On the fundamental mode

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Abstract. The asynchronous systems are multivalued applications f from $\mathbf{R} \to \{0,1\}^m$ functions, called (admissible) inputs, to sets of $\mathbf{R} \to \{0,1\}^n$ functions, called (possible) states. The fundamental (operating) mode of f consists in the existence of an input u and of a sequence $(\mu^k)_{k \in \mathbf{N}} \in \{0,1\}^n$ of binary vectors so that $\mu^0, \mu^1, \mu^2, ...$ are accessed by all the states $x \in f(u)$ simultaneously in this order, μ^0 is the initial state and $(\mu^k)_{k \geq 1}$ are 'steady states'.

1. Introduction

The concept of asynchronous system has its origin in the modeling of the asynchronous circuits from digital electrical engineering and the asynchronous systems theory is the theory of modeling such circuits. The uncertainties that govern these circuits can be surpassed in at least two ways: by using a three-valued logic and respectively by using a two-valued logic but many-valued functions (i.e. non-deterministic systems), that give for each cause all the possible effects, our choice.

Several works exist in this moment containing equations and inequalities written with $\mathbf{R} \to \{0,1\}$ functions that model the behavior of the asynchronous circuits. In [**1**] we present a method of modeling where the fundamental circuit is the 'delay element', i.e. the circuit that computes (inertially, in real time) the identical function $\{0,1\} \to \{0,1\}$. The technique of modeling is called delay theory. The 'delays', i.e. the models of the delay elements are one dimensional asynchronous systems that fulfill a certain requirement of stability. They were generalized later in our works [**2**] and [**3**].

Let the asynchronous system f that associates to some input $u : \mathbf{R} \to \{0,1\}^m$ the set of states $x \in f(u)$, where $x : \mathbf{R} \to \{0,1\}^n$. The fundamental operating mode of f asks the existence of a sequence $(\mu^k)_{k \in \mathbf{N}} \in \{0,1\}^n$ so that all $x \in f(u)$ run simultaneously through the values $\mu^0, \mu^1, \mu^2, ...$ in this order, where μ^0 is the initial state and $\mu^1, \mu^2, ...$ are final states (steady states). This concept is mentioned in many works under a non-formalized manner. We quote [**4**] where its characterization is the next one: 'inputs are constrained to change only when all the delay elements are stable (i.e. they have the input value equal with the output value)'. 'Note that the fundamental mode excludes' the existence of 'a cycle of oscillations', that is instability. Elsewhere the author refers to the fundamental

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mode where 'the designer has to make sure that the circuit inputs can change only when the circuit itself is stable and ready to accept them'. The characterization given by L. Lavagno to the fundamental mode, that agrees with other opinions, corresponds to our special case from Section 11.

2. Preliminaries

Definition 1. *We note with* $\mathbf{B}$ *the set* $\{0,1\}$ *together with the order* $0<1$, *the discrete topology and the laws: the complement* $'\overline{}'$, *the intersection* $'\cdot'$, *the reunion* $'\cup'$ *and the modulo 2 sum* $'\oplus'$. $\mathbf{B}$ *is called the binary Boole algebra or the Boole algebra with two elements.*

Notation 1. *For some interval* $I\subset\mathbf{R}$ *and* $x:\mathbf{R}\rightarrow\mathbf{B}^n$, *we note with* $x_{|I}$ *the restriction of* x *at* I.

Notation 2. *If* x *is constant on the interval* I *and equal with* $\mu\in\mathbf{B}^n$, *we write* $x_{|I}=\mu$, *by identifying the function with the constant.*

Definition 2. *Let* $x:\mathbf{R}\rightarrow\mathbf{B}^n$ *some function. We define the initial value of* x, *noted* $x(-\infty+0)$ *or* $\lim\limits_{t\rightarrow-\infty}x(t)$ *to be that vector from* $\mathbf{B}^n$ *satisfying one of the equivalent statements*

$$\exists t_0\in\mathbf{R},\forall t<t_0,x(t)=x(-\infty+0)$$

$$\exists t_0\in\mathbf{R},x_{|(-\infty,t_0)}=x(-\infty+0)$$

Dually, the final value of x *is noted with* $x(\infty-0)$ *or* $\lim\limits_{t\rightarrow\infty}x(t)$ *and it is the vector from* $\mathbf{B}^n$ *satisfying one of*

$$\exists t_f\in\mathbf{R},\forall t\geq t_f,x(t)=x(\infty-0)$$

$$\exists t_f\in\mathbf{R},x_{|[t_f,\infty)}=x(\infty-0)$$

Notation 3. *For some* $d\in\mathbf{R}$, *we note with* $\tau^d:\mathbf{R}\rightarrow\mathbf{R}$ *the translation* $\forall t\in\mathbf{R},\tau^d(t)=t-d$.

Remark 1. *For any* x, $x(-\infty+0)$ *and* $x(\infty-0)$ *are uniquely defined since* x *is a function. On the other hand, the initial and the final value of* x *and* $x\circ\tau^d$ *coincide (we have* $\forall t\in\mathbf{R},(x\circ\tau^d)(t)=x(t-d)$*).*

Definition 3. *For any set* $A\subset\mathbf{R}$, *we define the characteristic function of* A *by* $\chi_A:\mathbf{R}\rightarrow\mathbf{B}$,

$$\forall t\in\mathbf{R},\chi_A(t)=\begin{cases}1, if\ t\in A\\ 0, if\ t\notin A\end{cases}$$

Notation 4. *We note with Seq the set of the real unbounded strictly increasing sequences* $t_0<t_1<t_2<...$ *The elements of Seq will generally be noted with* (t_k).

Definition 4. *The function* $x:\mathbf{R}\rightarrow\mathbf{B}^n$ *is called* $n-$*dimensional signal if* $x(-\infty+0)\in\mathbf{B}^n$ *and* $(t_k)\in Seq$ *exist so that*

$$x(t)=x(-\infty+0)\cdot\chi_{(-\infty,t_0)}(t)\oplus x(t_0)\cdot\chi_{[t_0,t_1)}(t)\oplus x(t_1)\cdot\chi_{[t_1,t_2)}(t)\oplus...$$

Notation 5. *The set of the* $n-$*dimensional signals is noted* $S^{(n)}$.
We note with $S_c^{(n)}$ *the set of these* $x\in S^{(n)}$ *for which* $x(\infty-0)$ *exists.*
For some function $F:\mathbf{B}^m\rightarrow\mathbf{B}^n$ *we note*

$$S_{F,c}^{(m)}=\{u|u\in S^{(m)},\lim_{t\rightarrow\infty}F(u(t))\ exists\}$$

Definition 5. *Let* $x \in S^{(n)}$ *and the numbers* $t_0, t_1 \in \mathbf{R}$ *so that* $t_0 < t_1$. *The restrictions* $\gamma = x_{|(-\infty,t_1]}$, $\gamma' = x_{|[t_0,t_1]}$ *are called transitions of* x *from the value* $x(-\infty + 0)$ *to the value* $x(t_1)$*, respectively from* $x(t_0)$ *to* $x(t_1)$*. The intervals* $(-\infty, t_1]$, $[t_0, t_1]$ *are called the support intervals of the transitions* γ, γ'*. The number* $t_1 - t_0$ *is called the duration of the transition* γ'*.*

Notation 6. *We note with* $P^*(S^{(n)}) = \{X | X \subset S^{(n)}, X \neq \emptyset\}$ *the set of the non-empty subsets of* $S^{(n)}$*.*

Definition 6. *A function* $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$ *is called (asynchronous) system, given under the explicit form. It associates to the functions* $u \in U$ *called (admissible) inputs, sets of functions* $x \in f(u)$*, called (possible) states.*

Under the implicit form, the asynchronous system consists in one or several equations and/or inequalities where the unknown $x \in S^{(n)}$ *depends on* $u \in U$*.*

Definition 7. *The system* f *is non-anticipatory if* $\forall t \in \mathbf{R}$, $\forall u \in U$, $\forall v \in U$,

$$u_{|(-\infty,t)} = v_{|(-\infty,t)} \Longrightarrow \{x_{|(-\infty,t]} | x \in f(u)\} = \{y_{|(-\infty,t]} | y \in f(v)\}$$

3. Synchronous access

Definition 8. *By the synchronous access of (the states of)* $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$*, under the input* $u \in U$*, to the value* $\mu \in \mathbf{B}^n$ *at* $t_0 \in \mathbf{R}$ *it is understood the next property*

$$\forall x \in f(u), x(t_0) = \mu \tag{3.1}$$

If it is fulfilled, μ *is called synchronously accessible value and* t_0 *is called the access time (instant) of (the states of)* f *under the input* u *to the value* μ*.*

Theorem 1. *Let the non-anticipatory system* f *and we fix* $t_0 \in \mathbf{R}, u \in U$, $\mu \in \mathbf{B}^n$*. (3.1) is equivalent with*

$$\exists v \in U, u_{|(-\infty,t_0)} = v_{|(-\infty,t_0)}, \forall y \in f(v), y(t_0) = \mu \tag{3.2}$$

Proof. (3.1)$\Longrightarrow$(3.2) is obvious.

(3.2)$\Longrightarrow$(3.1). From $u_{|(-\infty,t_0)} = v_{|(-\infty,t_0)}$ and the non-anticipation of f we infer that $\{x_{|(-\infty,t_0]} | x \in f(u)\} = \{y_{|(-\infty,t_0]} | y \in f(v)\}$ in particular we have

$$\{x(t_0) | x \in f(u)\} = \{y(t_0) | y \in f(v)\} = \mu$$

i.e. (3.1) holds. □

Definition 9. *The next special cases of fulfillment of (3.1)*

$$\forall x \in f(u), x_{|(-\infty,t_0)} = \mu \tag{3.3}$$

$$\forall x \in f(u), x_{|[t_0,\infty)} = \mu \tag{3.4}$$

are called synchronous initial access, respectively synchronous final access of f*, under* u*, to* μ*. The vector* μ *is called (synchronously) accessible initial value, respectively (synchronously) accessible final value*[1] *and* $(-\infty, t_0), [t_0, \infty)$ *are called the access time intervals of (the states of)* f*, under* u*, to the initial value, respectively to the final value.*

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[1]Other terminologies are: final state of f, or steady state of f. We indicated in the Abstract and in the Introduction the concept of 'steady state' because it is the most popular, but we preffer the indicated terminology due to its precission and due to the fact that it highlights the duality initial-final.

