On the generalized technical condition of proper operation of the Boolean asynchronous systems

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Abstract. The coordinates $\Phi_1, ..., \Phi_n$ of the functions $\Phi : \{0,1\}^n \longrightarrow \{0,1\}^n$ are iterated independently on each other, in general, generating the so called Boolean asynchronous systems. The dynamics of these systems is unpredictable, since the computation durations of $\Phi_1, ..., \Phi_n$ are not known and variable. The generalized technical condition of proper operation of Φ gives conditions under which the behavior of these systems is, in some sense, predictable. We give several equivalent definitions of this concept and we characterize it versus duality, iterations, predecessors and successors, isomorphisms and antiisomorphisms.

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1 Introduction and preliminaries

The set $\mathbf{B} = \{0,1\}$ is a field relative to the modulo 2 sum $\oplus$ and the product $\cdot$ while $\mathbf{B}^n$ is a linear space over $\mathbf{B}$, with the sum of the vectors and the product with scalars defined coordinatewise. Then $\varepsilon^i = (0, ..., \underset{i}{1}, ..., 0)$, $i = \overline{1,n}$ are the vectors of the canonical basis of $\mathbf{B}^n$. We denote with $\underset{i\in I}{\Xi} a_i$ the modulo 2 summation of a family $a_i \in \mathbf{B}^n, i \in I$ where $\underset{i\in\varnothing}{\Xi} a_i = (0, ..., 0)$.

We define for $\mu, \lambda \in \mathbf{B}^n$ the sets $[\mu, \lambda] = \{\mu \oplus \underset{i\in A}{\Xi} \varepsilon^i | A \subset \{j | j \in \{1, ..., n\}, \mu_j \neq \lambda_j\}\}$, $[\mu, \lambda) = [\mu, \lambda] \setminus \{\lambda\}$, $(\mu, \lambda) = [\mu, \lambda] \setminus \{\mu, \lambda\}$. We can prove that $[\mu, \lambda]$ is an affine space. The function $h : \mathbf{B}^n \to \mathbf{B}^n$ is said to be compatible with the affine structure of $\mathbf{B}^n$ if $\forall \mu \in \mathbf{B}^n, \forall \lambda \in \mathbf{B}^n, h([\mu, \lambda]) = [h(\mu), h(\lambda)]$. In this case if h is bijective, then h^{-1} is compatible with the affine structure of $\mathbf{B}^n$ too.

Let $\Phi : \mathbf{B}^n \to \mathbf{B}^n$ a function, for which $\Phi^\lambda, \Phi^{(k)} : \mathbf{B}^n \to \mathbf{B}^n, \lambda \in \mathbf{B}^n, k \in \mathbf{N}$ are the functions given by: $\forall \mu \in \mathbf{B}^n$, $\forall i \in \{1, ..., n\}$, $\Phi_i^\lambda(\mu) = \begin{cases} \mu_i, \ if\ \lambda_i = 0, \\ \Phi_i(\mu), \ if\ \lambda_i = 1, \end{cases}$

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$$\Phi^{(k)}=\begin{cases}1_{\mathbf{B}^n},\ if\ k=0,\\ \Phi,\ if\ k=1,\\ \underbrace{\Phi\circ...\circ\Phi}_{k},\ if\ k\geq 2.\end{cases}$$

The computation durations of $\Phi_1,...,\Phi_n$ are unknown and variable. Asynchronicity means that the Boolean system generated by Φ iterates its coordinates independently on each other. We use the fact that, when μ runs in $\mathbf{B}^n$, the set $[\mu,\Phi(\mu)]$ contains all the intermediate partial computations of $\Phi(\mu)$ and our purpose is to state (at Definition (2.1)) a property that, when fulfilled by Φ, insures a certain degree of predictability of the Boolean asynchronous system.

2 Definitions

Theorem 2.1. *For $\Phi:\mathbf{B}^n\longrightarrow\mathbf{B}^n$, the following statements (2.1),...,(2.4) are equivalent:*

$$\forall\mu\in\mathbf{B}^n,\forall\omega\in[\mu,\Phi(\mu)),\Phi(\mu)=\Phi(\omega),\tag{2.1}$$

$$\forall\mu\in\mathbf{B}^n,[\mu,\Phi(\mu))\subset\Phi^{-1}(\Phi(\mu)),\tag{2.2}$$

$$\forall\nu\in\mathbf{B}^n,\forall\mu\in\Phi^{-1}(\nu),[\mu,\nu)\subset\Phi^{-1}(\nu),\tag{2.3}$$

$$\begin{array}{c}\forall\mu\in\mathbf{B}^n,\forall k\in\{2,...,n\},\forall i_1\in\{1,...,n\},...,\forall i_k\in\{1,...,n\},\\ \Phi(\mu)=\mu\oplus\varepsilon^{i_1}\oplus...\oplus\varepsilon^{i_k}\\ \Longrightarrow\forall\lambda\in\mathbf{B}^k\smallsetminus\{(1,...,1)\},\Phi(\mu)=\Phi(\mu\oplus\lambda_1\varepsilon^{i_1}\oplus...\oplus\lambda_k\varepsilon^{i_k})\end{array}\tag{2.4}$$

and any of them is equivalent with: $\forall\nu\in\mathbf{B}^n$, one of the following properties

$$\Phi^{-1}(\nu)=\varnothing,\tag{2.5}$$

$$\Phi^{-1}(\nu)=\{\nu\},\tag{2.6}$$

$$\exists\lambda\in\mathbf{B}^n,\Phi^{-1}(\nu)=[\lambda,\nu),\tag{2.7}$$

$$\exists\lambda\in\mathbf{B}^n,\Phi^{-1}(\nu)=[\lambda,\nu],\tag{2.8}$$

$$\exists\lambda^1\in\mathbf{B}^n,\exists\lambda^2\in\mathbf{B}^n,\Phi^{-1}(\nu)=[\lambda^1,\nu)\cup[\lambda^2,\nu),\tag{2.9}$$

$$\exists\lambda^1\in\mathbf{B}^n,\exists\lambda^2\in\mathbf{B}^n,\Phi^{-1}(\nu)=[\lambda^1,\nu]\cup[\lambda^2,\nu],\tag{2.10}$$

