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The double eventual periodicity of the asynchronous flows

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Abstract

Let $\Phi : \{0,1\}^n \to \{0,1\}^n$ a function whose coordinates $\Phi_i, i \in \{1,...,n\}$ are iterated independently on each other, in discrete time or real time. The resulted flows, called asynchronous, model the asynchronous circuits from the digital electrical circuits. The concept of double eventual periodicity refers to two eventually periodic simultaneous phenomena, one of the function (so called computation function) indicating when and how Φ is iterated and the other one of the flow itself. The paper introduces the double eventual periodicity of the asynchronous flows and gives two important results on them.

Keywords: computation function, asynchronous flows, circuits.

1 Preliminaries

We denote with $\mathbf{B} = \{0,1\}$ the binary Boole algebra, together with the usual laws ' $\overline{}$', ' $\cdot$', ' $\cup$', ' $\oplus$'. These laws induce laws that are denoted with the same symbols on $\mathbf{B}^n, n \geq 1$. Both sets $\mathbf{B}$ and $\mathbf{B}^n$ are organized as topological spaces by the discrete topology. $\mathbf{N}_{\_} = \mathbf{N} \cup \{-1\}$ is the notation of the discrete time set. $\chi_A : \mathbf{R} \to \mathbf{B}$ is the notation of the characteristic function of the set $A \subset \mathbf{R}$:

$$\forall t \in \mathbf{R}, \chi_A(t) = \begin{cases} 1, if\ t \in A, \\ 0, otherwise \end{cases} .$$

We denote with Seq the set of the real, strictly increasing sequences (t_k) which are unbounded from above.

Definition 1. *The **discrete time signals** are by definition the functions $\widehat{x}:\mathbf{N}\_\to\mathbf{B}^n$. Their set is denoted with $\widehat{S}^{(n)}$. The **continuous time signals** are the functions $x:\mathbf{R}\to\mathbf{B}^n$ of the form $\forall t\in\mathbf{R}$, $x(t)=\mu\cdot\chi_{(-\infty,t_0)}(t)\oplus x(t_0)\cdot\chi_{[t_0,t_1)}(t)\oplus...\oplus x(t_k)\cdot\chi_{[t_k,t_{k+1})}(t)\oplus...$ where $\mu\in\mathbf{B}^n$ and $(t_k)\in Seq$. We denote their set with $S^{(n)}$.*

Remark. The signals are the 'nice functions' that model the electrical signals from the digital electrical engineering.

Definition 2. *The **discrete time computation functions** are by definition the sequences $\alpha:\mathbf{N}\to\mathbf{B}^n$. Their set is denoted by $\widehat{\Pi}'_n$. In general, we write α^k instead of $\alpha(k),k\in\mathbf{N}$. The **real time computation functions** $\rho:\mathbf{R}\to\mathbf{B}^n$ are by definition the functions of the form $\rho(t)=\alpha^0\cdot\chi_{\{t_0\}}(t)\oplus\alpha^1\cdot\chi_{\{t_1\}}(t)\oplus...\oplus\alpha^k\cdot\chi_{\{t_k\}}(t)\oplus...$ where $\alpha\in\widehat{\Pi}'_n$ and $(t_k)\in Seq$. Their set is denoted with Π'_n.*

Definition 3. *The discrete time computation function $\alpha\in\widehat{\Pi}'_n$ is called **progressive** if $\forall i\in\{1,...,n\}$,the set $\{k|k\in\mathbf{N},\alpha_i^k=1\}$ is infinite. The set of the discrete time progressive computation functions is denoted by $\widehat{\Pi}_n$. The real time computation function $\rho\in\Pi'_n$ is called **progressive** if $\forall i\in\{1,...,n\}$,the set $\{t|t\in\mathbf{R},\rho_i(t)=1\}$ is infinite. The set of the real time progressive computation functions is denoted by Π_n.*