Definition 10. *If*

$$\forall x\in f(u), x=\mu \tag{3.5}$$

we say that f has, under u, a point of equilibrium and μ is called point of equilibrium of f. The access time interval of (the states of) f, under u, to μ is by definition $\mathbf{R}$.

Remark 2. *The word 'synchronous' in the previous definitions means the fact that the number t_0 does not depend on the choice of $x\in f(u)$.*

The point of equilibrium is a special case of both the synchronously accessible initial value and the synchronously accessible final value of f.

On the other hand we try to extend, when f is non-anticipatory, the result of Theorem 1 to the equivalencies between (3.3), (3.4) and respectively

$$\exists v\in U, u_{|(-\infty,t_0)}=v_{|(-\infty,t_0)}, \forall y\in f(v), y_{|(-\infty,t_0)}=\mu \tag{3.6}$$

$$\exists v\in U, u_{|(-\infty,t_0)}=v_{|(-\infty,t_0)}, \forall y\in f(v), y_{|[t_0,\infty)}=\mu \tag{3.7}$$

We observe that

- (3.3)$\Longleftrightarrow$(3.6) is true

- (3.4)$\Longleftrightarrow$(3.7) is not true. While (3.4) shows that all $x\in f(u)$, starting from the time instant t_0, become equal with μ, (3.7) states that all $x\in f(u)$, starting from t_0, may become equal with μ, if for example $u=v$.

It is not the case to try such reasoning for the points of equilibrium too.

4. Synchronous consecutive accesses

Definition 11. *We suppose that the system f is non-anticipatory. By the synchronous consecutive accesses of (the states of) f, under $u\in U$, to the values $\mu,\mu'\in\mathbf{B}^n$ at the time instants $t_0<t_1$ it is understood the property*

$$\forall x\in f(u), x(t_0)=\mu \text{ and } x(t_1)=\mu' \tag{4.1}$$

Remark 3. *In the present work we are interested in two special cases of synchronous consecutive accesses, the one when in (4.1) μ is initial value and μ' is final value*

$$\forall x\in f(u), x_{|(-\infty,t_0)}=\mu \text{ and } x_{|[t_1,\infty)}=\mu' \tag{4.2}$$

and respectively when in (4.1) μ,μ' are both final values

$$\forall x\in f(u), x_{|[t_0,\infty)}=\mu \text{ and } x_{|[t_1,\infty)}=\mu' \tag{4.3}$$

Let us replace in (4.2) and (4.3) the synchronous accesses of x to the final values by (3.7). After some computations that take into account the non-anticipation of f we get the properties

$$\exists v\in U, u_{|(-\infty,t_1)}=v_{|(-\infty,t_1)}, \forall y\in f(v), y_{|(-\infty,t_0)}=\mu \text{ and } y_{|[t_1,\infty)}=\mu' \tag{4.4}$$

$$\begin{gathered}\exists v\in U, u_{|(-\infty,t_0)}=v_{|(-\infty,t_0)}, \forall y\in f(v), y_{|[t_0,\infty)}=\mu \text{ and}\\ \text{and } \exists v'\in U, u_{|(-\infty,t_1)}=v'_{|(-\infty,t_1)}, \forall y'\in f(v'), y'_{|[t_1,\infty)}=\mu'\end{gathered} \tag{4.5}$$

The non-equivalent statements (4.2) and (4.4) describe, each of them, the accesses of f first to the initial value μ, then to the final value μ', with the difference that in the first case all $x\in f(u)$ stabilize at μ' and in the second case all $x\in f(u)$ may stabilize at μ', for example if $u=v$.

The non-equivalent statements (4.3) and (4.5) give two completely different manners of accessing synchronously first the final value μ, *then the final value* μ', *in the sense that at (4.3) we have necessarily the triviality* $\mu = \mu'$ *while at (4.5)* $\mu \neq \mu'$ *is possible.*

In the properties (4.1),...,(4.5) the possibility $\mu = \mu' =$ *point of equilibrium exists, with the trivialities that follow from this situation.*

5. Transfers

Definition 12. *We suppose that the non-anticipatory system* f *accesses synchronously, under the input* $u \in U$, *the values* $\mu, \mu' \in \mathbf{B}^n$ *at the time instants* $t_0 < t_1$, *i.e. (4.1) is fulfilled. Then we note*

$$\Gamma = \{x_{|[t_0,t_1]} | x \in f(u)\} \tag{5.1}$$

Γ *is called the (synchronous) transfer of (the states of)* f, *that is made under the input* u *from* μ *to* μ'.

Conversely, if we say that Γ *that is defined by (5.1) represents a transfer of* f, *made under the input* u, *from* $\mu = x(t_0)$ *to* $\mu' = x(t_1)$ *and* μ, μ' *are independent on the choice of* $x \in f(u)$, *then we mean that (4.1) is true.*

Definition 13. *We ask that (4.4) is true and we note*

$$\mu \overset{u_{|(-\infty,t_1)}}{\rightarrow} \mu' = \{x_{|(-\infty,t_1]} | x \in f(u)\} \tag{5.2}$$

$\mu \overset{u_{|(-\infty,t_1)}}{\rightarrow} \mu'$ *is called initial fundamental transfer of (the states of)* f, *under* u, *from the initial value* μ *to the final value* μ'.

Conversely, the statement that $\mu \overset{u_{|(-\infty,t_1)}}{\rightarrow} \mu'$ *defined by (5.2) is an initial fundamental transfer refers to the existence of* $t_0 < t_1$ *so that (4.4) is satisfied.*

Definition 14. *If (4.5) is true, we note*

$$\mu \overset{u_{|[t_0,t_1)}}{\rightarrow} \mu' = \{x_{|[t_0,t_1]} | x \in f(u)\} \tag{5.3}$$

$\mu \overset{u_{|[t_0,t_1)}}{\rightarrow} \mu'$ *is called non-initial fundamental transfer of (the states of)* f, *made under the input* u, *from the final value* μ *to the final value* μ'.

Conversely, the statement that $\mu \overset{u_{|[t_0,t_1)}}{\rightarrow} \mu'$ *defined by (5.3) is a non-initial fundamental transfer means the satisfaction of (4.5).*

Definition 15. *If (3.5) is fulfilled, we note*

$$(\mu \overset{u}{=} \mu) = \{\mu\} \tag{5.4}$$

$\mu \overset{u}{=} \mu$ *is called trivial fundamental transfer of (the states of)* f, *made under the input* u, μ *being a point of equilibrium.*

Conversely, when we state that $\mu \overset{u}{=} \mu$ *defined by (5.4) is a trivial fundamental transfer, this means the truth of (3.5).*

Definition 16. *If the synchronous transfer* Γ *satisfies* $\forall \gamma \in \Gamma$, γ *is coordinatewise monotonous then it is called hazard-free.*

Remark 4. *At (4.4), the synchronism of the access of the states to the initial value* μ *is not necessary in many situations and it was asked for the symmetry of the exposure only.*

At the hazard-free transfers, the condition of monotony seems one of economy and normalization, the coordinates of x *do not switch more than necessary, but it has rather a functional meaning.*

The trivial fundamental transfers are hazard-free.

6. Some simple properties of the fundamental transfers and an example

THEOREM 2. *Let the non-anticipatory system f and we fix $t_0, t_1 \in \mathbf{R}, t_0 < t_1, u \in U, \mu, \mu' \in \mathbf{B}^n$. If (4.2) is true then $\mu \overset{u_{|(-\infty,t_1)}}{\rightarrow} \mu'$ is an initial fundamental transfer and if*

$$\exists v \in U, u_{|(-\infty,t_0)} = v_{|(-\infty,t_0)}, \forall y \in f(v), y_{|[t_0,\infty)} = \mu \text{ and } \forall x \in f(u), x_{|[t_1,\infty)} = \mu'$$

then $\mu \overset{u_{|[t_0,t_1)}}{\longrightarrow} \mu'$ is a non-initial fundamental transfer.

PROOF. The first hypothesis makes (4.4) true for $v = u$ and the second statement makes (4.5) true for $v' = u$. □

THEOREM 3. *Let f non-anticipatory and $\mu \overset{u_{|I}}{\rightarrow} \mu'$ a fundamental transfer, where $I \subset \mathbf{R}$ is an interval of the form $(-\infty, t_1)$ or $[t_0, t_1)$.*

a) If $I = (-\infty, t_1)$ and $u' \in U$ is arbitrary with $u_{|(-\infty,t_1)} = u'_{|(-\infty,t_1)}$, then $\mu \overset{u'_{|I}}{\rightarrow} \mu'$ is an initial fundamental transfer equal with $\mu \overset{u_{|I}}{\rightarrow} \mu'$.

b) If $I = [t_0, t_1)$, then $\forall u' \in U$, $u_{|(-\infty,t_1)} = u'_{|(-\infty,t_1)}$ implies that $\mu \overset{u'_{|I}}{\rightarrow} \mu'$ is a non-initial fundamental transfer equal with $\mu \overset{u_{|I}}{\rightarrow} \mu'$.