...

$$\exists k\geq 2,\exists\lambda^1\in\mathbf{B}^n,...,\exists\lambda^k\in\mathbf{B}^n,\Phi^{-1}(\nu)=[\lambda^1,\nu)\cup...\cup[\lambda^k,\nu),\tag{2.11}$$

$$\exists k\geq 2,\exists\lambda^1\in\mathbf{B}^n,...,\exists\lambda^k\in\mathbf{B}^n,\Phi^{-1}(\nu)=[\lambda^1,\nu]\cup...\cup[\lambda^k,\nu]\tag{2.12}$$

is true.

Proof. The scheme of the proof is:

$$(2.1) \Longrightarrow (2.2) \Longrightarrow (2.3) \Longrightarrow (2.4) \Longrightarrow (2.1),$$

$$(2.3) \Longrightarrow \forall \nu \in \mathbf{B}^n, ((2.5) \text{ or } (2.6) \text{ or...or } (2.12)) \Longrightarrow (2.3).$$

(2.1)$\Longrightarrow$(2.2) We take $\mu \in \mathbf{B}^n$ arbitrary. If $\Phi(\mu) = \mu$, then (2.2) is trivially true with $[\mu, \Phi(\mu)) = \varnothing$, thus we suppose that $\Phi(\mu) \neq \mu$ and let $\omega \in [\mu, \Phi(\mu))$ arbitrary. As $\Phi(\mu) = \Phi(\omega)$, we obtain $\omega \in \Phi^{-1}(\Phi(\mu))$.

(2.2)$\Longrightarrow$(2.3) We take an arbitrary $\nu \in \mathbf{B}^n$. If $\Phi^{-1}(\nu) = \varnothing$, then (2.3) is trivially true, thus we can suppose that $\Phi^{-1}(\nu) \neq \varnothing$ and we take an arbitrary $\mu \in \Phi^{-1}(\nu)$. In the situation when $\mu = \nu$, $[\mu, \nu) = \varnothing$ and the property (2.3) holds trivially again, therefore we can suppose that $\mu \neq \nu$ and let $\omega \in [\mu, \nu) = [\mu, \Phi(\mu))$ arbitrary. We get $\omega \in \Phi^{-1}(\Phi(\mu)) = \Phi^{-1}(\nu)$.

(2.3)$\Longrightarrow$(2.4) Let $\mu \in \mathbf{B}^n$ arbitrary. We denote $\nu = \Phi(\mu)$ and we suppose that for $k \in \{2, ..., n\}$, $i_1 \in \{1, ..., n\}, ..., i_k \in \{1, ..., n\}$ we have $\Phi(\mu) = \mu \oplus \varepsilon^{i_1} \oplus ... \oplus \varepsilon^{i_k}$. We fix an arbitrary $\lambda \in \mathbf{B}^k \setminus \{(1, ..., 1)\}$. Then $\mu \oplus \lambda_1 \varepsilon^{i_1} \oplus ... \oplus \lambda_k \varepsilon^{i_k} \in [\mu, \mu \oplus \varepsilon^{i_1} \oplus ... \oplus \varepsilon^{i_k}) = [\mu, \Phi(\mu))$, thus

$$\nu = \Phi(\mu) \overset{(2.3)}{=} \Phi(\mu \oplus \lambda_1 \varepsilon^{i_1} \oplus ... \oplus \lambda_k \varepsilon^{i_k}).$$

(2.4)$\Longrightarrow$(2.1) We take an arbitrary $\mu \in \mathbf{B}^n$.

Case $\Phi(\mu) = \mu$

We have $[\mu, \Phi(\mu)) = \varnothing$ and the property

$$\forall \omega \in [\mu, \Phi(\mu)), \Phi(\mu) = \Phi(\omega) \tag{2.13}$$

is trivially true.

Case $\Phi(\mu) = \mu \oplus \varepsilon^i, i \in \{1, ..., n\}$

In this case $[\mu, \Phi(\mu)) = [\mu, \mu \oplus \varepsilon^i) = \{\mu\}$ and (2.13) is trivially true once again.

Case $\Phi(\mu) = \mu \oplus \varepsilon^{i_1} \oplus ... \oplus \varepsilon^{i_k}, k \in \{2, ..., n\}, i_1, ..., i_k \in \{1, ..., n\}$

Let $\omega \in [\mu, \Phi(\mu))$ arbitrary and fixed. We get the existence of $\lambda \in \mathbf{B}^k \setminus \{(1, ..., 1)\}$ with the property $\omega = \mu \oplus \lambda_1 \varepsilon^{i_1} \oplus ... \oplus \lambda_k \varepsilon^{i_k}$ and we can write

$$\Phi(\mu) \overset{(2.4)}{=} \Phi(\mu \oplus \lambda_1 \varepsilon^{i_1} \oplus ... \oplus \lambda_k \varepsilon^{i_k}) = \Phi(\omega).$$

We show now that (2.3) implies the fact that $\forall \nu \in \mathbf{B}^n$, the disjunction of (2.5),..., (2.12) holds and let ν arbitrary, fixed.

a) Case $\Phi(\nu) \neq \nu$

If $\Phi^{-1}(\nu) = \varnothing$ then (2.5) is true and the implication holds, thus we can suppose that $\Phi^{-1}(\nu) \neq \varnothing$.

