1) = (1,1), \tag{6}$$

$$x(4) = \Phi^{\alpha^3}(x(3)) = \Phi^{(0,1)}(1,1) = (1,0), \tag{7}$$

$$x(5) = \Phi^{\alpha^4}(x(4)) = \Phi^{(1,0)}(1,0) = (0,0), \tag{8}$$

$$\ldots$$

2. Semi-tensor product

Notation 2.1. We use the notation $M_{m\times n}$ for the set of the matrices with binary entries that have m rows and n columns.

Remark 2.1. In the following Definitions 2.1 and 2.2, the operations with matrices are induced by the field structure of $\mathbf{B}$ relative to $\oplus, \cdot$.

Definition 2.1. The *Kronecker product* $\otimes$ of the matrices $A \in M_{m\times n}$ and $B \in M_{p\times q}$ is

$$A \otimes B = \begin{pmatrix} a_{11}B & \ldots & a_{1n}B \\ & \ldots & \\ a_{m1}B & \ldots & a_{mn}B \end{pmatrix} \in M_{mp\times nq}.$$

Definition 2.2. The *semi-tensor product* $\ltimes$ of $A \in M_{m\times n}$ and $B \in M_{p\times q}$ is by definition

$$A \ltimes B = (A \otimes I_{\frac{c}{n}})(B \otimes I_{\frac{c}{p}}) \in M_{\frac{mc}{n}\times\frac{qc}{p}},$$

where I_k is the $k \times k$ identity matrix and c is the least common multiple of n and p.

Remark 2.2. At Definition 2.2, $A \otimes I_{\frac{c}{n}}$ has $n\frac{c}{n}$ columns and $B \otimes I_{\frac{c}{p}}$ has $p\frac{c}{p}$ rows, thus the product of the matrices $A \otimes I_{\frac{c}{n}}$, $B \otimes I_{\frac{c}{p}}$ makes sense.

Remark 2.3. If $n = p$, the semi-tensor product coincides with the usual product of the matrices. This happens because we get $c = n = p$, $A \otimes I_1 = A$, and $B \otimes I_1 = B$.

Example 2.1. We have the following examples of Kronecker product

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \end{pmatrix} = \begin{pmatrix} 1\begin{pmatrix} 1 & 1 \end{pmatrix} \\ 0\begin{pmatrix} 1 & 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

and semi-tensor product

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \ltimes \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes I_2\right)\left(\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \otimes I_1\right)$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Remark 2.4. The semi-tensor product is associative, and for this reason we shall omit writing brackets when it is used repeatedly.

3. Replacement of B with D

Notation 3.1. We denote with $\delta_n^i \in M_{n\times 1}$ the columns of the identity matrix of dimension n:

$$\delta_n^i = \begin{pmatrix} 0 \\ ... \\ 1 \\ ... \\ 0 \end{pmatrix} - i,$$

where $n \geq 1$ and $i \in \{1, ..., n\}$.

Notation 3.2. We use also the notations

$$\mathbf{D} = \{\delta_2^1, \delta_2^2\},$$

$$\mathbf{D}^{(n)} = \{\delta_{2^n}^1, ..., \delta_{2^n}^{2^n}\}.$$

Remark 3.1. $\mathbf{D}$ and $\mathbf{D}^{(n)}$ do not have a name and an algebraical structure of their own, but they will act as $\mathbf{B}$ and $\mathbf{B}^n$ in the following. Obviously, $card(\mathbf{B}) = card(\mathbf{D}) = 2$ and $card(\mathbf{B}^n) = card(\mathbf{D}^{(n)}) = 2^n$.

Notation 3.3. We use the notations $\zeta : \mathbf{B} \to \mathbf{D}$, $\zeta_n : \mathbf{B}^n \to \mathbf{D}^{(n)}$ for the following functions: $\forall \mu \in \mathbf{B}, \forall \lambda \in \mathbf{B}^n$,

$$\zeta(\mu) = \begin{pmatrix} \mu \\ \overline{\mu} \end{pmatrix},$$

$$\zeta_n(\lambda) = \begin{pmatrix} \lambda_1 ... \lambda_{n-1}\lambda_n \\ \lambda_1 ... \lambda_{n-1}\overline{\lambda_n} \\ \lambda_1 ... \overline{\lambda_{n-1}}\lambda_n \\ ... \\ \overline{\lambda_1} ... \overline{\lambda_{n-1}}\,\overline{\lambda_n} \end{pmatrix}.$$

We denote in general $\underline{\underline{\mu}} = \zeta(\mu)$ and $\underline{\underline{\lambda}} = \zeta_n(\lambda)$.