PROOF. a) $\mu \overset{u'_{|(-\infty,t_1)}}{\rightarrow} \mu'$ is an initial fundamental transfer i.e.

$$\exists t_0 < t_1, \exists v \in U, u'_{|(-\infty,t_1)} = v_{|(-\infty,t_1)},$$
$$\forall y \in f(v), y_{|(-\infty,t_0)} = \mu \text{ and } y_{|[t_1,\infty)} = \mu'$$

takes place because the hypothesis (4.4) is true as well as $u_{|(-\infty,t_1)} = u'_{|(-\infty,t_1)}$. We take into account the non-anticipation of f and we get the second statement of the Theorem

$$\mu \overset{u_{|(-\infty,t_1)}}{\rightarrow} \mu' = \{x_{|(-\infty,t_1]} | x \in f(u)\} = \{x'_{|(-\infty,t_1]} | x' \in f(u')\} = \mu \overset{u'_{|(-\infty,t_1)}}{\rightarrow} \mu'$$

b) is proved similarly with a). □

EXAMPLE 1. *The system $f : S \to P^*(S)$ that is defined by the double inequality*

$$\bigcap_{\xi \in [t-1,t)} \overline{u(\xi)} \leq x(t) \leq \bigcup_{\xi \in [t-1,t)} \overline{u(\xi)} \tag{6.1}$$

models the computation of the logical complement of u, made with a delay of one time unit. We suppose that it is non-anticipatory and we note $u = \chi_{[0,2)}$, $v = \chi_{[0,\infty)}$ for which the inequalities $\bigcap_{\xi \in [t-1,t)} \overline{u(\xi)} \leq x(t) \leq \bigcup_{\xi \in [t-1,t)} \overline{u(\xi)}$, $\bigcap_{\xi \in [t-1,t)} \overline{v(\xi)} \leq y(t) \leq \bigcup_{\xi \in [t-1,t)} \overline{v(\xi)}$ become

$$\chi_{(-\infty,0] \cup [3,\infty)}(t) \leq x(t) \leq \chi_{(-\infty,1) \cup (2,\infty)}(t) \tag{6.2}$$

$$\chi_{(-\infty,0]}(t) \leq y(t) \leq \chi_{(-\infty,1)}(t) \tag{6.3}$$

From (6.3) we infer that

$$\forall y \in f(v), y_{|(-\infty,0)} = 1 \text{ and } y_{|[1,\infty)} = 0$$

and because

$$u_{|(-\infty,1)} = v_{|(-\infty,1)}$$

we have that $(1 \stackrel{u_{|(-\infty,1)}}{\to} 0) = (1 \stackrel{v_{|(-\infty,1)}}{\to} 0)$ *is an initial fundamental transfer ((4.4) is true). From the inequalities (6.2), (6.3) we also infer that*

$$\forall y \in f(v), y_{|[1,\infty)} = 0$$

$$\forall x \in f(u), x_{|[3,\infty)} = 1$$

i.e. $0 \stackrel{u_{|[1,3)}}{\to} 1$ *is non-initial fundamental transfer (from Theorem 2).*

The transitions $\gamma \in 1 \stackrel{u_{|(-\infty,1)}}{\to} 0$ *and* $\gamma' \in 0 \stackrel{u_{|[1,3)}}{\to} 1$ *are not monotonous in general. We ask in what conditions, if we add the (absolute inertia) requests*

$$\overline{x(t-0)} \cdot x(t) \leq \bigcap_{\xi \in [t,t+\delta]} x(\xi) \tag{6.4}$$

$$x(t-0) \cdot \overline{x(t)} \leq \bigcap_{\xi \in [t,t+\delta]} \overline{x(\xi)} \tag{6.5}$$

-where $\delta \geq 0$*- to (6.1) with* $u = \chi_{[0,2)}$*, i.e. to (6.2), monotony is true. Monotony means that x switches from 1 to 0 in the interval* $(0,1]$ *and that it cannot switch from 0 to 1 and then from 1 to 0 again in this interval. Let* $0 < t_1 < t_2 < t_3 \leq 1$ *so that*

$$x(t_1-0) \cdot \overline{x(t_1)} = \overline{x(t_2-0)} \cdot x(t_2) = x(t_3-0) \cdot \overline{x(t_3)} = 1$$

We have then $t_2 - t_1 > \delta, t_3 - t_2 > \delta$ *-from the satisfaction of (6.4) and (6.5)- meaning that* $1 > t_3 - t_1 > 2\delta$*. Thus if* $\delta \geq \frac{1}{2}$*, such* t_1, t_2, t_3 *do not exist and any* $\gamma \in 1 \stackrel{u_{|(-\infty,1)}}{\to} 0$ *is a monotonous transition. Similarly* $\delta \geq \frac{1}{2}$ *implies the fact that any* $\gamma' \in 0 \stackrel{u_{|[1,3)}}{\to} 1$ *is monotonous.*

Another condition is also required here: after having switched from 1 to 0 in the interval $(0,1]$*, x is also allowed to switch from 0 to 1 in the interval* $(2,3]$*. This gives* $\delta < 3$.

The conclusion is the following: for $\delta \in [\frac{1}{2}, 3)$*, the system g that is obtained by intersecting (6.1), (6.4), (6.5) where* $u = \chi_{[0,2)}$ *has the transfers* $1 \stackrel{u_{|(-\infty,1)}}{\to} 0$, $0 \stackrel{u_{|[1,3)}}{\to} 1$ *hazard-free.*

7. The composition of the fundamental transfers

THEOREM 4. *Let the non-anticipatory system* $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$ *satisfying the conditions:*

i) U is closed under translations and under 'concatenation'

$$\forall d \in \mathbf{R}, \forall u \in U, u \circ \tau^d \in U$$

$$\forall t \in \mathbf{R}, \forall u \in U, \forall v \in U, u \cdot \chi_{(-\infty,t)} \oplus v \cdot \chi_{[t,\infty)} \in U$$

*ii) non-anticipation** $\forall t \in \mathbf{R}$, $\forall u \in U$, $\forall v \in U$,

$$(u_{|[t,\infty)} = v_{|[t,\infty)} \text{ and } \{x(t)|x \in f(u)\} = \{y(t)|y \in f(v)\}) \Longrightarrow$$

$$\Longrightarrow \{x_{|[t,\infty)}|x \in f(u)\} = \{y_{|[t,\infty)}|y \in f(v)\}$$

iii) time invariance

$$\forall d \in \mathbf{R}, \forall u \in U, f(u \circ \tau^d) = \{x \circ \tau^d | x \in f(u)\}$$

a) We suppose that $t_0 < t_1$, $t_2 < t_3$, $u^0, u^1, v^1 \in U$ *and* $\mu, \mu', \mu" \in \mathbf{B}^n$ *are arbitrary with*

$$\forall x \in f(u^0), x_{|(-\infty,t_0)} = \mu$$
$$\forall x \in f(u^0), x_{|[t_1,\infty)} = \mu'$$
$$u^1_{|(-\infty,t_2)} = v^1_{|(-\infty,t_2)}$$
$$\forall y' \in f(v^1), y'_{|[t_2,\infty)} = \mu'$$
$$\forall x' \in f(u^1), x'_{|[t_3,\infty)} = \mu"$$

We note with $d = t_1 - t_2$ *and*

$$\widetilde{u}_\varepsilon = u^0 \cdot \chi_{(-\infty,t_1+\varepsilon)} \oplus (u^1 \circ \tau^{d+\varepsilon}) \cdot \chi_{[t_1+\varepsilon,\infty)}$$

for $\varepsilon \geq 0$. *We have*

$$\forall \widetilde{x} \in f(\widetilde{u}_\varepsilon), \widetilde{x}_{|(-\infty,t_0)} = \mu$$
$$\forall \widetilde{x} \in f(\widetilde{u}_\varepsilon), \widetilde{x}_{|[t_3+d+\varepsilon,\infty)} = \mu"$$

meaning that if $\mu \overset{u^0_{|(-\infty,t_1)}}{\rightarrow} \mu'$ *is initial fundamental and* $\mu' \overset{u^1_{|[t_2,t_3)}}{\rightarrow} \mu"$ *is non-initial fundamental, then* $\mu \overset{\widetilde{u}_{\varepsilon|(-\infty,t_3+d+\varepsilon)}}{\rightarrow} \mu"$ *is initial fundamental. In other words if* $f(u^0)$ *transfers synchronously the initial value* μ *in the final value* μ' *and if* $f(u^1)$ *transfers synchronously the final value* μ' *in the final value* $\mu"$ *then* $f(\widetilde{u}_\varepsilon)$ *transfers synchronously the initial value* μ *in the final value* $\mu"$.

b) Let us suppose that $t_0 < t_1$, $t_2 < t_3$, $u^0, v^0, u^1, v^1 \in U$ *and* $\mu, \mu', \mu" \in \mathbf{B}^n$ *are given so that*

$$u^0_{|(-\infty,t_0)} = v^0_{|(-\infty,t_0)} \tag{7.1}$$

$$\forall y \in f(v^0), y_{|[t_0,\infty)} = \mu \tag{7.2}$$

$$\forall x \in f(u^0), x_{|[t_1,\infty)} = \mu' \tag{7.3}$$

$$u^1_{|(-\infty,t_2)} = v^1_{|(-\infty,t_2)} \tag{7.4}$$

$$\forall y' \in f(v^1), y'_{|[t_2,\infty)} = \mu' \tag{7.5}$$

$$\forall x' \in f(u^1), x'_{|[t_3,\infty)} = \mu" \tag{7.6}$$

With the notations $d = t_1 - t_2$, $\widetilde{v} = v^0$ *and*

$$\widetilde{u}_\varepsilon = u^0 \cdot \chi_{(-\infty,t_1+\varepsilon)} \oplus (u^1 \circ \tau^{d+\varepsilon}) \cdot \chi_{[t_1+\varepsilon,\infty)} \tag{7.7}$$

$\varepsilon \geq 0$, *we have*

$$\widetilde{u}_{\varepsilon|(-\infty,t_0)} = \widetilde{v}_{|(-\infty,t_0)} \tag{7.8}$$

$$\forall \widetilde{y} \in f(\widetilde{v}), \widetilde{y}_{|[t_0,\infty)} = \mu \tag{7.9}$$

$$\forall \widetilde{x} \in f(\widetilde{u}_\varepsilon), \widetilde{x}_{|[t_3+d+\varepsilon,\infty)} = \mu" \tag{7.10}$$

This means that if $\mu \overset{u^0_{|[t_0,t_1)}}{\rightarrow} \mu'$, $\mu' \overset{u^1_{|[t_2,t_3)}}{\rightarrow} \mu"$ *are non-initial fundamental, then* $\mu \overset{\widetilde{u}_{\varepsilon|[t_0,t_3+d+\varepsilon)}}{\rightarrow} \mu"$ *is non-initial fundamental (if* $f(u^0)$ *transfers synchronously the final value* μ *in the final value* μ' *and if* $f(u^1)$ *transfers synchronously the final value* μ' *in the final value* $\mu"$ *then* $f(\widetilde{u}_\varepsilon)$ *transfers synchronously the final value* μ *in the final value* $\mu"$*).*

PROOF. b) Let us observe first that $\widetilde{u}_\varepsilon$ given by (7.7) belongs to U, from i).