Let $\mu^1 \in \Phi^{-1}(\nu)$ arbitrary, thus $[\mu^1, \nu) \subset \Phi^{-1}(\nu)$. We infer from the hypothesis the existence of $p \geq 1$ and $\mu^2, ..., \mu^p \in \mathbf{B}^n$ such that

$$[\mu^1, \nu) \subsetneq [\mu^2, \nu) \subsetneq ... \subsetneq [\mu^p, \nu) \subset \Phi^{-1}(\nu),$$

$$\forall \mu \in \mathbf{B}^n, \text{not } ([\mu^p, \nu) \subsetneq [\mu, \nu) \subset \Phi^{-1}(\nu)).$$

In such conditions we define $\lambda^1 = \mu^p$. If $\Phi^{-1}(\nu) = [\lambda^1, \nu)$, then the implication holds, thus we can suppose that $\Phi^{-1}(\nu) \neq [\lambda^1, \nu)$ and let $\omega^1 \in \Phi^{-1}(\nu) \setminus [\lambda^1, \nu)$ arbitrary.

We get $[\omega^1, \nu) \subset \Phi^{-1}(\nu)$ and, moreover, we infer from the hypothesis the existence of $p' \geq 1$ and $\omega^2, ..., \omega^{p'} \in \mathbf{B}^n$ such that

$$[\omega^1, \nu) \subsetneq [\omega^2, \nu) \subsetneq ... \subsetneq [\omega^{p'}, \nu) \subset \Phi^{-1}(\nu),$$

$$\forall \mu \in \mathbf{B}^n, \text{not } ([\omega^{p'}, \nu) \subsetneq [\mu, \nu) \subset \Phi^{-1}(\nu)),$$

therefore we can define $\lambda^2 = \omega^{p'}$. If $\Phi^{-1}(\nu) = [\lambda^1, \nu) \cup [\lambda^2, \nu)$ then the implication holds, thus we can suppose that $\Phi^{-1}(\nu) \neq [\lambda^1, \nu) \cup [\lambda^2, \nu)$ and let $\delta^1 \in \Phi^{-1}(\nu) \setminus ([\lambda^1, \nu) \cup [\lambda^2, \nu))$ arbitrary. We get $[\delta^1, \nu) \subset \Phi^{-1}(\nu)$...

In finitely many steps we get the existence of $\lambda^k \in \mathbf{B}^n$ such that $\Phi^{-1}(\nu) = [\lambda^1, \nu) \cup ... \cup [\lambda^k, \nu)$ and the implication holds.

b) Case $\Phi(\nu) = \nu$

If $\Phi^{-1}(\nu) = \{\nu\}$ then (2.6) is true and the implication holds, thus we can suppose that $\Phi^{-1}(\nu) \neq \{\nu\}$.

Let $\mu^1 \in \Phi^{-1}(\nu) \setminus \{\nu\}$ arbitrary, thus $[\mu^1, \nu] \subset \Phi^{-1}(\nu)$. The hypothesis shows the existence of $p \geq 1$ and $\mu^2, ..., \mu^p \in \mathbf{B}^n$ such that

$$[\mu^1, \nu] \subsetneq [\mu^2, \nu] \subsetneq ... \subsetneq [\mu^p, \nu] \subset \Phi^{-1}(\nu),$$

$$\forall \mu \in \mathbf{B}^n, \text{not } ([\mu^p, \nu] \subsetneq [\mu, \nu] \subset \Phi^{-1}(\nu))$$

and we define $\lambda^1 = \mu^p$... The proof continues similarly with Case a), until we get all of $\lambda^1 \in \mathbf{B}^n, ..., \lambda^k \in \mathbf{B}^n$ such that $\Phi^{-1}(\nu) = [\lambda^1, \nu] \cup ... \cup [\lambda^k, \nu]$. The implication is proved.

We show that $\forall \nu \in \mathbf{B}^n$, the disjunction of (2.5),...,(2.12) implies (2.3). Let for this $\nu \in \mathbf{B}^n$ arbitrary, fixed. If (2.5) is true, then the implication

$$\forall \mu \in \varnothing, [\mu, \nu) \subset \varnothing$$

is trivially true.

We suppose that (2.6) is true, when the only choice of $\mu \in \Phi^{-1}(\nu)$ is $\mu = \nu$ and (2.3) is true under the form

$$\varnothing \subset \{\nu\}.$$

The rest of the possibilities is represented by the disjunction of (2.7),...,(2.12), when we choose $\mu \in \Phi^{-1}(\nu)$ arbitrarily. In this case $\lambda \in \mathbf{B}^n$ exists such that $\mu \in [\lambda, \nu)$ and

$$[\mu, \nu) \subset [\lambda, \nu) \subset \Phi^{-1}(\nu)$$

are true, thus (2.3) holds. □

Definition 2.1. If one of the equivalent statements from Theorem 2.1 holds, we say that Φ fulfills the **generalized technical condition of proper operation** (gtcpo).

Example 2.2. The functions $1_{\mathbf{B}^2}, \Phi, \Gamma : \mathbf{B}^2 \to \mathbf{B}^2$, where $\forall \mu \in \mathbf{B}^2$,

$$\Phi(\mu_1, \mu_2) = (\mu_1, \mu_1 \oplus \mu_2),$$

$$\Gamma(\mu_1, \mu_2) = (0, 0)$$

fulfill gtcpo.