Remark 3.2. We notice that for any $\mu \in \mathbf{B}$, respectively $\lambda \in \mathbf{B}^n$, exactly one of $\mu, \overline{\mu}$ is 1, respectively exactly one of $\lambda_1...\lambda_{n-1}\lambda_n$, $\lambda_1...\lambda_{n-1}\overline{\lambda_n}$, $\lambda_1...\overline{\lambda_{n-1}}\lambda_n$, ..., $\overline{\lambda_1}...\overline{\lambda_{n-1}}\,\overline{\lambda_n}$ is 1, meaning that $\underline{\underline{\mu}} \in \mathbf{D}$, respectively that $\underline{\underline{\lambda}} \in \mathbf{D}^{(n)}$ indeed.

Theorem 3.4. *(a) ζ and ζ_n are bijections;*

(b) $\forall \lambda \in \mathbf{B}^n$,

$$\underline{\underline{\lambda}} = \underline{\underline{\lambda_1}} \ltimes ... \ltimes \underline{\underline{\lambda_n}}.$$

Proof. (a) When $\lambda \in \mathbf{B}^n$ takes the distinct 2^n values $(1, ..., 1, 1)$, $(1, ..., 1, 0)$, $(1, ..., 0, 1)$, ..., $(0, ..., 0, 0)$, $\underline{\underline{\lambda}}$ takes the distinct 2^n values $\delta_{2^n}^1, \delta_{2^n}^2, \delta_{2^n}^3, ..., \delta_{2^n}^{2^n}$.

(b) For $n = 2$ and arbitrary $\lambda \in \mathbf{B}^2$, we obtain

$$\underline{\underline{\lambda_1}} \ltimes \underline{\underline{\lambda_2}} = \begin{pmatrix} \lambda_1 \\ \overline{\lambda_1} \end{pmatrix} \ltimes \begin{pmatrix} \lambda_2 \\ \overline{\lambda_2} \end{pmatrix} = (\begin{pmatrix} \lambda_1 \\ \overline{\lambda_1} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})(\begin{pmatrix} \lambda_2 \\ \overline{\lambda_2} \end{pmatrix} \otimes 1)$$

$$= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \\ \overline{\lambda_1} & 0 \\ 0 & \overline{\lambda_1} \end{pmatrix} \begin{pmatrix} \lambda_2 \\ \overline{\lambda_2} \end{pmatrix} = \begin{pmatrix} \lambda_1\lambda_2 \\ \lambda_1\overline{\lambda_2} \\ \overline{\lambda_1}\lambda_2 \\ \overline{\lambda_1}\,\overline{\lambda_2} \end{pmatrix} = \underline{\underline{\lambda}}.$$

The property is supposed to be true for n and the proof is made for $n + 1$. □

4. Structure matrix

Notation 4.1. The notation of the $i-th$ column of an arbitrary binary matrix A is $col_i(A)$.

Definition 4.1. A matrix A with n rows and m columns is called *logical* if $\forall j \in \{1, ..., m\}, col_j(A) \in \{\delta_n^1, ..., \delta_n^n\}$. The set of the logical matrices with n rows and m columns is denoted with $L_{n\times m}$.

Definition 4.2. Let $f : \mathbf{B}^n \to \mathbf{B}$, $\Phi : \mathbf{B}^n \to \mathbf{B}^n$ and $\widetilde{\Phi} : \mathbf{B}^n \times \mathbf{B}^n \to \mathbf{B}^n$, as defined at (1). We denote with $M_f \in L_{2\times 2^n}$ the matrix

$$M_f = \begin{pmatrix} f(1, ..., 1, 1), & f(1, ..., 1, 0), & f(1, ..., 0, 1), & ... & f(0, ..., 0, 0) \\ \overline{f(1, ..., 1, 1)}, & \overline{f(1, ..., 1, 0)}, & \overline{f(1, ..., 0, 1)}, & ... & \overline{f(0, ..., 0, 0)} \end{pmatrix},$$