(7.8) is satisfied because for any $\varepsilon \geq 0$ we have $t_1 + \varepsilon \geq t_1 > t_0$ and also from the definition of $\widetilde{v}$:

$$\widetilde{u}_{\varepsilon|(-\infty,t_0)} \overset{(7.7)}{=} u^0_{|(-\infty,t_0)} \overset{(7.1)}{=} v^0_{|(-\infty,t_0)} = \widetilde{v}_{|(-\infty,t_0)}$$

(7.9) is true because it coincides with the hypothesis (7.2).

We prove (7.10). From (7.4) and from the non-anticipation of f we infer

$$\{y'_{|(-\infty,t_2]}|y' \in f(v^1)\} = \{x'_{|(-\infty,t_2]}|x' \in f(u^1)\}$$

and if we take into account (7.5) also, we can see that

$$\{y'(t_2)|y' \in f(v^1)\} = \{x'(t_2)|x' \in f(u^1)\} = \mu' \tag{7.11}$$

The time invariance of f implies that

$$\{x"|x" \in f(u^1 \circ \tau^{d+\varepsilon})\} = \{x' \circ \tau^{d+\varepsilon}|x' \in f(u^1)\}$$

thus

$$\{x'(t_2)|x' \in f(u^1)\} = \{x"(t_1+\varepsilon)|x" \in f(u^1 \circ \tau^{d+\varepsilon})\} \tag{7.12}$$

From

$$\widetilde{u}_{\varepsilon|(-\infty,t_1+\varepsilon)} = u^0_{|(-\infty,t_1+\varepsilon)}$$

and from the non-anticipation we get

$$\{\widetilde{x}_{|(-\infty,t_1+\varepsilon]}|\widetilde{x} \in f(\widetilde{u}_\varepsilon)\} = \{x_{|(-\infty,t_1+\varepsilon]}|x \in f(u^0)\}$$

in particular we have

$$\{\widetilde{x}(t_1+\varepsilon)|\widetilde{x} \in f(\widetilde{u}_\varepsilon)\} = \{x(t_1+\varepsilon)|x \in f(u^0)\} \tag{7.13}$$

Then

$$\{\widetilde{x}(t_1+\varepsilon)|\widetilde{x} \in f(\widetilde{u}_\varepsilon)\} \overset{(7.13)}{=} \{x(t_1+\varepsilon)|x \in f(u^0)\} \overset{(7.3)}{=} \mu' =$$

$$\overset{(7.5)}{=} \{y'(t_2)|y' \in f(v^1)\} \overset{(7.11)}{=} \{x'(t_2)|x' \in f(u^1)\} =$$

$$\overset{(7.12)}{=} \{x"(t_1+\varepsilon)|x" \in f(u^1 \circ \tau^{d+\varepsilon})\} \tag{7.14}$$

Because

$$\widetilde{u}_{\varepsilon|[t_1+\varepsilon,\infty)} = (u^1 \circ \tau^{d+\varepsilon})_{|[t_1+\varepsilon,\infty)} \tag{7.15}$$

(7.14), (7.15) and the non-anticipation* of f show that

$$\{\widetilde{x}_{|[t_1+\varepsilon,\infty)}|\widetilde{x} \in f(\widetilde{u}_\varepsilon)\} = \{x"_{|[t_1+\varepsilon,\infty)}|x" \in f(u^1 \circ \tau^{d+\varepsilon})\} \tag{7.16}$$

But the fact that $t_3 + d + \varepsilon > t_1 + \varepsilon$ and

$$\{x"_{|[t_1+\varepsilon,\infty)}|x" \in f(u^1 \circ \tau^{d+\varepsilon})\} = \{(x' \circ \tau^{d+\varepsilon})_{|[t_1+\varepsilon,\infty)}|x' \in f(u^1)\} \tag{7.17}$$

indicate the truth of

$$\{\widetilde{x}_{|[t_3+d+\varepsilon,\infty)}|\widetilde{x} \in f(\widetilde{u}_\varepsilon)\} \overset{(7.16)}{=} \{x"_{|[t_3+d+\varepsilon,\infty)}|x" \in f(u^1 \circ \tau^{d+\varepsilon})\} =$$

$$\overset{(7.17)}{=} \{(x' \circ \tau^{d+\varepsilon})_{|[t_3+d+\varepsilon,\infty)}|x' \in f(u^1)\} = \{x'_{|[t_3,\infty)}|x' \in f(u^1)\} \overset{(7.6)}{=} \mu"$$

(7.10) is proved. $\square$

Definition 17. *We use the notations from the previous theorem and we suppose that the requests stated there are fulfilled. We have the next partial law of composition of the fundamental transfers:*

$$(\mu \overset{u^0_{|(-\infty,t_1)}}{\to} \mu') \vee (\mu' \overset{u^1_{|[t_2,t_3)}}{\to} \mu") = \mu \overset{\widetilde{u}_{\varepsilon|(-\infty,t_3+d+\varepsilon)}}{\to} \mu"$$

$$(\mu \overset{u^0_{|[t_0,t_1)}}{\to} \mu') \vee (\mu' \overset{u^1_{|[t_2,t_3)}}{\to} \mu") = \mu \overset{\widetilde{u}_{\varepsilon|[t_0,t_3+d+\varepsilon)}}{\to} \mu"$$

8. The composition of the fundamental transfers, special case

Theorem 5. *We know about the system f that it is non-anticipatory. The next statements are true:*

a) For any $t_1 < t_2$, $u \in U$ and $\mu, \mu', \mu" \in \mathbf{B}^n$ so that the transfers $\mu \overset{u_{|(-\infty,t_1)}}{\to} \mu'$, $\mu' \overset{u_{|[t_1,t_2)}}{\to} \mu"$ are fundamental, the transfer $\mu \overset{u_{|(-\infty,t_2)}}{\to} \mu"$ is fundamental.

b) We suppose that $t_1 < t_2 < t_3$, $u \in U$ and $\mu, \mu', \mu" \in \mathbf{B}^n$ are arbitrary and satisfy the property that the transfers $\mu \overset{u_{|[t_1,t_2)}}{\to} \mu'$, $\mu' \overset{u_{|[t_2,t_3)}}{\to} \mu"$ are fundamental. In this situation the transfer $\mu \overset{u_{|[t_1,t_3)}}{\to} \mu"$ is fundamental.

Proof. a) The hypothesis states the existence of $t_0 < t_1, v \in U$ and $v' \in U$ so that

$$u_{|(-\infty,t_1)} = v_{|(-\infty,t_1)}, \forall y \in f(v), y_{|(-\infty,t_0)} = \mu \; and \; y_{|[t_1,\infty)} = \mu'$$

$$u_{|(-\infty,t_2)} = v'_{|(-\infty,t_2)}, \forall y' \in f(v'), y'_{|[t_2,\infty)} = \mu"$$

Because $v_{|(-\infty,t_0)} = v'_{|(-\infty,t_0)}$, from the non-anticipation of f we have

$$\{y_{|(-\infty,t_0)} | y \in f(v)\} = \{y'_{|(-\infty,t_0)} | y' \in f(v')\} = \mu$$

thus

$$u_{|(-\infty,t_2)} = v'_{|(-\infty,t_2)}, \forall y' \in f(v'), y'_{|(-\infty,t_0)} = \mu \; and \; y'_{|[t_2,\infty)} = \mu"$$

i.e. the transfer $\mu \overset{u_{|(-\infty,t_2)}}{\to} \mu"$ is fundamental.

b) is made similarly with a). □

Definition 18. *In the conditions and with the notations from the previous theorem, we have the next partial law of composition of the fundamental transfers:*

$$(\mu \overset{u_{|(-\infty,t_1)}}{\to} \mu') \vee (\mu' \overset{u_{|[t_1,t_2)}}{\to} \mu") = \mu \overset{u_{|(-\infty,t_2)}}{\to} \mu"$$

$$(\mu \overset{u_{|[t_1,t_2)}}{\to} \mu') \vee (\mu' \overset{u_{|[t_2,t_3)}}{\to} \mu") = \mu \overset{u_{|[t_1,t_3)}}{\to} \mu"$$

Remark 5. *Theorem 5 restates the results from Theorem 4 under a simplified form, for example at Theorem 4 a) we have $u^0 = u^1$ and for this reason the requests of closure of U under the concatenation of the inputs and of non-anticipation* disappear, respectively $t_1 = t_2$ and for this reason the requests of closure of U under translations and of time invariance disappear too.*

9. The fundamental mode

THEOREM 6. *We consider the system f that is supposed to be non-anticipatory and let $u \in U$ a fixed input. The next statements are equivalent*

a) $(t_k) \in Seq$, $(u^k) \in U$ *and* $(\mu^k) \in \mathbf{B}^n$ *exist so that*

$$\forall x \in f(u^0), x_{|(-\infty,t_0)} = \mu^0 \text{ and } x_{|[t_1,\infty)} = \mu^1$$

$$u_{|(-\infty,t_1)} = u^0_{|(-\infty,t_1)},\ u_{|(-\infty,t_2)} = u^1_{|(-\infty,t_2)},\ u_{|(-\infty,t_3)} = u^2_{|(-\infty,t_3)}, ...$$

$$\forall x \in f(u^1), x_{|[t_2,\infty)} = \mu^2,\ \forall x \in f(u^2), x_{|[t_3,\infty)} = \mu^3,\ \forall x \in f(u^3), x_{|[t_4,\infty)} = \mu^4, ...$$

b) $(t_k) \in Seq$ *and* $(\mu^k) \in \mathbf{B}^n$ *exist so that the transfers* $\mu^0 \overset{u_{|(-\infty,t_1)}}{\longrightarrow} \mu^1$, $\mu^1 \overset{u_{|[t_1,t_2)}}{\longrightarrow} \mu^2$, $\mu^2 \overset{u_{|[t_2,t_3)}}{\longrightarrow} \mu^3$,... *are fundamental.*

c) $(t_k) \in Seq$ *and* $(\mu^k) \in \mathbf{B}^n$ *exist so that the transfers* $\mu^0 \overset{u_{|(-\infty,t_1)}}{\longrightarrow} \mu^1$, $\mu^0 \overset{u_{|(-\infty,t_2)}}{\longrightarrow} \mu^2$, $\mu^0 \overset{u_{|(-\infty,t_3)}}{\longrightarrow} \mu^3, ...$ *are initial fundamental.*