Remark 2.3. For any μ, we can prove that a unique λ exists in (2.7), (2.8) such that $\Phi^{-1}(\mu) = [\lambda, \mu), \Phi^{-1}(\mu) = [\lambda, \mu]$ take place. The same is true in (2.9),...,(2.12) also, if $\lambda^1, \lambda^2, ..., \lambda^k$ are taken distinct, modulo their order.

Remark 2.4. For any μ, gtcpo refers to the situation when μ and $\nu = \Phi(\mu)$ differ on $k \geq 2$ coordinates, $i_1, ..., i_k$; then the value $\Phi(\mu)$ is asked to be equal with the value of Φ in any intermediate value $\omega = \mu \oplus \lambda_1 \varepsilon^{i_1} \oplus ... \oplus \lambda_k \varepsilon^{i_k}$, $\lambda \neq (1, ..., 1) \in \mathbf{B}^k$ that might result by the computation of $\leq k-1$ coordinates $\mu_i \neq \Phi_i(\mu)$, $i \in \{i_1, ..., i_k\}$.

Remark 2.5. If μ and $\Phi(\mu)$ differ on 0 or 1 coordinates, then the hypothesis of (2.4) is false and gtcpo is fulfilled.

Remark 2.6. Statement (2.5) is a special case of (2.7), when $\lambda = \nu$ and $[\lambda, \nu) = \varnothing$; similarly, (2.6) is a special case of (2.8) when $\lambda = \nu$ and $[\lambda, \nu] = \{\nu\}$. Such remarks may continue, since (2.7) is a special case of (2.9) when $\lambda^1 = \lambda^2$ etc. We have written (2.5),...,(2.12) under that form in order to state gtcpo in a most intuitive manner.

Theorem 2.2. *Φ fulfills gtcpo if and only if its dual Φ^* fulfills gtcpo. By definition $\forall \mu \in \mathbf{B}^n, \Phi^*(\mu) = \overline{\Phi(\overline{\mu})}$, where the logical complement $\overline{}$ of $\mathbf{B}$ is made coordinatewise.*

Proof. Only if. $\forall \mu \in \mathbf{B}^n$, we see that $\{\overline{\omega} | \omega \in [\mu, \Phi(\mu))\}$, denoted $\overline{[\mu, \Phi(\mu))}$, satisfies

$$\begin{aligned} \overline{[\mu, \Phi(\mu))} &= \overline{[\mu, \Phi(\mu)] \setminus \{\Phi(\mu)\}} = \overline{[\mu, \Phi(\mu)]} \setminus \overline{\{\Phi(\mu)\}} \\ &= [\overline{\mu}, \overline{\Phi(\mu)}] \setminus \{\overline{\Phi(\mu)}\} = [\overline{\mu}, \overline{\Phi(\mu)}) = [\overline{\mu}, \Phi^*(\overline{\mu})). \end{aligned}$$

We take $\mu \in \mathbf{B}^n$, $\omega \in [\mu, \Phi(\mu))$ arbitrary, fixed and we have

$$\Phi^*(\overline{\omega}) = \overline{\Phi(\omega)} = \overline{\Phi(\mu)} = \Phi^*(\overline{\mu}).$$

Moreover, when μ runs in $\mathbf{B}^n$ and ω runs in $[\mu, \Phi(\mu))$, $\overline{\mu}$ runs in $\mathbf{B}^n$ and $\overline{\omega}$ runs in $[\overline{\mu}, \Phi^*(\overline{\mu}))$. Φ^* fulfills gtcpo.

If. The inverse reasoning is clear now. □

3 Iterates

Remark 3.1. If $\Phi : \mathbf{B}^n \longrightarrow \mathbf{B}^n$ fulfills gtcpo, then $\Phi \circ \Phi$ might not fulfill the same property.

Theorem 3.1. *Φ fulfills gtcpo if and only if for any $\lambda \in \mathbf{B}^n, \Phi^\lambda$ fulfills gtcpo.*

Proof. Only if. We fix $\lambda \in \mathbf{B}^n$, $\mu \in \mathbf{B}^n$ arbitrarily and we prove that

$$\forall \omega \in [\mu, \Phi(\mu)), \Phi(\mu) = \Phi(\omega) \tag{3.1}$$

implies

$$\forall \omega \in [\mu, \Phi^\lambda(\mu)), \Phi^\lambda(\mu) = \Phi^\lambda(\omega). \tag{3.2}$$

We suppose that $p \in \{1, ..., n\}, i_1 \in \{1, ..., n\}, ..., i_p \in \{1, ..., n\}$ exist such that

$$\Phi(\mu) = \mu \oplus \varepsilon^{i_1} \oplus ... \oplus \varepsilon^{i_p}$$

and we get

$$\Phi^{\lambda}(\mu) = \mu \oplus \lambda_{i_1}\varepsilon^{i_1} \oplus ... \oplus \lambda_{i_p}\varepsilon^{i_p}.$$

In order that $\Phi^{\lambda}(\mu) \neq \mu$, for non triviality, we have the existence of $j \in \{1, ..., p\}$ with $\lambda_{i_j} = 1$. An element $\omega \in [\mu, \Phi^{\lambda}(\mu))$ fulfills

$$\omega = \mu \oplus \delta_{i_1}\lambda_{i_1}\varepsilon^{i_1} \oplus ... \oplus \delta_{i_p}\lambda_{i_p}\varepsilon^{i_p}, \tag{3.3}$$

where $\delta \in \mathbf{B}^n$ and at least a $j \in \{1, ..., p\}$ exists such that $\delta_{i_j} = 0, \lambda_{i_j} = 1$. As $\omega \in [\mu, \mu \oplus \varepsilon^{i_1} \oplus ... \oplus \varepsilon^{i_p})$ we can apply (3.1) and we infer: $\forall k \in \{1, ..., n\}$,

$$\Phi_k^{\lambda}(\omega) = \begin{cases} \omega_k, \ if\ \lambda_k = 0, \\ \Phi_k(\omega), \ if\ \lambda_k = 1 \end{cases} \overset{(3.1),(3.3)}{=} \begin{cases} \mu_k, \ if\ \lambda_k = 0, \\ \Phi_k(\mu), \ if\ \lambda_k = 1 \end{cases} = \Phi_k^{\lambda}(\mu).$$