with $M_\Phi \in L_{2^n\times 2^n}$ the matrix whose columns are

$$col_1(M_\Phi) = \begin{pmatrix} \Phi_1(1, ..., 1, 1)...\Phi_{n-1}(1, ..., 1, 1)\Phi_n(1, ..., 1, 1) \\ \Phi_1(1, ..., 1, 1)...\Phi_{n-1}(1, ..., 1, 1)\overline{\Phi_n(1, ..., 1, 1)} \\ \Phi_1(1, ..., 1, 1)...\overline{\Phi_{n-1}(1, ..., 1, 1)}\Phi_n(1, ..., 1, 1) \\ ... \\ \overline{\Phi_1(1, ..., 1, 1)}...\overline{\Phi_{n-1}(1, ..., 1, 1)}\,\overline{\Phi_n(1, ..., 1, 1)} \end{pmatrix},$$

$$\mathrm{col}_2(M_\Phi) = \begin{pmatrix} \Phi_1(1,...,1,0)...\Phi_{n-1}(1,...,1,0)\Phi_n(1,...,1,0) \\ \Phi_1(1,...,1,0)...\Phi_{n-1}(1,...,1,0)\overline{\Phi_n(1,...,1,0)} \\ \Phi_1(1,...,1,0)...\overline{\Phi_{n-1}(1,...,1,0)}\Phi_n(1,...,1,0) \\ ... \\ \overline{\Phi_1(1,...,1,0)}...\overline{\Phi_{n-1}(1,...,1,0)}\ \overline{\Phi_n(1,...,1,0)} \end{pmatrix},$$

...

$$\mathrm{col}_{2^n}(M_\Phi) = \begin{pmatrix} \Phi_1(0,...,0,0)...\Phi_{n-1}(0,...,0,0)\Phi_n(0,...,0,0) \\ \Phi_1(0,...,0,0)...\Phi_{n-1}(0,...,0,0)\overline{\Phi_n(0,...,0,0)} \\ \Phi_1(0,...,0,0)...\overline{\Phi_{n-1}(0,...,0,0)}\Phi_n(0,...,0,0) \\ ... \\ \overline{\Phi_1(0,...,0,0)}...\overline{\Phi_{n-1}(0,...,0,0)}\ \overline{\Phi_n(0,...,0,0)} \end{pmatrix}$$

and with $M_{\widetilde{\Phi}} \in L_{2^n \times 2^{2n}}$ the matrix

$$\mathrm{col}_1(M_{\widetilde{\Phi}}) = \begin{pmatrix} \Phi_1^{(1,...,1,1)}(1,...,1)...\Phi_{n-1}^{(1,...,1,1)}(1,...,1)\Phi_n^{(1,...,1,1)}(1,...,1) \\ \Phi_1^{(1,...,1,1)}(1,...,1)...\Phi_{n-1}^{(1,...,1,1)}(1,...,1)\overline{\Phi_n^{(1,...,1,1)}(1,...,1)} \\ \Phi_1^{(1,...,1,1)}(1,...,1)...\overline{\Phi_{n-1}^{(1,...,1,1)}(1,...,1)}\Phi_n^{(1,...,1,1)}(1,...,1) \\ ... \\ \overline{\Phi_1^{(1,...,1,1)}(1,...,1)}...\overline{\Phi_{n-1}^{(1,...,1,1)}(1,...,1)}\ \overline{\Phi_n^{(1,...,1,1)}(1,...,1)} \end{pmatrix},$$

$$\mathrm{col}_2(M_{\widetilde{\Phi}}) = \begin{pmatrix} \Phi_1^{(1,...,1,0)}(1,...,1)...\Phi_{n-1}^{(1,...,1,0)}(1,...,1)\Phi_n^{(1,...,1,0)}(1,...,1) \\ \Phi_1^{(1,...,1,0)}(1,...,1)...\Phi_{n-1}^{(1,...,1,0)}(1,...,1)\overline{\Phi_n^{(1,...,1,0)}(1,...,1)} \\ \Phi_1^{(1,...,1,0)}(1,...,1)...\overline{\Phi_{n-1}^{(1,...,1,0)}(1,...,1)}\Phi_n^{(1,...,1,0)}(1,...,1) \\ ... \\ \overline{\Phi_1^{(1,...,1,0)}(1,...,1)}...\overline{\Phi_{n-1}^{(1,...,1,0)}(1,...,1)}\ \overline{\Phi_n^{(1,...,1,0)}(1,...,1)} \end{pmatrix},$$