PROOF. *a)* $\Longrightarrow$ *b)* Let (t_k), (u^k) and (μ^k) like at *a*). Because

$$u_{|(-\infty,t_1)} = u^0_{|(-\infty,t_1)},\ \forall x \in f(u^0), x_{|(-\infty,t_0)} = \mu^0 \text{ and } x_{|[t_1,\infty)} = \mu^1 \tag{9.1}$$

is true, $\mu^0 \overset{u_{|(-\infty,t_1)}}{\longrightarrow} \mu^1$ is an initial fundamental transfer. The fact that

$$u_{|(-\infty,t_1)} = u^0_{|(-\infty,t_1)},\ \forall x \in f(u^0), x_{|[t_1,\infty)} = \mu^1 \tag{9.2}$$

$$u_{|(-\infty,t_2)} = u^1_{|(-\infty,t_2)},\ \forall x \in f(u^1), x_{|[t_2,\infty)} = \mu^2 \tag{9.3}$$

implies that $\mu^1 \overset{u_{|[t_1,t_2)}}{\longrightarrow} \mu^2$ is non-initial fundamental etc.

b) $\Longrightarrow$ *c)* (t_k) and (μ^k) exist so that $\mu^0 \overset{u_{|(-\infty,t_1)}}{\longrightarrow} \mu^1$, $\mu^1 \overset{u_{|[t_1,t_2)}}{\longrightarrow} \mu^2$, $\mu^2 \overset{u_{|[t_2,t_3)}}{\longrightarrow} \mu^3, ...$ are fundamental. Like in Theorem 5 and Definition 18

$$\mu^0 \overset{u_{|(-\infty,t_2)}}{\to} \mu^2 = (\mu^0 \overset{u_{|(-\infty,t_1)}}{\to} \mu^1) \vee (\mu^1 \overset{u_{|[t_1,t_2)}}{\to} \mu^2)$$

$$\mu^0 \overset{u_{|(-\infty,t_3)}}{\to} \mu^3 = (\mu \overset{u_{|(-\infty_1,t_2)}}{\to} \mu^2) \vee (\mu^2 \overset{u_{|[t_2,t_3)}}{\to} \mu^3)$$

$$...$$

are initial fundamental.

c) $\Longrightarrow$ *a)* We consider the sequences (t_k) and (μ^k) like at *c*). The fact that $\mu^0 \overset{u_{|(-\infty,t_1)}}{\longrightarrow} \mu^1$ is initial fundamental shows the existence of $u^0 \in U$ so that (9.1) holds and because $\mu^0 \overset{u_{|(-\infty,t_2)}}{\longrightarrow} \mu^2$ is initial fundamental we obtain the existence of $u^1 \in U$ with (9.3) true etc. The statement from a) is true. □

DEFINITION 19. *We say that f is, under the input u, in the fundamental (operating) mode if one of the previous properties a), b), c) from Theorem 6 is satisfied.*

THEOREM 7. *If f is non-anticipatory and $t_0 < t_1, u \in U, \mu, \mu' \in \mathbf{B}^n$ are fixed, then the fact that*

$$\forall x \in f(u), x_{|(-\infty,t_0)} = \mu \text{ and } x_{|[t_1,\infty)} = \mu'$$

implies that f is, under u, in the fundamental mode.

PROOF. The sequences $(t'_k) \in Seq$ and $(\mu^k) \in \mathbf{B}^n$ exist satisfying

$$t'_0 = t_0, t'_1 = t_1, t'_k, k \geq 2 \text{ arbitrary}$$

$$\mu^0 = \mu, \mu^1 = \mu^2 = ... = \mu'$$

We observe that $\mu \overset{u_{|(-\infty,t'_1)}}{\longrightarrow} \mu', \mu \overset{u_{|(-\infty,t'_2)}}{\longrightarrow} \mu', \mu \overset{u_{|(-\infty,t'_3)}}{\longrightarrow} \mu', ...$ are initial fundamental transfers. □

REMARK 6. *The fundamental mode may be interpreted as a discrete time symbolic evolution of a deterministic system (i.e. f is uni-valued) of the form*

$$\mu^0 = x(0) \overset{u^0}{\to} \mu^1 = x(1) \overset{u^1}{\to} ... \overset{u^k}{\to} \mu^{k+1} = x(k+1) \overset{u^{k+1}}{\to} ...$$

where the initial fundamental transfer $\mu^0 \overset{u^0_{|(-\infty,t_1)}}{\longrightarrow} \mu^1$ is identified with the symbolic transfer $x(0) \overset{u^0}{\to} x(1)$ and a non-initial fundamental transfer of rank $k \geq 1$, $\mu^k \overset{u^k_{|[t_k,t_{k+1})}}{\longrightarrow} \mu^{k+1}$ is identified with the symbolic transfer $x(k) \overset{u^k}{\to} x(k+1)$.

If we are in the hypothesis of the previous Theorem, then the symbolic evolution may be considered to be given by a finite sequence

$$\mu^0 = x(0) \overset{u^0}{\to} \mu^1 = x(1) \overset{u^1}{\to} ... \overset{u^k}{\to} \mu^{k+1} = x(k+1)$$

where k can be 0.

EXAMPLE 2. *In Example 1 both systems f, g are in the fundamental mode under the inputs u and v.*

EXAMPLE 3. *The deterministic system $f : S \to S$,*

$$\forall u \in S, \ f(u) = \begin{cases} 1, u = \chi_{[0,1)\cup[2,3)\cup[4,5)\cup...} \\ 0, otherwise \end{cases}$$

satisfies the next properties: $u = \chi_{[0,1)\cup[2,3)\cup[4,5)\cup...}$, the unbounded sequence $0 < 2 < 4 < ...$ of real numbers, the family

$$u^0 = \chi_{[0,1)}, \ u^1 = \chi_{[0,1)\cup[2,3)}, \ u^2 = \chi_{[0,1)\cup[2,3)\cup[4,5)}, ...$$

of inputs and the binary null sequence $0_k \in \mathbf{B}, k \in \mathbf{N}$ exist so that

$$f(u^0)_{|(-\infty,0)} = 0 \text{ and } f(u^0)_{|[2,\infty)} = 0$$

$$u_{|(-\infty,2)} = u^0_{|(-\infty,2)}, \ u_{|(-\infty,4)} = u^1_{|(-\infty,4)}, ...$$

$$f(u^1)_{|[4,\infty)} = 0, \ f(u^2)_{|[6,\infty)} = 0, ...$$

The statements

$$f(u)_{|(-\infty,2]} = f(u^0)_{|(-\infty,2]}, \ f(u)_{|(-\infty,4]} = f(u^1)_{|(-\infty,4]}, ...$$

are false, since f is anticipatory. f is not in the fundamental mode under u.

THEOREM 8. *Let the non-anticipatory system f be in the fundamental mode under u. Then the families $(t_k) \in Seq$ and $(u^k) \in U$ exist so that*

$$\forall k \in \mathbf{N}, u_{|(-\infty,t_{k+1})} = u^k_{|(-\infty,t_{k+1})}$$

and for all $k \in \mathbf{N}, f$ is in the fundamental mode under u^k.

PROOF. From Theorem 6 item c), $(t_k) \in Seq$ and $(\mu^k) \in \mathbf{B}^n$ exist so that the transfers $\mu^0 \overset{u_{|(-\infty,t_1)}}{\longrightarrow} \mu^1$, $\mu^0 \overset{u_{|(-\infty,t_2)}}{\longrightarrow} \mu^2$, $\mu^0 \overset{u_{|(-\infty,t_3)}}{\longrightarrow} \mu^3, ...$ are initial fundamental, i.e. the sequence $(u^k) \in U$ exists with

$$u_{|(-\infty,t_1)} = u^0_{|(-\infty,t_1)},\ \forall x \in f(u^0), x_{|(-\infty,t_0)} = \mu^0 \ and\ x_{|[t_1,\infty)} = \mu^1$$

$$u_{|(-\infty,t_2)} = u^1_{|(-\infty,t_2)},\ \forall x \in f(u^1), x_{|(-\infty,t_0)} = \mu^0 \ and\ x_{|[t_2,\infty)} = \mu^2$$

$$u_{|(-\infty,t_3)} = u^2_{|(-\infty,t_3)},\ \forall x \in f(u^2), x_{|(-\infty,t_0)} = \mu^0 \ and\ x_{|[t_3,\infty)} = \mu^3$$

...

thus $\mu^0 \overset{u^0_{|(-\infty,t_1)}}{\longrightarrow} \mu^1$, $\mu^0 \overset{u^1_{|(-\infty,t_2)}}{\longrightarrow} \mu^2$, $\mu^0 \overset{u^2_{|(-\infty,t_3)}}{\longrightarrow} \mu^3, ...$ are initial fundamental and from Theorem 7 we obtain that f is in the fundamental mode under all u^k, $k \in \mathbf{N}$. □

THEOREM 9. *Let f non-anticipatory and we suppose that U has the next closure property: for any $u \in S^{(m)}$ and any sequences $(t_k) \in Seq$, $(u^k) \in U$, from*

$$\forall k \in \mathbf{N}, u_{|(-\infty,t_{k+1})} = u^k_{|(-\infty,t_{k+1})} \tag{9.4}$$

we infer $u \in U$. Then the next statement is true: for any $(u^k) \in U$ so that f is in the fundamental mode under all u^k, a sequence $(t_k) \in Seq$ exists so that

$$\forall k \in \mathbf{N}, u^k_{|(-\infty,t_{k+1})} = u^{k+1}_{|(-\infty,t_{k+1})} \tag{9.5}$$

implies that f is in the fundamental mode under the unique u satisfying (9.4).