If. This implication is obvious if we take $\lambda = (1, ..., 1) \in \mathbf{B}^n$. □

4 The sets of predecessors and successors

Definition 4.1. Let $\Phi : \mathbf{B}^n \longrightarrow \mathbf{B}^n$ and we denote for any $\mu \in \mathbf{B}^n$, the following sets of **predecessors** $\mu^- = \{\nu | \nu \in \mathbf{B}^n, \exists \lambda \in \mathbf{B}^n, \Phi^{\lambda}(\nu) = \mu\}$, $O^-(\mu) = \{\nu | \nu \in \mathbf{B}^n, \exists \lambda \in \mathbf{B}^n, ..., \exists \lambda' \in \mathbf{B}^n, (\Phi^{\lambda} \circ ... \circ \Phi^{\lambda'})(\nu) = \mu\}$ and **successors** $\mu^+ = \{\Phi^{\lambda}(\mu) | \lambda \in \mathbf{B}^n\}$, $O^+(\mu) = \{(\Phi^{\lambda} \circ ... \circ \Phi^{\lambda'})(\mu) | \lambda \in \mathbf{B}^n, ..., \lambda' \in \mathbf{B}^n\}$.

Theorem 4.1. *If* Φ *fulfills gtcpo, then*

$$\forall \mu \in \mathbf{B}^n, \forall \nu \in \mu^-, [\nu, \mu] \subset \mu^-.$$

Proof. Let $\mu \in \mathbf{B}^n, \nu \in \mu^-$ arbitrary and fixed. Some $\lambda \in \mathbf{B}^n$ exists with

$$\Phi^{\lambda}(\nu) = \mu. \tag{4.1}$$

If

$$\Phi(\nu) = \nu \tag{4.2}$$

then

$$\nu = \Phi(\nu) = \Phi^{\lambda}(\nu) \overset{(4.1)}{=} \mu$$

and the inclusion to be proved

$$[\nu, \mu] = [\nu, \nu] = \{\nu\} \subset \nu^-$$

is trivial, thus we can suppose from now the falsity of (4.2). In other words, $p, i_1, ..., i_p \in \{1, ..., n\}$ exist such that

$$\Phi(\nu) = \nu \oplus \varepsilon^{i_1} \oplus ... \oplus \varepsilon^{i_p} \tag{4.3}$$

and we infer the truth of

$$\Phi^{\lambda}(\nu) \overset{(4.3)}{=} \nu \oplus \lambda_{i_1}\varepsilon^{i_1} \oplus ... \oplus \lambda_{i_p}\varepsilon^{i_p} \overset{(4.1)}{=} \mu. \tag{4.4}$$

The satisfaction of gtcpo means that

$$\forall\omega \in [\nu, \Phi(\nu)), \Phi(\nu) = \Phi(\omega) \tag{4.5}$$

and the inclusion to be proved is, from (4.4):

$$[\nu, \nu \oplus \lambda_{i_1}\varepsilon^{i_1} \oplus ... \oplus \lambda_{i_p}\varepsilon^{i_p}] \subset \mu^-. \tag{4.6}$$

We take an arbitrary $\omega \in [\nu, \nu \oplus \lambda_{i_1}\varepsilon^{i_1} \oplus ... \oplus \lambda_{i_p}\varepsilon^{i_p}]$, i.e. $\delta \in \mathbf{B}^n$ exists with

$$\omega = \nu \oplus \delta_{i_1}\lambda_{i_1}\varepsilon^{i_1} \oplus ... \oplus \delta_{i_p}\lambda_{i_p}\varepsilon^{i_p} \tag{4.7}$$

and we must prove the existence of $\rho \in \mathbf{B}^n$ such that

$$\Phi^\rho(\omega) = \mu. \tag{4.8}$$

If $\omega = \nu \oplus \lambda_{i_1}\varepsilon^{i_1} \oplus ... \oplus \lambda_{i_p}\varepsilon^{i_p} \overset{(4.4)}{=} \mu$, then equation (4.8) takes place for $\rho = (0, ..., 0)$, thus we can suppose that $\omega \neq \mu$, in other words $\exists k \in \{1, ..., p\}$ with $\delta_{i_k} = 0, \lambda_{i_k} = 1$. In these conditions $\omega \in [\nu, \Phi^\lambda(\nu)) \subset [\nu, \Phi(\nu))$ and we can apply (4.5). We have $\forall k \in \{1, ..., n\}$,

$$\Phi_k^\rho(\omega) = \begin{cases} \omega_k, \ if\ \rho_k = 0, \\ \Phi_k(\omega), \ if\ \rho_k = 1 \end{cases}$$

$$\overset{(4.5),(4.7)}{=} \begin{cases} \nu_k, \ if\ k \in \{1, ..., n\} \smallsetminus \{i_1, ..., i_p\}, \rho_k = 0, \\ \nu_k \oplus \delta_k\lambda_k, \ if\ k \in \{i_1, ..., i_p\}, \rho_k = 0, \\ \Phi_k(\nu), \ if\ \rho_k = 1 \end{cases}$$