...

$$\mathrm{col}_{2^n}(M_{\widetilde{\Phi}}) = \begin{pmatrix} \Phi_1^{(0,...,0,0)}(1,...,1)...\Phi_{n-1}^{(0,...,0,0)}(1,...,1)\Phi_n^{(0,...,0,0)}(1,...,1) \\ \Phi_1^{(0,...,0,0)}(1,...,1)...\Phi_{n-1}^{(0,...,0,0)}(1,...,1)\overline{\Phi_n^{(0,...,0,0)}(1,...,1)} \\ \Phi_1^{(0,...,0,0)}(1,...,1)...\overline{\Phi_{n-1}^{(0,...,0,0)}(1,...,1)}\Phi_n^{(0,...,0,0)}(1,...,1) \\ ... \\ \overline{\Phi_1^{(0,...,0,0)}(1,...,1)}...\overline{\Phi_{n-1}^{(0,...,0,0)}(1,...,1)}\ \overline{\Phi_n^{(0,...,0,0)}(1,...,1)} \end{pmatrix},$$

...

$$\mathrm{col}_{2^{2n}}(M_{\widetilde{\Phi}}) = \begin{pmatrix} \Phi_1^{(0,...,0,0)}(0,...,0)...\Phi_{n-1}^{(0,...,0,0)}(0,...,0)\Phi_n^{(0,...,0,0)}(0,...,0) \\ \Phi_1^{(0,...,0,0)}(0,...,0)...\Phi_{n-1}^{(0,...,0,0)}(0,...,0)\overline{\Phi_n^{(0,...,0,0)}(0,...,0)} \\ \Phi_1^{(0,...,0,0)}(0,...,0)...\overline{\Phi_{n-1}^{(0,...,0,0)}(0,...,0)}\Phi_n^{(0,...,0,0)}(0,...,0) \\ ... \\ \overline{\Phi_1^{(0,...,0,0)}(0,...,0)}...\overline{\Phi_{n-1}^{(0,...,0,0)}(0,...,0)}\ \overline{\Phi_n^{(0,...,0,0)}(0,...,0)} \end{pmatrix}.$$

$M_f, M_\Phi, M_{\widetilde{\Phi}}$ are called the *structure matrices* of $f, \Phi, \widetilde{\Phi}$.

Theorem 4.2. *We consider* $f : \mathbf{B}^n \to \mathbf{B}$, $\Phi : \mathbf{B}^n \to \mathbf{B}^n$ *and* $\widetilde{\Phi} : \mathbf{B}^n \times \mathbf{B}^n \to \mathbf{B}^n$ *like previously. The assignments*

$$\underline{\underline{\mu}} \mapsto M_f \ltimes \underline{\underline{\mu}},$$

$$\underline{\underline{\mu}} \mapsto M_\Phi \ltimes \underline{\underline{\mu}},$$

$$(\underline{\underline{\mu}}, \underline{\underline{\lambda}}) \mapsto M_{\widetilde{\Phi}} \ltimes \underline{\underline{\mu}} \ltimes \underline{\underline{\lambda}},$$

with $\mu \in \mathbf{B}^n, \lambda \in \mathbf{B}^n$, *define the functions* $M(f) : \mathbf{D}^{(n)} \to \mathbf{D}$, $M(\Phi) : \mathbf{D}^{(n)} \to \mathbf{D}^{(n)}$, $M(\widetilde{\Phi}) : \mathbf{D}^{(n)} \times \mathbf{D}^{(n)} \to \mathbf{D}^{(n)}$ *in the following way:* $\forall \underline{\underline{\mu}} \in \mathbf{D}^{(n)}, \forall \underline{\underline{\lambda}} \in \mathbf{D}^{(n)}$,

$$M(f)(\underline{\underline{\mu}}) = M_f \ltimes \underline{\underline{\mu}}, \tag{9}$$

$$M(\Phi)(\underline{\underline{\mu}}) = M_\Phi \ltimes \underline{\underline{\mu}}, \tag{10}$$

$$M(\widetilde{\Phi})(\underline{\underline{\mu}}, \underline{\underline{\lambda}}) = M_{\widetilde{\Phi}} \ltimes \underline{\underline{\mu}} \ltimes \underline{\underline{\lambda}}. \tag{11}$$