Definition 4. *Let $\widehat{x}\in\widehat{S}^{(n)},\alpha\in\widehat{\Pi}_n$. For $p\geq1,p'\geq1$ and $k'\in\mathbf{N}\_$, $k''\in\mathbf{N}$, if $\forall k\geq k',\widehat{x}(k)=\widehat{x}(k+p)$, $\forall k\geq k'',\alpha^k=\alpha^{k+p'}$, we say that $\widehat{x},\alpha$ are **eventually periodic** with the **periods** p,p' and the **limits of periodicity** k',k''. We consider $x\in S^{(n)},\rho\in\Pi_n,T>0,T'>0$, $t'\in\mathbf{R}$,$t''\in\mathbf{R}$. If $\forall t\geq t',x(t)=x(t+T)$, $\forall t\geq t'',\rho(t)=\rho(t+T')$, we use to say that x,ρ are **eventually periodic** with the **periods** T,T' and the **limits of periodicity** t',t''.*

Definition 5. *For the function $\Phi:\mathbf{B}^n\to\mathbf{B}^n$ and $\lambda\in\mathbf{B}^n$, we define $\Phi^\lambda:\mathbf{B}^n\to\mathbf{B}^n$ by $\forall\mu\in\mathbf{B}^n,\Phi^\lambda(\mu)=(\overline{\lambda_1}\cdot\mu_1\oplus\lambda_1\cdot\Phi_1(\mu),...,\overline{\lambda_n}\cdot\mu_n\oplus\lambda_n\cdot\Phi_n(\mu))$.*

Definition 6. *Let $\alpha^0,...,\alpha^k,\alpha^{k+1}\in\mathbf{B}^n,k\geq0$. We define iteratively the function $\Phi^{\alpha^0...\alpha^k\alpha^{k+1}}:\mathbf{B}^n\to\mathbf{B}^n$ by $\forall\mu\in\mathbf{B}^n,\Phi^{\alpha^0...\alpha^k\alpha^{k+1}}(\mu)=\Phi^{\alpha^{k+1}}(\Phi^{\alpha^0...\alpha^k}(\mu))$.*

Definition 7. *The function $\mathbf{B}^n\times\mathbf{N}\_\times\widehat{\Pi}_n\ni(\mu,k,\alpha)\longmapsto\widehat{\Phi}^\alpha(\mu,k)\in$*

$\mathbf{B}^n$ *defined by* $\forall k \in \mathbf{N}_{\_}$, $\widehat{\Phi}^\alpha(\mu,k) = \begin{cases} \mu, if\ k=-1, \\ \Phi^{\alpha^0...\alpha^k}(\mu), if\ k \geq 0 \end{cases}$ *is called (discrete time)* ***evolution function****. We define the function* $\mathbf{B}^n \times \mathbf{R} \times \Pi_n \ni (\mu,t,\rho) \longmapsto \Phi^\rho(\mu,t) \in \mathbf{B}^n$ *in the following way. Let* $\forall t \in \mathbf{R}$, $\rho(t) = \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus \alpha^1 \cdot \chi_{\{t_1\}}(t) \oplus ... \oplus \alpha^k \cdot \chi_{\{t_k\}}(t) \oplus ...$ *where* $\alpha \in \widehat{\Pi}_n$ *and* $(t_k) \in Seq$*. Then* $\Phi^\rho(\mu,t) = \widehat{\Phi}^\alpha(\mu,-1) \cdot \chi_{(-\infty,t_0)}(t) \oplus \widehat{\Phi}^\alpha(\mu,0) \cdot \chi_{[t_0,t_1)}(t) \oplus ... \oplus \widehat{\Phi}^\alpha(\mu,k) \cdot \chi_{[t_k,t_{k+1})}(t) \oplus ...$ *is called (real time)* ***evolution function****.*

Definition 8. *We fix in the argument of the discrete time evolution function* $\mu \in \mathbf{B}^n$ *and* $\alpha \in \widehat{\Pi}_n$*. The signal* $\widehat{\Phi}^\alpha(\mu,\cdot) \in \widehat{S}^{(n)}$ *is called (discrete time)* ***flow (through*** μ, ***under*** α*). We fix in the argument of the real time evolution function* $\mu \in \mathbf{B}^n$ *and* $\rho \in \Pi_n$*. The signal* $\Phi^\rho(\mu,\cdot) \in S^{(n)}$ *is called (real time)* ***flow (through*** μ, ***under*** ρ*).*