$$\overset{(4.3)}{=} \begin{cases} \nu_k, \ if\ k \in \{1, ..., n\} \smallsetminus \{i_1, ..., i_p\}, \rho_k = 0, \\ \nu_k \oplus \delta_k\lambda_k, \ if\ k \in \{i_1, ..., i_p\}, \rho_k = 0, \\ \nu_k, \ if\ k \in \{1, ..., n\} \smallsetminus \{i_1, ..., i_p\}, \rho_k = 1, \\ \nu_k \oplus 1, \ if\ k \in \{i_1, ..., i_p\}, \rho_k = 1 \end{cases}$$

$$= \begin{cases} \nu_k, \ if\ k \in \{1, ..., n\} \smallsetminus \{i_1, ..., i_p\}, \\ \nu_k \oplus (\rho_k \cup \delta_k\lambda_k), \ if\ k \in \{i_1, ..., i_p\} \end{cases} \overset{(4.8)}{=} \mu_k.$$

From (4.4), the last equality is true if we take $\rho = \lambda$. The inclusion (4.6) is proved. □

Theorem 4.2. *We suppose that $\Phi : \mathbf{B}^n \longrightarrow \mathbf{B}^n$ fulfills gtcpo and we take an arbitrary $\mu \in \mathbf{B}^n$. Then one of*

$$\mu^- = \{\mu\}, \tag{4.9}$$

$$\exists\lambda \in \mathbf{B}^n, \mu^- = [\lambda, \mu], \tag{4.10}$$

$$\exists k \in \{2, ..., 2^n\}, \exists\lambda^1 \in \mathbf{B}^n, ..., \exists\lambda^k \in \mathbf{B}^n, \mu^- = [\lambda^1, \mu] \cup ... \cup [\lambda^k, \mu] \tag{4.11}$$

holds.

Proof. The proof is similar with the proof of (2.3) implies that $\forall\nu \in \mathbf{B}^n$, the disjunction of (2.5),...,(2.12) holds from Theorem 2.1. □

Remark 4.2. The statement referring to the form of μ^+ when gtcpo is fulfilled is trivial, since $\mu^+ = [\mu, \Phi(\mu)]$ is true irrespective of the fact that gtcpo holds or not.

Theorem 4.3. *If Φ fulfills gtcpo then $\forall \mu \in \mathbf{B}^n$ we have*

$$O^-(\mu) \supset \{\mu\} \cup \Phi^{-1}(\mu) \cup \Phi^{-1}(\Phi^{-1}(\mu)) \cup ..., \tag{4.12}$$

$$O^+(\mu) = [\mu, \Phi(\mu)] \cup [\Phi(\mu), \Phi^{(2)}(\mu)] \cup ... \tag{4.13}$$

Proof. (4.12). Let $\nu \in \{\mu\} \cup \Phi^{-1}(\mu) \cup \Phi^{-1}(\Phi^{-1}(\mu)) \cup ...$ arbitrary. If $\nu = \mu$, then $\nu \in O^-(\mu)$, thus we can take $k \geq 1$ and $\nu \in \underbrace{\Phi^{-1}(\Phi^{-1}(...(\Phi^{-1}(\mu))...))}_{k}$, for which $\Phi^{(k)}(\nu) = \mu$, hence $\nu \in O^-(\mu)$.

(4.13). Let $\nu \in O^+(\mu), \nu = (\Phi^{\lambda^p} \circ ... \circ \Phi^{\lambda^0})(\mu)$, where $\lambda^0, ..., \lambda^p \in \mathbf{B}^n$ and we prove

$$O^+(\mu) \subset [\mu, \Phi(\mu)] \cup [\Phi(\mu), \Phi^{(2)}(\mu)] \cup ... \tag{4.14}$$

If $k \in \mathbf{N}$ exists with $\nu = \Phi^{(k)}(\mu)$ the inclusion holds, thus we can suppose that $\forall k \in \mathbf{N}, \nu \neq \Phi^{(k)}(\mu)$. We define

$$H = \{i | i \in \{0, 1, ..., p\}, \exists k \in \mathbf{N}, (\Phi^{\lambda^i} \circ ... \circ \Phi^{\lambda^0})(\mu) = \Phi^{(k)}(\mu)\}.$$

Case $H = \emptyset$.

This means that $\nu \in (\mu, \Phi(\mu))$, thus (4.14) is true.

Case $H \neq \emptyset$.

We define $k_1 = \max H$, for which $\nu \in (\Phi^{(k_1)}(\mu), \Phi^{(k_1+1)}(\mu))$ and (4.14) is true again.

The inclusion

$$O^+(\mu) \supset [\mu, \Phi(\mu)] \cup [\Phi(\mu), \Phi^{(2)}(\mu)] \cup ...$$

is obvious. □

5 Source, isolated fixed point, transient point, sink

Definition 5.1. The function $\Phi : \mathbf{B}^n \longrightarrow \mathbf{B}^n$ is given. A point $\mu \in \mathbf{B}^n$ is called **source**, if $\mu^- = \{\mu\}, \mu^+ \neq \{\mu\}$; **isolated fixed point**, if $\mu^- = \{\mu\}, \mu^+ = \{\mu\}$; **transient point**, if $\mu^- \neq \{\mu\}, \mu^+ \neq \{\mu\}$; and **sink**, if $\mu^- \neq \{\mu\}, \mu^+ = \{\mu\}$.