We have

$$M_f \ltimes \underline{\underline{\mu}} = M_f \cdot \underline{\underline{\mu}}, \tag{12}$$

$$M_\Phi \ltimes \underline{\underline{\mu}} = M_\Phi \cdot \underline{\underline{\mu}}, \tag{13}$$

$$M_{\widetilde{\Phi}} \ltimes \underline{\underline{\mu}} \ltimes \underline{\underline{\lambda}} = M_{\widetilde{\Phi}} \cdot (\underline{\underline{\mu}} \ltimes \underline{\underline{\lambda}}), \tag{14}$$

where $'\cdot'$ *is the product of the matrices*.

Proof. We note first that

$$\mathbf{D} = L_{2\times 1},$$

$$\mathbf{D}^{(n)} = L_{2^n \times 1}$$

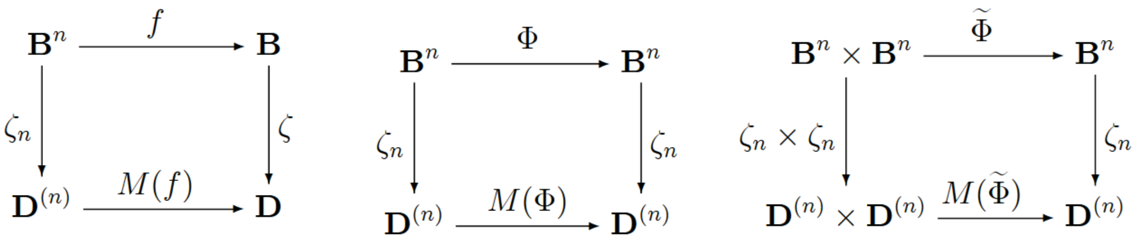
are true. As far as $\underline{\underline{\mu}} \in L_{2^n\times 1}$ and $M_f \in L_{2\times 2^n}$, we infer from Remark 2.3 that (12) holds. On the other hand, $\underline{\underline{\mu}} \in L_{2^n\times 1}$ makes $M_f \cdot \underline{\underline{\mu}}$ coincide with one of $col_1(M_f), ..., col_{2^n}(M_f)$ and we know that $col_1(M_f), ..., col_{2^n}(M_f) \in L_{2\times 1}$, thus we can define $M(f)$ as

$$\mathbf{D}^{(n)} \ni \underline{\underline{\mu}} \mapsto M(f)(\underline{\underline{\mu}}) = M_f \ltimes \underline{\underline{\mu}} \in \mathbf{D}.$$

The other statements are proved similarly. □

Notation 4.3. We denote $F_{n,m} = \{h | h : \mathbf{B}^n \to \mathbf{B}^m\}$.

Theorem 4.4. *(a) The following diagrams*



commute.

(b) The assignments $F_{n,1} \ni f \longmapsto M_f \in L_{2\times 2^n}$, $F_{n,n} \ni \Phi \longmapsto M_\Phi \in L_{2^n\times 2^n}$, $F_{2n,n} \ni \widetilde{\Phi} \longmapsto M_{\widetilde{\Phi}} \in L_{2^n\times 2^{2n}}$ *are bijective.*

Proof. We fix $\mu \in \mathbf{B}^n$ and $\lambda \in \mathbf{B}^n$ arbitrary.

(a) In order to prove the commutativity of the first diagram, we use the fact that

$$f(\mu) = f(1,...,1,1)\mu_1...\mu_{n-1}\mu_n \oplus f(1,...,1,0)\mu_1...\mu_{n-1}\overline{\mu_n}$$

$$\oplus f(1,...,0,1)\mu_1...\overline{\mu_{n-1}}\mu_n \oplus ... \oplus f(0,...,0,0)\overline{\mu_1}...\overline{\mu_{n-1}}\ \overline{\mu_n},$$

$$\overline{f(\mu)} = \overline{f(1,...,1,1)}\mu_1...\mu_{n-1}\mu_n \oplus \overline{f(1,...,1,0)}\mu_1...\mu_{n-1}\overline{\mu_n}$$

$$\oplus\overline{f(1,...,0,1)}\mu_1...\overline{\mu_{n-1}}\mu_n \oplus ... \oplus \overline{f(0,...,0,0)}\overline{\mu_1}...\overline{\mu_{n-1}}\ \overline{\mu_n},$$

wherefrom

$$\underline{\underline{f(\mu)}} = \begin{pmatrix} f(\mu) \\ \overline{f(\mu)} \end{pmatrix} = M_f \cdot \underline{\underline{\mu}}. \tag{15}$$