PROOF. Let $(u^k) \in U$ a sequence of inputs having the property that f is in the fundamental mode under all u^k and let us take an arbitrary $\delta > 0$. We have the existence of $t_0, t_1 \in \mathbf{R}$ and $\mu^0, \mu^1 \in \mathbf{B}^n$ so that $t_0 + \delta < t_1$ and

$$\forall x \in f(u^0), x_{|(-\infty,t_0)} = \mu^0$$

$$\forall x \in f(u^0), x_{|[t_1,\infty)} = \mu^1$$

of $t_2 \in \mathbf{R}$ and $\mu^2 \in \mathbf{B}^n$ so that $t_1 + \delta < t_2$ and

$$\forall x \in f(u^1), x_{|[t_2,\infty)} = \mu^2$$

of $t_3 \in \mathbf{R}$ and $\mu^3 \in \mathbf{B}^n$ so that $t_2 + \delta < t_3$ and

$$\forall x \in f(u^2), x_{|[t_3,\infty)} = \mu^3$$

...

Obviously $\mu^0 \overset{u^0_{|(-\infty,t_1)}}{\longrightarrow} \mu^1$, $\mu^1 \overset{u^1_{|[t_1,t_2)}}{\longrightarrow} \mu^2$, $\mu^2 \overset{u^2_{|[t_2,t_3)}}{\longrightarrow} \mu^3, ...$ are fundamental transfers (see Theorem 2) and $(t_k) \in Seq$. The relation (9.4) may be written for some $u \in S^{(m)}$ due to (9.5); it defines a unique function $u \in S^{(m)}$ and moreover we have that $u \in U$. From Theorem 3, the transfers $\mu^0 \overset{u_{|(-\infty,t_1)}}{\longrightarrow} \mu^1$, $\mu^1 \overset{u_{|[t_1,t_2)}}{\longrightarrow} \mu^2$, $\mu^2 \overset{u_{|[t_2,t_3)}}{\longrightarrow} \mu^3, ...$ are fundamental, thus f is in the fundamental mode under u. □

10. A property of existence

THEOREM 10. *Let the non-anticipatory system* f*. We suppose that the next properties are fulfilled*

a) for any $(t_k) \in Seq$ *and any sequence* $(u^k) \in U$ *of inputs we have* $u^0 \cdot \chi_{(-\infty,t_0)} \oplus u^1 \cdot \chi_{[t_0,t_1)} \oplus u^2 \cdot \chi_{[t_1,t_2)} \oplus ... \in U$

b) f *satisfies the next property of race-free initialization with bounded initial time:*

$$\forall u \in U, \exists \mu \in \mathbf{B}^n, \exists t_0 \in \mathbf{R}, \forall x \in f(u), x_{|(-\infty,t_0)} = \mu$$

c) f *is absolutely race-free stable with bounded final time, i.e.*

$$\forall u \in U, \exists \mu' \in \mathbf{B}^n, \exists t_1 \in \mathbf{R}, \forall x \in f(u), x_{|[t_1,\infty)} = \mu'$$

Then for any sequence $(u^k) \in U$ *of inputs, the time instants* $(t_k) \in Seq$ *exist so that* f *is in the fundamental mode under the input*

$$\widetilde{u} = u^0 \cdot \chi_{(-\infty,t_1)} \oplus u^1 \cdot \chi_{[t_1,t_2)} \oplus ... \oplus u^k \cdot \chi_{[t_k,t_{k+1})} \oplus ...$$

PROOF. We consider some real number $\delta > 0$ and the arbitrary sequence $(u^k) \in U$ of inputs. From b) we infer the existence of $\mu^0 \in \mathbf{B}^n$ and $t_0 \in \mathbf{R}$ so that

$$\forall x \in f(u^0), x_{|(-\infty,t_0)} = \mu^0$$

and from c) we have the existence of $\mu^1 \in \mathbf{B}^n$ and $t_1 \in \mathbf{R}$ with $t_1 > t_0 + \delta$ and

$$\forall x \in f(u^0), x_{|[t_1,\infty)} = \mu^1$$

Furthermore, from a) we have that $u^0 \cdot \chi_{(-\infty,t_1)} \oplus u^1 \cdot \chi_{[t_1,\infty)} \in U$ and from c) the existence of $\mu^2 \in \mathbf{B}^n$ and $t_2 \in \mathbf{R}$ so that $t_2 > t_1 + \delta$ and

$$\forall x \in f(u^0 \cdot \chi_{(-\infty,t_1)} \oplus u^1 \cdot \chi_{[t_1,\infty)}), x_{|[t_2,\infty)} = \mu^2$$

is inferred. The construction of (t_k) and the fact that $(t_k) \in Seq$ are obvious. On the other hand $\widetilde{u}$ obtained this way belongs to U taking in consideration a). The statement that f is in the fundamental mode under the input $\widetilde{u}$ is inferred from the equalities

$$\widetilde{u}_{|(-\infty,t_1)} = u^0_{|(-\infty,t_1)}, \widetilde{u}_{|(-\infty,t_2)} = (u^0 \cdot \chi_{(-\infty,t_1)} \oplus u^1 \cdot \chi_{[t_1,\infty)})_{|(-\infty,t_2)}, ...$$

□

THEOREM 11. *If the non-anticipatory system* f *satisfies the next properties:*

a) race-free initialization with bounded initial time:

$$\forall u \in U, \exists \mu \in \mathbf{B}^n, \exists t_0 \in \mathbf{R}, \forall x \in f(u), x_{|(-\infty,t_0)} = \mu$$

b) absolute race-free stability with bounded final time

$$\forall u \in U, \exists \mu' \in \mathbf{B}^n, \exists t_1 \in \mathbf{R}, \forall x \in f(u), x_{|[t_1,\infty)} = \mu'$$

then

$$\forall u \in U, \exists \mu \in \mathbf{B}^n, \exists \mu' \in \mathbf{B}^n, \exists t_0 \in \mathbf{R}, \exists t_1 > t_0,$$
$$\forall x \in f(u), x_{|(-\infty,t_0)} = \mu \text{ and } x_{|[t_1,\infty)} = \mu'$$

i.e. for any u*, some* μ, μ' *and* $t_0 < t_1$ *exist so that* $\mu \overset{u_{|(-\infty,t_1)}}{\rightarrow} \mu'$ *is initial fundamental.*

PROOF. From the first part of the proof of Theorem 10, where $u^0 = u$. □

THEOREM 12. *We suppose that the non-anticipatory system f is absolutely race-free stable with bounded final time, i.e.*

$$\forall u \in U, \exists \mu \in \mathbf{B}^n, \exists t \in \mathbf{R}, \forall x \in f(u), x_{|[t,\infty)} = \mu$$

Then $\forall u \in U$, the vectors $\mu, \mu' \in \mathbf{B}^n$ and the numbers $t_0 < t_1$ exist so that the transfer $\mu \overset{u_{|[t_0,t_1)}}{\longrightarrow} \mu'$ is non-initial fundamental.

PROOF. It is sufficient to consider the next property: for any $u \in U$, μ and t_0 exist so that $\forall x \in f(u), x_{|[t_0,\infty)} = \mu$; then $\mu' = \mu$ and $t_1 > t_0$ arbitrary make the conclusion of the theorem be fulfilled. □

11. Fundamental mode, special case

DEFINITION 20. *For any $t_1 \in \mathbf{R}$, the prefix of $u \in S^{(m)}$ is the function $u_{t_1} \in S^{(m)}$ given by*

$$u_{t_1}(t) = \begin{cases} u(t), t < t_1 \\ u(t_1 - 0), t \geq t_1 \end{cases}$$

THEOREM 13. *Let the non-anticipatory system f and the input $u \in U$. For any $(t_k) \in Seq$ and $(\mu^k) \in \mathbf{B}^n$ so that $u_{t_1}, u_{t_2}, u_{t_3}, ... \in U$ and*

$$\forall x \in f(u_{t_1}), x_{|(-\infty,t_0)} = \mu^0 \text{ and } x_{|[t_1,\infty)} = \mu^1$$

$$\forall x \in f(u_{t_2}), x_{|[t_2,\infty)} = \mu^2,\ \forall x \in f(u_{t_3}), x_{|[t_3,\infty)} = \mu^3,\ \forall x \in f(u_{t_4}), x_{|[t_4,\infty)} = \mu^4, ...$$

f is, under the input u, in the fundamental mode.

PROOF. We define the sequence $(u^k) \in U$ by $u^k = u_{t_{k+1}}, k \in \mathbf{N}$. Because for any $k \geq 0$ we have $u_{|(-\infty,t_{k+1})} = u^k_{|(-\infty,t_{k+1})}$, the statement from Theorem 6 a) is true. □

COROLLARY 1. *We suppose that the non-anticipatory system f and the input $u \in U$ are given. If the sequences $(t_k) \in Seq$, $(\mu^k) \in \mathbf{B}^n$ and $(\lambda^k) \in \mathbf{B}^m$ satisfy*

$$u(t) = \lambda^0 \cdot \chi_{(-\infty,t_1)}(t) \oplus \lambda^1 \cdot \chi_{[t_1,t_2)}(t) \oplus \lambda^2 \cdot \chi_{[t_2,t_3)}(t) \oplus ...$$

$$\lambda^0, \lambda^0 \cdot \chi_{(-\infty,t_1)} \oplus \lambda^1 \cdot \chi_{[t_1,\infty)}, \lambda^0 \cdot \chi_{(-\infty,t_1)} \oplus \lambda^1 \cdot \chi_{[t_1,t_2)} \oplus \lambda^2 \cdot \chi_{[t_2,\infty)}, ... \in U \text{ and}$$

$$\forall x \in f(\lambda^0), x_{|(-\infty,t_0)} = \mu^0 \text{ and } x_{|[t_1,\infty)} = \mu^1$$

$$\forall x \in f(\lambda^0 \cdot \chi_{(-\infty,t_1)} \oplus \lambda^1 \cdot \chi_{[t_1,\infty)}), x_{|[t_2,\infty)} = \mu^2$$

$$\forall x \in f(\lambda^0 \cdot \chi_{(-\infty,t_1)} \oplus \lambda^1 \cdot \chi_{[t_1,t_2)} \oplus \lambda^2 \cdot \chi_{[t_2,\infty)}), x_{|[t_3,\infty)} = \mu^3$$

$$...$$

then f is, under the input u, in the fundamental mode.