Theorem 5.1. *We suppose that gtcpo holds and let $\mu \in \mathbf{B}^n$ be arbitrary, fixed. The following exclusive possibilities exist.*

i) If $\Phi^{-1}(\mu) = \emptyset$, then μ is either a source, or a transient point;

ii) if $\Phi^{-1}(\mu) = \{\mu\}$, then μ is an isolated fixed point;

iii) if $\exists p \in \{1, ..., 2^n\}, \exists \lambda^1 \in \mathbf{B}^n, ..., \exists \lambda^p \in \mathbf{B}^n, \Phi^{-1}(\mu) = [\lambda^1, \mu) \cup ... \cup [\lambda^p, \mu)$, then μ is a transient point;

iv) if $\exists p \in \{1, ..., 2^n\}, \exists \lambda^1 \in \mathbf{B}^n, ..., \exists \lambda^p \in \mathbf{B}^n, \Phi^{-1}(\mu) = [\lambda^1, \mu] \cup ... \cup [\lambda^p, \mu]$, then μ is a sink.

Proof. Case i) The constant function $\Phi : \mathbf{B}^2 \to \mathbf{B}^2$ equal with $(0,0)$ satisfies gtcpo. The point $\mu = (1,1)$ is a source with $\Phi^{-1}(1,1) = \varnothing$, $(1,1)^- = \{(1,1)\}$, $(1,1)^+ = \mathbf{B}^2$ and the point $\mu = (0,1)$ is transient, with $\Phi^{-1}(0,1) = \varnothing$, $(0,1)^- = \{(1,1),(0,1)\}$, $(0,1)^+ = \{(0,1),(0,0)\}$. The first assertion of the theorem results from the fact that the isolated fixed points and the sinks μ of an arbitrary function Φ satisfy $\Phi(\mu) = \mu$, thus $\mu \in \Phi^{-1}(\mu)$.

Case ii) As $\Phi(\mu) = \mu$, we have $\mu^+ = \{\mu\}$ and we must still prove that $\mu^- = \{\mu\}$. We suppose against all reason that this is false, i.e. $\omega \neq \mu, \omega \in \mu^-$ exists, in other words we get the existence of $\nu \in \mathbf{B}^n$ such that $\Phi^\nu(\omega) = \mu$. We infer $\Phi(\omega) \neq \mu$ (otherwise $\Phi(\omega) = \mu$, resulting the contradiction $\omega \in \Phi^{-1}(\mu)$). The conclusion is $\mu \in (\omega, \Phi(\omega))$, but

$$\Phi(\omega) \overset{gtcpo}{=} \Phi(\mu) = \mu$$

is a contradiction. This proves that $\mu^- = \{\mu\}$.

Case iii) As $\{\mu\} \cup \Phi^{-1}(\mu) \subset \mu^-$ is always true, we get

$$\{\mu\} \neq [\lambda^1, \mu) \cup ... \cup [\lambda^p, \mu) \cup \{\mu\} \subset \mu^-.$$

In addition, $\Phi(\mu) \neq \mu$ and $\mu^+ \neq \{\mu\}$ are clear.

Case iv) The inclusion $\{\mu\} \cup \Phi^{-1}(\mu) \subset \mu^-$ gives

$$\{\mu\} \neq [\lambda^1, \mu] \cup ... \cup [\lambda^p, \mu] \cup \{\mu\} \subset \mu^-.$$

Moreover, we infer $\Phi(\mu) = \mu$ and $\mu^+ = \{\mu\}$. □

Remark 5.2. In Theorem 5.1, the situation $\Phi^{-1}(\mu) = [\mu, \Phi(\mu))$ is impossible. Indeed, there are two possibilities:

a) Case $[\mu, \Phi(\mu)) = \varnothing$, when $\Phi(\mu) = \mu$. This implies that $\mu \in \Phi^{-1}(\mu)$, contradiction.

b) Case $[\mu, \Phi(\mu)) \neq \varnothing$. As $\mu \in [\mu, \Phi(\mu))$, we obtain $\Phi(\mu) = \mu$, but this shows that $[\mu, \Phi(\mu)) = \varnothing$, contradiction.

Remark 5.3. If Φ is bijective and satisfies gtcpo, we can prove that one of the next statements is true for any μ:

j) $\Phi^{-1}(\mu) = \{\mu\}$, when μ is an isolated fixed point;

jj) $\exists i \in \{1, ..., n\}, \Phi^{-1}(\mu) = \{\mu \oplus \varepsilon^i\}$, when μ is a transient point.

6 Isomorphisms vs gtcpo

Definition 6.1. We consider the functions $\Phi, \Psi : \mathbf{B}^n \to \mathbf{B}^n$. If $h, h' : \mathbf{B}^n \to \mathbf{B}^n$ exist such that $\forall \nu \in \mathbf{B}^n$, the diagram

$$\begin{array}{ccc} \mathbf{B}^n & \xrightarrow{\Phi^\nu} & \mathbf{B}^n \\ h \downarrow & & \downarrow h \\ \mathbf{B}^n & \xrightarrow{\Psi^{h'(\nu)}} & \mathbf{B}^n \end{array}$$

is commutative, then we denote $(h, h') : \Phi \to \Psi$ and we say that the **morphism** (h, h') is defined, from Φ to Ψ. If h, h' are both bijections, then (h, h') is called an **isomorphism** from Φ to Ψ.