We conclude that

$$(M(f)\circ\zeta_n)(\mu) = M(f)(\zeta_n(\mu)) = M(f)(\underline{\underline{\mu}}) \overset{(9)}{=} M_f \ltimes \underline{\underline{\mu}}$$

$$\overset{(12)}{=} M_f \cdot \underline{\underline{\mu}} \overset{(15)}{=} \underline{\underline{f(\mu)}} = \zeta(f(\mu)) = (\zeta\circ f)(\mu),$$

i.e. the first diagram is commutative.

As far as the second diagram is concerned, we can prove that

$$\underline{\underline{\Phi(\mu)}} = M_\Phi \cdot \underline{\underline{\mu}}, \tag{16}$$

which is analogue with (15), and we obtain

$$(M(\Phi)\circ\zeta_n)(\mu) = M(\Phi)(\zeta_n(\mu)) = M(\Phi)(\underline{\underline{\mu}}) \overset{(10)}{=} M_\Phi \ltimes \underline{\underline{\mu}}$$

$$\overset{(13)}{=} M_\Phi \cdot \underline{\underline{\mu}} \overset{(16)}{=} \underline{\underline{\Phi(\mu)}} = \zeta_n(\Phi(\mu)) = (\zeta_n\circ\Phi)(\mu).$$

For the commutativity of the third diagram, we have

$$\underline{\underline{\widetilde{\Phi}(\mu,\lambda)}} = \begin{pmatrix} \Phi_1^{\lambda}(\mu)...\Phi_{n-1}^{\lambda}(\mu)\Phi_n^{\lambda}(\mu) \\ \Phi_1^{\lambda}(\mu)...\Phi_{n-1}^{\lambda}(\mu)\overline{\Phi_n^{\lambda}(\mu)} \\ \Phi_1^{\lambda}(\mu)...\overline{\Phi_{n-1}^{\lambda}(\mu)}\Phi_n^{\lambda}(\mu) \\ ... \\ \overline{\Phi_1^{\lambda}(\mu)}...\overline{\Phi_{n-1}^{\lambda}(\mu)}\ \overline{\Phi_n^{\lambda}(\mu)} \end{pmatrix},$$

$$\underline{\underline{\mu}} \ltimes \underline{\underline{\lambda}} = \begin{pmatrix} \mu_1...\mu_n\lambda_1...\lambda_{n-1}\lambda_n \\ \mu_1...\mu_n\lambda_1...\lambda_{n-1}\overline{\lambda_n} \\ \mu_1...\mu_n\lambda_1...\overline{\lambda_{n-1}}\lambda_n \\ ... \\ \overline{\mu_1}...\overline{\mu_n}\ \overline{\lambda_1}...\overline{\lambda_{n-1}}\ \overline{\lambda_n} \end{pmatrix}$$

and we note that

$$\underline{\underline{\widetilde{\Phi}(\mu,\lambda)}} = M_{\widetilde{\Phi}} \cdot (\underline{\underline{\mu}} \ltimes \underline{\underline{\lambda}}). \tag{17}$$