Remark. The function Φ applied to the argument μ is computed on all its coordinates: $\Phi(\mu) = (\Phi_1(\mu), \Phi_2(\mu), ..., \Phi_n(\mu))$. The function Φ^λ applied to μ computes those coordinates Φ_i of Φ for which $\lambda_i = 1$ and it does not compute those coordinates Φ_i for which $\lambda_i = 0 : \forall i \in \{1,2,...,n\}$, $\Phi_i^\lambda(\mu) = \begin{cases} \Phi_i(\mu), \lambda_i = 1, \\ \mu_i, \lambda_i = 0 \end{cases}$. Unlike the usual computations from the dynamical systems theory that take place synchronously on all the coordinates: $\Phi(\mu)$, $(\Phi \circ \Phi)(\mu)$, $(\Phi \circ \Phi \circ \Phi)(\mu), ...$ here things happen on some coordinates only. The asynchronous flows represent a generalization of the computations from the dynamical systems theory, since the constant sequence $\alpha^k = (1,...,1) \in \mathbf{B}^n, k \in \mathbf{N}$ belongs to $\widehat{\Pi}_n$, and it gives for any $\mu \in \mathbf{B}^n$ that $\Phi^{\alpha^0}(\mu) = \Phi(\mu)$, $\Phi^{\alpha^0\alpha^1}(\mu) = (\Phi \circ \Phi)(\mu)$, $\Phi^{\alpha^0\alpha^1\alpha^2}(\mu) = (\Phi \circ \Phi \circ \Phi)(\mu), ...$ So, the functions $\alpha \in \widehat{\Pi}_n, \rho \in \Pi_n$ show when and how the coordinates $\Phi_i, i = \overline{1,n}$ are computed.

Remark. The progressiveness of α, ρ means that $\widehat{\Phi}^\alpha(\mu,\cdot)$, $\Phi^\rho(\mu,\cdot)$ compute each coordinate $\Phi_i, i = \overline{1,n}$ infinitely many times as $k \to \infty, t \to \infty$. In electrical engineering, this corresponds to the so called **unbounded delay model** of computation of the Boolean functions, stating basically that each coordinate i of Φ is computed independently on the other coordinates, in finite time.

2 Double eventual periodicity

Definition 9. *Let* $\mu \in \mathbf{B}^n, \alpha \in \widehat{\Pi}_n$ *and* $\rho \in \Pi_n$. *If* $p \geq 1, p' \geq 1$ *and* $k' \in \mathbf{N}$ *exist such that*

(1) $\forall k \geq k', \alpha^k = \alpha^{k+p}$ *and* **(2)** $\forall k \geq k', \widehat{\Phi}^{\alpha}(\mu, k) = \widehat{\Phi}^{\alpha}(\mu, k+p')$ *are true, then* $\widehat{\Phi}^{\alpha}(\mu, \cdot)$ *is called* ***double eventually periodic***. *And if* $T > 0, T' > 0, t' \in \mathbf{R}$ *exist with*

(3) $\forall t \geq t', \rho(t) = \rho(t+T)$ *and* **(4)** $\forall t \geq t', \Phi^{\rho}(\mu, t) = \Phi^{\rho}(\mu, t+T')$ *fulfilled, then* $\Phi^{\rho}(\mu, \cdot)$ *is called* ***double eventually periodic***.

Theorem 10. *Let* $\alpha \in \widehat{\Pi}_n$. *We ask that* $t_0 \in \mathbf{R}$ *and* $h > 0$ *exist such that*

(5) $\rho(t) = \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus \alpha^1 \cdot \chi_{\{t_0+h\}}(t) \oplus ... \oplus \alpha^k \cdot \chi_{\{t_0+kh\}}(t) \oplus ...$

Then the equivalence ((**1**) *and* (**2**)) $\Longleftrightarrow$ ((**3**) *and* (**4**)) *is true.*

Theorem 11. *Let* $\mu \in \mathbf{B}^n$. *a) If* $\alpha \in \widehat{\Pi}_n$ *is eventually periodic, then* $\widehat{\Phi}^{\alpha}(\mu, \cdot)$ *is double eventually periodic. b) If* $\rho \in \Pi_n$ *is eventually periodic, then* $\Phi^{\rho}(\mu, \cdot)$ *is double eventually periodic.*

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