PROOF. This is a special case of the previous theorem when $u_{t_1} = \lambda^0$, $u_{t_2} = \lambda^0 \cdot \chi_{(-\infty,t_1)} \oplus \lambda^1 \cdot \chi_{[t_1,\infty)}$, $u_{t_3} = \lambda^0 \cdot \chi_{(-\infty,t_1)} \oplus \lambda^1 \cdot \chi_{[t_1,t_2)} \oplus \lambda^2 \cdot \chi_{[t_2,\infty)}, ...$ □

REMARK 7. *Theorem 13 gives a new perspective on the fundamental mode, when $\forall k \geq 1$ the stabilization of x to the value $x(t_k)$ is a direct consequence of the fact that u has stabilized before t_k to the value $u(t_k - 0)$. Thus at the time instants $t_1, t_2, t_3, ...$ u and all $x \in f(u)$ are in equilibrium*

$$\forall k \geq 1, \forall t \geq t_k, u_{t_k}(t) = u(t_k - 0) \text{ and } \forall x \in f(u_{t_k}), x(t) = x(t_k)$$

and we consider the equilibrium be true at the time instant t_0 also under the form

$$\forall t < t_0, u(t) = u(t_0 - 0) \text{ and } \forall x \in f(u_{t_1}), x(t) = x(t_0 - 0)$$

by a suitable choice of t_0.

The situation that is described in Theorem 13 includes the possibilities $\exists k \geq 1, u_{t_k} = u_{t_{k+1}}$ *and respectively* $\exists k \geq 1, u = u_{t_k}$.

Corollary 1 represents that special case of Theorem 13, when u is constant in the intervals $(-\infty, t_1)$, $[t_1, t_2)$, $[t_2, t_3), ...$

The next theorem is an adaptation of Theorem 10 for the present context.

THEOREM 14. *The non-anticipatory system* f *is given and let* $H \subset \mathbf{B}^m$ *a non-empty set. If*

a) $U = \{\lambda^0 \cdot \chi_{(-\infty,t_1)} \oplus \lambda^1 \cdot \chi_{[t_1,t_2)} \oplus \lambda^2 \cdot \chi_{[t_2,t_3)} \oplus ...|(\lambda^k) \in H, (t_k) \in Seq\}$

b) f *has race-free initial states with bounded initial time*

$$\forall u \in U, \exists \mu \in \mathbf{B}^n, \exists t_0 \in \mathbf{R}, \forall x \in f(u), x_{|(-\infty,t_0)} = \mu$$

c) f *is relatively race-free stable with bounded final time*

$$\forall u \in U \cap S_c^{(m)}, \exists \mu' \in \mathbf{B}^n, \exists t_1 \in \mathbf{R}, \forall x \in f(u), x_{|[t_1,\infty)} = \mu'$$

then for any $(\lambda^k) \in H$, *the time instants* $(t_k) \in Seq$ *exist so that* f *is in the fundamental mode under the input*

$$u = \lambda^0 \cdot \chi_{(-\infty,t_1)} \oplus \lambda^1 \cdot \chi_{[t_1,t_2)} \oplus \lambda^2 \cdot \chi_{[t_2,t_3)} \oplus ...$$

PROOF. We just remark that the closure property from Theorem 10 a) is fulfilled and that $\lambda^0, \lambda^0 \cdot \chi_{(-\infty,t_1)} \oplus \lambda^1 \cdot \chi_{[t_1,\infty)}, \lambda^0 \cdot \chi_{(-\infty,t_1)} \oplus \lambda^1 \cdot \chi_{[t_1,t_2)} \oplus \lambda^2 \cdot \chi_{[t_2,\infty)}, ... \in U \cap S_c^{(m)}$ for any $(\lambda^k) \in H$ and any $(t_k) \in Seq$. The proof is similar with that of Theorem 10. □

12. Accessibility vs. fundamental mode

THEOREM 15. *Let the non-anticipatory system* $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$ *and we suppose that the next requests are fulfilled:*

a) for any $(t_k) \in Seq$ *and any* $(u^k) \in U$ *we have* $u^0 \cdot \chi_{(-\infty,t_0)} \oplus u^1 \cdot \chi_{[t_0,t_1)} \oplus u^2 \cdot \chi_{[t_1,t_2)} \oplus ... \in U$

b) f *has race-free initial states and bounded initial time, i.e.*

$$\forall u \in U, \exists \mu \in \mathbf{B}^n, \exists t \in \mathbf{R}, \forall x \in f(u), x_{|(-\infty,t)} = \mu$$

c) any vector from $\mathbf{B}^n$ *is final state under an input having arbitrary initial segment*

$$\forall \mu \in \mathbf{B}^n, \forall u \in U, \forall t \in \mathbf{R}, \exists v \in U, \exists t' > t,$$
$$u_{|(-\infty,t)} = v_{|(-\infty,t)} \text{ and } \forall y \in f(v), y_{|[t',\infty)} = \mu$$

Then some $\mu^0 \in \mathbf{B}^n$ *exists so that for any sequence* $\mu^k \in \mathbf{B}^n, k \geq 1$ *of binary vectors, a sequence* $(t_k) \in Seq$ *and an input* $\widetilde{u} \in U$ *exist having the property that* $\mu^0 \overset{\widetilde{u}_{|(-\infty,t_1)}}{\longrightarrow} \mu^1$, $\mu^1 \overset{\widetilde{u}_{|[t_1,t_2)}}{\longrightarrow} \mu^2$, $\mu^2 \overset{\widetilde{u}_{|[t_2,t_3)}}{\longrightarrow} \mu^3, ...$ *are fundamental transfers.*

PROOF. Let an arbitrary input $v^0 \in U$. From b) we get the existence of $\mu^0 \in \mathbf{B}^n$ and $t_0 \in \mathbf{R}$ depending on v^0 so that

$$\forall x \in f(v^0), x_{|(-\infty,t_0)} = \mu^0 \tag{12.1}$$

We fix the sequence $\mu^k \in \mathbf{B}^n, k \geq 1$ and an arbitrary number $\delta > 0$. The property c) implies in this moment the existence of $u^0 \in U$ and $t_1 > t_0 + \delta$ so that

$$v^0_{|(-\infty,t_0)} = u^0_{|(-\infty,t_0)} \text{ and } \forall x \in f(u^0), x_{|[t_1,\infty)} = \mu^1$$

of $u^1 \in U$ and $t_2 > t_1 + \delta$ so that

$$u^0_{|(-\infty,t_1)} = u^1_{|(-\infty,t_1)} \text{ and } \forall x \in f(u^1), x_{|[t_2,\infty)} = \mu^2$$

of $u^2 \in U$ and $t_3 > t_2 + \delta$ so that

$$u^1_{|(-\infty,t_2)} = u^2_{|(-\infty,t_2)} \text{ and } \forall x \in f(u^2), x_{|[t_3,\infty)} = \mu^3$$

$$...$$

The transfers $\mu^0 \overset{u^0_{|(-\infty,t_1)}}{\longrightarrow} \mu^1$, $\mu^1 \overset{u^1_{|[t_1,t_2)}}{\longrightarrow} \mu^2$, $\mu^2 \overset{u^2_{|[t_2,t_3)}}{\longrightarrow} \mu^3, ...$ are obviously fundamental.

The way that (t_k) was constructed guarantees the fact that this sequence belongs to Seq, thus the input $\widetilde{u}$ defined in the next manner

$$\widetilde{u} = u^0 \cdot \chi_{(-\infty,t_1)} \oplus u^1 \cdot \chi_{[t_1,t_2)} \oplus u^2 \cdot \chi_{[t_2,t_3)} \oplus ...$$

belongs to U, from a). We have

$$\widetilde{u}_{|(-\infty,t_1)} = u^0_{|(-\infty,t_1)}, \; \widetilde{u}_{|(-\infty,t_2)} = u^1_{|(-\infty,t_2)}, \; \widetilde{u}_{|(-\infty,t_3)} = u^2_{|(-\infty,t_3)}, ...$$

from where we infer that the transfers $\mu^0 \overset{\widetilde{u}_{|(-\infty,t_1)}}{\longrightarrow} \mu^1$, $\mu^1 \overset{\widetilde{u}_{|[t_1,t_2)}}{\longrightarrow} \mu^2$, $\mu^2 \overset{\widetilde{u}_{|[t_2,t_3)}}{\longrightarrow} \mu^3, ...$ equal with $\mu^0 \overset{u^0_{|(-\infty,t_1)}}{\longrightarrow} \mu^1$, $\mu^1 \overset{u^1_{|[t_1,t_2)}}{\longrightarrow} \mu^2$, $\mu^2 \overset{u^2_{|[t_2,t_3)}}{\longrightarrow} \mu^3, ...$ (see Theorem 3) are fundamental. □

THEOREM 16. *The non-anticipatory system $f : U \to P^*(S^{(n)})$ is given and we suppose that the conditions*

a) for any $(t_k) \in Seq$ and any sequence $(u^k) \in U$ of inputs we have $u^0 \cdot \chi_{(-\infty,t_0)} \oplus u^1 \cdot \chi_{[t_0,t_1)} \oplus u^2 \cdot \chi_{[t_1,t_2)} \oplus ... \in U$

b) f has race-free initial states and bounded initial time

$$\forall u \in U, \exists \mu \in \mathbf{B}^n, \exists t \in \mathbf{R}, \forall x \in f(u), x_{|(-\infty,t)} = \mu$$

c) the vectors from $\mathbf{B}^n$ are accessible final states in the next manner

$$\forall \mu \in \mathbf{B}^n, \forall u \in U, \forall t \in \mathbf{R}, \exists \lambda \in U, \exists t' > t,$$

$$\forall y \in f(u \cdot \chi_{(-\infty,t)} \oplus \lambda \cdot \chi_{[t,\infty)}), y_{|[t',\infty)} = \mu$$