For example, the second row in (17) is proved like this:

$$\begin{aligned} &\Phi_1^{\lambda}(\mu)...\Phi_{n-1}^{\lambda}(\mu)\overline{\Phi_n^{\lambda}(\mu)} \\ &= \Phi_1^{(1,...,1,1)}(1,...,1)...\Phi_{n-1}^{(1,...,1,1)}(1,...,1)\overline{\Phi_n^{(1,...,1,1)}(1,...,1)}\mu_1...\mu_n\lambda_1...\lambda_{n-1}\lambda_n \\ &\oplus\Phi_1^{(1,...,1,0)}(1,...,1)...\Phi_{n-1}^{(1,...,1,0)}(1,...,1)\overline{\Phi_n^{(1,...,1,0)}(1,...,1)}\mu_1...\mu_n\lambda_1...\lambda_{n-1}\overline{\lambda_n} \\ &... \\ &\oplus\Phi_1^{(0,...,0,0)}(1,...,1)...\Phi_{n-1}^{(0,...,0,0)}(1,...,1)\overline{\Phi_n^{(0,...,0,0)}(1,...,1)}\mu_1...\mu_n\overline{\lambda_1}...\overline{\lambda_{n-1}}\ \overline{\lambda_n} \\ &... \\ &\oplus\Phi_1^{(0,...,0,0)}(0,...,0)...\Phi_{n-1}^{(0,...,0,0)}(0,...,0)\overline{\Phi_n^{(0,...,0,0)}(0,...,0)}\overline{\mu_1}...\overline{\mu_n}\overline{\lambda_1}...\overline{\lambda_{n-1}}\ \overline{\lambda_n}. \end{aligned}$$

We infer

$$(M(\widetilde{\Phi}) \circ (\zeta_n \times \zeta_n))(\mu,\lambda) = M(\widetilde{\Phi})(\zeta_n(\mu),\zeta_n(\lambda)) = M(\widetilde{\Phi})(\underline{\underline{\mu}},\underline{\underline{\lambda}}) \overset{(11)}{=} M_{\widetilde{\Phi}} \ltimes \underline{\underline{\mu}} \ltimes \underline{\underline{\lambda}}$$

$$\overset{(14)}{=} M_{\widetilde{\Phi}} \cdot (\underline{\underline{\mu}} \ltimes \underline{\underline{\lambda}}) \overset{(17)}{=} \underline{\underline{\widetilde{\Phi}(\mu,\lambda)}} = \zeta_n(\widetilde{\Phi}(\mu,\lambda)) = (\zeta_n \circ \widetilde{\Phi})(\mu,\lambda).$$

(b) For example we suppose against all reason that $f, f' : \mathbf{B}^n \to \mathbf{B}$ exist, $f \neq f'$, with the property that $M_f = M_{f'}$. The hypothesis states the existence of $\mu \in \mathbf{B}^n$ such that $f(\mu) \neq f'(\mu)$ thus, from Theorem 3.4, $\underline{\underline{f(\mu)}} \neq \underline{\underline{f'(\mu)}}$. We have:

$$\begin{aligned} \underline{\underline{f(\mu)}} &= \zeta(f(\mu)) = (\zeta \circ f)(\mu) = (M(f) \circ \zeta_n)(\mu) = M(f)(\zeta_n(\mu)) \\ &= M(f)(\underline{\underline{\mu}}) = M_f \ltimes \mu = M_{f'} \ltimes \mu = M(f')(\underline{\underline{\mu}}) = M(f')(\zeta_n(\mu)) \\ &= (M(f') \circ \zeta_n)(\mu) = (\zeta \circ f')(\mu) = \zeta(f'(\mu)) = \underline{\underline{f'(\mu)}}, \end{aligned}$$

contradiction, showing that the assignment $F_{n,1} \ni f \longmapsto M_f \in L_{2\times 2^n}$ is injective. Due to the fact that $card(F_{n,1}) = card(L_{2\times 2^n}) = 2^{2^n}$, injectivity and bijectivity coincide. □

5. Equations of evolution

Remark 5.1. Daizhan Cheng's theory adapted to asynchornicity replaces the equation of evolution (2) where $\Phi : \mathbf{B}^n \to \mathbf{B}^n$, $x : \mathbf{N} \to \mathbf{B}^n$, $\alpha \in \Pi_n$, $k \in \mathbf{N}$, with the equation

$$\underline{\underline{x(k+1)}} = \underline{\underline{\Phi^{\alpha^k}(x(k))}} = \underline{\underline{\widetilde{\Phi}(x(k),\alpha^k)}} \overset{(17)}{=} M_{\widetilde{\Phi}} \cdot (\underline{\underline{x(k)}} \ltimes \underline{\underline{\alpha^k}}), \tag{18}$$

which is easier to be studied. The price to pay is the increase of the dimension of the system from n to 2^n.