Theorem 6.1. *We consider the functions* $\Phi, \Psi : \mathbf{B}^n \longrightarrow \mathbf{B}^n$ *and the isomorphism* $(h, h') : \Phi \to \Psi$. *If* Φ *fulfills gtcpo and* h *is compatible with the affine structure of* $\mathbf{B}^n$ *then* Ψ *fulfills gtcpo.*

Proof. For $\nu \in \mathbf{B}^n$ arbitrary, fixed the diagram

$$\begin{array}{ccc} \mathbf{B}^n & \xrightarrow{\Phi^\nu} & \mathbf{B}^n \\ h\downarrow & & \downarrow h \\ \mathbf{B}^n & \xrightarrow{\Psi^{h'(\nu)}} & \mathbf{B}^n \end{array}$$

is commutative. Let $\mu' \in \mathbf{B}^n$ arbitrary. If $\mu' = \Psi^{h'(\nu)}(\mu')$, then gtcpo:

$$\forall \omega' \in [\mu', \Psi^{h'(\nu)}(\mu')), \Psi^{h'(\nu)}(\mu') = \Psi^{h'(\nu)}(\omega')$$

is trivially fulfilled, so that we can suppose from now that $\mu' \neq \Psi^{h'(\nu)}(\mu')$ and we take $\omega' \in [\mu', \Psi^{h'(\nu)}(\mu'))$ arbitrary itself. We define $\mu = h^{-1}(\mu')$ and $\omega = h^{-1}(\omega')$. As h^{-1} is compatible with the affine structure of $\mathbf{B}^n$:

$$\begin{aligned} h^{-1}([\mu', \Psi^{h'(\nu)}(\mu')]) &= [h^{-1}(\mu'), h^{-1}(\Psi^{h'(\nu)}(\mu'))] = [\mu, h^{-1}(\Psi^{h'(\nu)}(h(\mu)))] \\ &= [\mu, h^{-1}(h(\Phi^\nu(\mu)))] = [\mu, \Phi^\nu(\mu)], \end{aligned}$$

in particular $\omega \in [\mu, \Phi^\nu(\mu))$. But Φ^ν fulfills gtcpo

$$\Phi^\nu(\mu) = \Phi^\nu(\omega)$$

from Theorem 3.1 and we infer

$$\Psi^{h'(\nu)}(\mu') = \Psi^{h'(\nu)}(h(\mu)) = h(\Phi^\nu(\mu)) = h(\Phi^\nu(\omega)) = \Psi^{h'(\nu)}(h(\omega)) = \Psi^{h'(\nu)}(\omega').$$

The previous property holds for any ν and any μ', with h' bijective, thus Ψ^ν fulfill all of them gtcpo and we can apply Theorem 3.1 again in order to conclude that Ψ fulfills gtcpo. □

7 Antiisomorphisms vs gtcpo

Definition 7.1. Let us consider the functions $\Phi, \Psi : \mathbf{B}^n \to \mathbf{B}^n$ for which $h, h' : \mathbf{B}^n \to \mathbf{B}^n$ exist with the property that $\forall \nu \in \mathbf{B}^n$, the diagram

$$\begin{array}{ccc} \mathbf{B}^n & \xrightarrow{\Phi^\nu} & \mathbf{B}^n \\ h\downarrow & & \downarrow h \\ \mathbf{B}^n & \xleftarrow{\Psi^{h'(\nu)}} & \mathbf{B}^n \end{array}$$

is commutative. We denote then $(h, h')^\sim : \Phi \to \Psi$ and we say that the **antimorphism** $(h, h')^\sim$ is defined, from Φ to Ψ. If the functions h, h' are both bijections and $(h^{-1}, h'^{-1})^\sim : \Psi \longrightarrow \Phi$ is antimorphism, then $(h, h')^\sim$ is called **antiisomorphism** from Φ to Ψ.

Theorem 7.1. *Let* $\Phi, \Psi : \mathbf{B}^n \longrightarrow \mathbf{B}^n$ *and the antiisomorphism* $(h, h')^{\sim} : \Phi \to \Psi$. *We suppose that* Φ *fulfills gtcpo, that in addition* h *is compatible with the affine structure of* $\mathbf{B}^n$ *and that the property*

$$\forall \nu \in \mathbf{B}^n, \Phi^{\nu}(\mathbf{B}^n) = \mathbf{B}^n$$

holds. Then Ψ *satisfies*

$$\forall \nu \in \mathbf{B}^n, \Psi^{\nu}(\mathbf{B}^n) = \mathbf{B}^n$$

and also gtcpo.

Proof. ommitted. □

8 Other properties

Theorem 8.1. *If* Φ *satisfies gtcpo then* $\forall \mu \in \mathbf{B}^n, \forall \mu' \in \mu^+$,

$$\Phi(\mu') = \mu' \Longrightarrow \mu' = \Phi(\mu). \tag{8.1}$$

Proof. We suppose against all reason that the property is false, thus $\mu \in \mathbf{B}^n$ and $\mu' \in [\mu, \Phi(\mu)]$ exist such that

$$\Phi(\mu') = \mu' \text{ and } \mu' \neq \Phi(\mu). \tag{8.2}$$

As $\mu' \in [\mu, \Phi(\mu))$, we apply gtcpo and we infer

$$\mu' \overset{(8.2)}{=} \Phi(\mu') = \Phi(\mu),$$

contradiction with (8.2). □

Theorem 8.2. *We suppose that* Φ *fulfills gtcpo. Then* $\forall \mu \in \mathbf{B}^n, \forall \omega \in \mathbf{B}^n$,

$$\Phi(\mu) \neq \Phi(\omega) \Longrightarrow [\mu, \Phi(\mu)) \cap [\omega, \Phi(\omega)) = \varnothing.$$

Proof. We suppose against all reason that this is not true, thus μ and ω exist such that

$$\Phi(\mu) \neq \Phi(\omega) \text{ and } [\mu, \Phi(\mu)) \cap [\omega, \Phi(\omega)) \neq \varnothing.$$

Let $\lambda \in [\mu, \Phi(\mu)) \cap [\omega, \Phi(\omega))$. We obtain $\Phi(\mu) = \Phi(\lambda) = \Phi(\omega)$, contradiction. □

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