(we have identified $\lambda \in \mathbf{B}^m$ with the constant input $\lambda \in U$). Then $\mu^0 \in \mathbf{B}^n$ exists so that for any sequence $\mu^k \in \mathbf{B}^n, k \geq 1$ of binary vectors, the time instants $(t_k) \in Seq$ and the constants $(\lambda^k) \in \mathbf{B}^m$ exist with the property that $\mu^0 \overset{\widetilde{u}_{|(-\infty,t_1)}}{\longrightarrow} \mu^1$, $\mu^1 \overset{\widetilde{u}_{|[t_1,t_2)}}{\longrightarrow} \mu^2$, $\mu^2 \overset{\widetilde{u}_{|[t_2,t_3)}}{\longrightarrow} \mu^3, ...$ are fundamental transfers and we have noted

$$\widetilde{u} = \lambda^0 \cdot \chi_{(-\infty,t_1)} \oplus \lambda^1 \cdot \chi_{[t_1,t_2)} \oplus \lambda^2 \cdot \chi_{[t_2,t_3)} \oplus ...$$

PROOF. Special case of Theorem 15. □

THEOREM 17. *We suppose that the non-anticipatory system f is given so that*

a) for any $(t_k) \in Seq$ and any $(u^k) \in U$ we have $u^0 \cdot \chi_{(-\infty,t_0)} \oplus u^1 \cdot \chi_{[t_0,t_1)} \oplus u^2 \cdot \chi_{[t_1,t_2)} \oplus ... \in U$

b) f has race-free initial states and bounded initial time

$$\forall u \in U, \exists \mu \in \mathbf{B}^n, \exists t \in \mathbf{R}, \forall x \in f(u), x_{|(-\infty,t)} = \mu$$

c) f has accessible final states in bounded time under the form

$$\exists \delta > 0, \forall \mu \in \mathbf{B}^n, \forall u \in U, \forall t \in \mathbf{R}, \exists v \in U, \exists t' \in (t, t+\delta),$$

$$u_{|(-\infty,t)} = v_{|(-\infty,t)} \text{ and } \forall y \in f(v), y_{|[t',\infty)} = \mu$$

Then $\delta > 0$ and $\mu^0 \in \mathbf{B}^n$ exist so that for any sequence $\mu^k \in \mathbf{B}^n, k \geq 1$, we have the existence of $t_0 \in \mathbf{R}$ and $\widetilde{u} \in U$ with the property that $\mu^0 \overset{\widetilde{u}_{|(-\infty,t_0+\delta)}}{\longrightarrow} \mu^1$, $\mu^1 \overset{\widetilde{u}_{|[t_0+\delta,t_0+2\delta)}}{\longrightarrow} \mu^2$, $\mu^2 \overset{\widetilde{u}_{|[t_0+2\delta,t_0+3\delta)}}{\longrightarrow} \mu^3, \ldots$ are fundamental transfers.

Proof. Similar with Theorem 15. □

Theorem 18. *The non-anticipatory system f satisfies the requests*

a) f has race-free initial states and bounded initial time

$$\forall u \in U, \exists \mu \in \mathbf{B}^n, \exists t \in \mathbf{R}, \forall x \in f(u), x_{|(-\infty,t)} = \mu$$

b) the vectors from $\mathbf{B}^n$ are accessible final states

$$\forall \mu \in \mathbf{B}^n, \exists u \in U, \exists t \in \mathbf{R}, \forall x \in f(u), x_{|[t,\infty)} = \mu$$

Then

$$\forall \mu' \in \mathbf{B}^n, \exists \mu \in \mathbf{B}^n, \exists u \in U, \exists t_0 \in \mathbf{R}, \exists t_1 > t_0,$$
$$\forall x \in f(u), x_{|(-\infty,t_0)} = \mu \text{ and } x_{|[t_1,\infty)} = \mu'$$

i.e. for any μ', we have the existence of μ, u, t_0 and $t_1 > t_0$ so that $\mu \overset{u_{|(-\infty,t_1)}}{\longrightarrow} \mu'$ is initial fundamental.

Proof. Let $\mu' \in \mathbf{B}^n$ arbitrary, fixed. *b*) shows the existence of $u \in U$ and $t_1 \in \mathbf{R}$ so that

$$\forall x \in f(u), x_{|[t_1,\infty)} = \mu'$$

Because of a) we infer the existence of $\mu \in \mathbf{B}^n$ and $t_0 \in \mathbf{R}$ that can be chosen $< t_1$ with

$$\forall x \in f(u), x_{|(-\infty,t_0)} = \mu$$

□

Remark 8. *In the theorems of this section the next accessibility properties occurred:*

a) any vector from $\mathbf{B}^n$ is final state under an input having arbitrary initial segment

$$\forall \mu \in \mathbf{B}^n, \forall u \in U, \forall t \in \mathbf{R}, \exists v \in U, \exists t' > t,$$
$$u_{|(-\infty,t)} = v_{|(-\infty,t)} \text{ and } \forall y \in f(v), y_{|[t',\infty)} = \mu$$

b) version of a) where the access in a final state is made under a constant input

$$\forall \mu \in \mathbf{B}^n, \forall u \in U, \forall t \in \mathbf{R}, \exists \lambda \in U, \exists t' > t,$$
$$\forall y \in f(u \cdot \chi_{(-\infty,t)} \oplus \lambda \cdot \chi_{[t,\infty)}), y_{|[t',\infty)} = \mu$$

c) version of a) where the access in a final state is made in the next way

$$\exists \delta > 0, \forall \mu \in \mathbf{B}^n, \forall u \in U, \forall t \in \mathbf{R}, \exists v \in U, \exists t' \in (t, t+\delta),$$
$$u_{|(-\infty,t)} = v_{|(-\infty,t)} \text{ and } \forall y \in f(v), y_{|[t',\infty)} = \mu$$

d) version of a) where the inputs under which the vectors from $\mathbf{B}^n$ are final states do not have an arbitrary initial segment

$$\forall \mu \in \mathbf{B}^n, \exists u \in U, \exists t \in \mathbf{R}, \forall x \in f(u), x_{|[t,\infty)} = \mu$$

We have the implications:

$$\begin{array}{ccccc} b) & \Longrightarrow & a) & \Longrightarrow & d) \\ & & \Uparrow & & \\ & & c) & & \end{array}$$

13. The fundamental mode relative to a function

Definition 21. *Let the system* $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$ *and the Boolean function* $F : \mathbf{B}^m \to \mathbf{B}^n$. *When the next property is fulfilled:* $\forall t \in \mathbf{R}$, $\forall u \in U$, $\forall v \in U$,

$$\forall \xi < t, F(u(\xi)) = F(v(\xi)) \Longrightarrow \{x_{|(-\infty,t]} | x \in f(u)\} = \{y_{|(-\infty,t]} | y \in f(v)\}$$

we say that f *is non-anticipatory relative to the function* F.

Definition 22. *We suppose that the system* f *is non-anticipatory relative to the function* F *and that* $(t_k) \in Seq$, $u, (u^k) \in U$ *and* $\mu^0 \in \mathbf{B}^n$ *exist satisfying the properties*

$$\forall x \in f(u^0), x_{|(-\infty,t_0)} = \mu^0 \text{ and } x_{|[t_1,\infty)} = F(u(t_1 - 0))$$

$$\forall k \in \mathbf{N}, \forall \xi \in \mathbf{R}, F(u^k(\xi)) = \begin{cases} F(u(\xi)), \xi < t_{k+1} \\ F(u(t_{k+1} - 0)), \xi \geq t_{k+1} \end{cases}$$

$$\forall k \geq 1, \forall x \in f(u^k), x_{|[t_{k+1},\infty)} = F(u(t_{k+1} - 0))$$

Then we say that f *is, under the input* u, *in the fundamental (operating) mode relative to* F.

Remark 9. *Let the function* $u \in U$ *and the number* $t \in \mathbf{R}$. *We make the observation that the functions* $v \in U$ *having the property that*

$$\forall \xi \in \mathbf{R}, F(v(\xi)) = \begin{cases} F(u(\xi)), \xi < t \\ F(u(t - 0)), \xi \geq t \end{cases}$$

act here as prefixes of u, *in other words* v *is the prefix of* u *relative to* F. *Definition 22 follows the idea from Theorem 13, where* $\mu^k = F(u(t_k - 0)), k \geq 1$ *and we observe that* $u^k = u_{t_{k+1}}, k \in \mathbf{N}$ *from there are prefixes of* u *relative to* F *also.*

We state now the version of Theorem 10 that is valid in this context.

Theorem 19. *Let the function* F *and the system* f *that is non-anticipatory relative to* F. *We suppose that the next properties are fulfilled:*

a) for any $(t_k) \in Seq$ *and any sequence* $(u^k) \in U$ *of inputs we have* $u^0 \cdot \chi_{(-\infty,t_0)} \oplus u^1 \cdot \chi_{[t_0,t_1)} \oplus u^2 \cdot \chi_{[t_1,t_2)} \oplus ... \in U$

b) race-free initialization with bounded initial time

$$\forall u \in U, \exists \mu \in \mathbf{B}^n, \exists t \in \mathbf{R}, \forall x \in f(u), x_{|(-\infty,t)} = \mu$$

c) $F-$*relative race-free stability with bounded final time*

$$\forall u \in U \cap S^{(m)}_{F,c}, \exists t \in \mathbf{R}, \forall x \in f(u), x_{|[t,\infty)} = \lim_{\xi \to \infty} F(u(\xi))$$

Then for any sequence $(u^k) \in U \cap S^{(m)}_{F,c}$ *of inputs, the time instants* $(t_k) \in Seq$ *exist so that* f *is in the fundamental mode relative to* F *under the input*

$$\widetilde{u} = u^0 \cdot \chi_{(-\infty,t_1)} \oplus u^1 \cdot \chi_{[t_1,t_2)} \oplus ... \oplus u^k \cdot \chi_{[t_k,t_{k+1})} \oplus ...$$

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