Example 5.1. We return to Example 1.1 now. Function

$$\widetilde{\Phi}(\mu_1,\mu_2,\lambda_1,\lambda_2) = (\overline{\lambda_1}\mu_1 \cup \lambda_1\mu_2, \overline{\lambda_2}\mu_2 \cup \lambda_2\overline{\mu_1})$$

defines the matrix

$$M_{\widetilde{\Phi}} = \begin{pmatrix} \widetilde{\Phi_1}(1,1,1,1)\widetilde{\Phi_2}(1,1,1,1) & & \widetilde{\Phi_1}(0,0,0,0)\widetilde{\Phi_2}(0,0,0,0) \\ \widetilde{\Phi_1}(1,1,1,1)\overline{\widetilde{\Phi_2}(1,1,1,1)} & \dots & \widetilde{\Phi_1}(0,0,0,0)\overline{\widetilde{\Phi_2}(0,0,0,0)} \\ \overline{\widetilde{\Phi_1}(1,1,1,1)}\widetilde{\Phi_2}(1,1,1,1) & & \overline{\widetilde{\Phi_1}(0,0,0,0)}\widetilde{\Phi_2}(0,0,0,0) \\ \overline{\widetilde{\Phi_1}(1,1,1,1)}\;\overline{\widetilde{\Phi_2}(1,1,1,1)} & & \overline{\widetilde{\Phi_1}(0,0,0,0)}\;\overline{\widetilde{\Phi_2}(0,0,0,0)} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

Equation (3) implies that $\forall k \in \mathbf{N}$, (18) is true, with $\underline{\underline{x}},\underline{\underline{\alpha}}: \mathbf{N} \to \mathbf{D}^{(2)}$. We can see that, via (18), equation (4) becomes

$$\underline{\underline{x(0)}} \ltimes \underline{\underline{\alpha^0}} = \underline{\underline{(0,0)}} \ltimes \underline{\underline{(1,0)}} = \delta_4^4 \ltimes \delta_4^2 = \delta_{16}^{14},$$

$$\underline{\underline{x(1)}} = M_{\widetilde{\Phi}} \cdot \delta_{16}^{14} = \delta_4^4 = \underline{\underline{(0,0)}},$$

while (5) becomes

$$\underline{\underline{x(1)}} \ltimes \underline{\underline{\alpha^1}} = \underline{\underline{(0,0)}} \ltimes \underline{\underline{(0,1)}} = \delta_4^4 \ltimes \delta_4^3 = \delta_{16}^{15},$$

$$\underline{\underline{x(2)}} = M_{\widetilde{\Phi}} \cdot \delta_{16}^{15} = \delta_4^3 = \underline{\underline{(0,1)}},$$

(6) becomes

$$\underline{\underline{x(2)}} \ltimes \underline{\underline{\alpha^2}} = \underline{\underline{(0,1)}} \ltimes \underline{\underline{(1,1)}} = \delta_4^3 \ltimes \delta_4^1 = \delta_{16}^{9},$$

$$\underline{\underline{x(3)}} = M_{\widetilde{\Phi}} \cdot \delta_{16}^{9} = \delta_4^1 = \underline{\underline{(1,1)}},$$

(7) becomes

$$\underline{\underline{x(3)}} \ltimes \underline{\underline{\alpha^3}} = \underline{\underline{(1,1)}} \ltimes \underline{\underline{(0,1)}} = \delta_4^1 \ltimes \delta_4^3 = \delta_{16}^3,$$

$$\underline{\underline{x(4)}} = M_{\widetilde{\Phi}} \cdot \delta_{16}^3 = \delta_4^2 = \underline{\underline{(1,0)}},$$

and (8) becomes

$$\underline{\underline{x(4)}} \ltimes \underline{\underline{\alpha^4}} = \underline{\underline{(1,0)}} \ltimes \underline{\underline{(1,0)}} = \delta_4^2 \ltimes \delta_4^2 = \delta_{16}^6,$$

$$\underline{\underline{x(5)}} = M_{\widetilde{\Phi}} \cdot \delta_{16}^6 = \delta_4^4 = \underline{\underline{(0,0)}},$$

$$\ldots$$

References

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(Vlad Serban E.) Oradea City Hall, Department of Computers, România

E-mail address: serban_e_vlad@yahoo.com