Proceedings of the 14th International Conference on Mathematics
and its Applications
Politehnica University of Timişoara
November 5-7, 2015

ON SOME INVARIANCE PROPERTIES OF THE ASYNCHRONOUS FLOWS

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Abstract

The asynchronous flows are defined by Boolean functions $\Phi : \{0,1\}^n \to \{0,1\}^n$ that iterate their coordinates $\Phi_1, ..., \Phi_n$ independently on each other. We define for the set $A \subset \{0,1\}^n$ the properties of invariance, connectedness, path connectedness and we initiate a study of these concepts. [1]

1 Introduction and preliminaries

The asynchronous circuits from electronics are modeled by Boolean functions $\Phi : \{0,1\}^n \to \{0,1\}^n$ that iterate their coordinates $\Phi_1, ..., \Phi_n$ independently on each other, giving the so-called asynchronous flows. The bibliography is mainly related to the (real, usual) dynamical systems theory and we use analogies. The invariance of a set $A \subset \{0,1\}^n$ is the most important property addressed in the paper and we define some concepts (connectedness, path connectedness) compatible with it. We initiate a study in this framework and we show in the end that several other possibilities of defining invariance exist, giving several other concepts (connectedness, path connectedness) compatible with them, that require their own study.

We denote in the following with $\mathbf{B}$ the Boolean algebra with two elements $\{0,1\}$ and with $\mathbf{N}_{\_} = \{-1, 0, 1, ...\}$ the discrete time set.

Definition 1 *For* $\Phi : \mathbf{B}^n \longrightarrow \mathbf{B}^n$ *and* $\lambda \in \mathbf{B}^n$, *we define the function* $\Phi^\lambda : \mathbf{B}^n \longrightarrow \mathbf{B}^n$ *by* $\forall \mu \in \mathbf{B}^n, \forall i \in \{1, ..., n\}, \Phi_i^\lambda(\mu) = \begin{cases} \mu_i, if\ \lambda_i = 0, \\ \Phi_i(\mu), if\ \lambda_i = 1. \end{cases}$

Definition 2 *The sequence* $\alpha : \{0,1,2,...\} \longrightarrow \mathbf{B}^n$, *whose terms are denoted in general with* α^k, *is called* ***progressive*** *if* $\forall i \in \{1, ..., n\}$, *the set* $\{k | k \in \{0,1,2,...\}, \alpha_i^k = 1\}$ *is infinite. The set of the progressive sequences is denoted by* $\widehat{\Pi}_n$.

[1]Mathematical Subject Classification(2010):*94C10*

Keywords and phrases:*Boolean function, asynchronous flow, invariance, connectedness, path connectedness*

Definition 3 *We define* $\forall k \in \mathbf{N}$ *the function* $\widehat{\sigma}^k : \widehat{\Pi}_n \longrightarrow \widehat{\Pi}_n$ *by* $\forall \alpha \in \widehat{\Pi}_n, \forall k' \in \mathbf{N}, (\widehat{\sigma}^k(\alpha))^{k'} = \alpha^{k+k'}$.

Definition 4 *Let* $\mu \in \mathbf{B}^n$ *and* $\alpha \in \widehat{\Pi}_n$. *The (**asynchronous) flow*** $\widehat{\Phi}^\alpha(\mu, \cdot) : \mathbf{N}_{\_} \longrightarrow \mathbf{B}^n$ *is defined by:* $\widehat{\Phi}^\alpha(\mu, -1) = \mu$ *and* $\forall k \in \mathbf{N}_{\_}, \widehat{\Phi}^\alpha(\mu, k+1) = \Phi^{\alpha^{k+1}}(\widehat{\Phi}^\alpha(\mu, k))$.

Definition 5 *The **orbit*** $\widehat{Or}^\alpha(\mu)$ *and the* $\omega-$***limit set*** $\widehat{\omega}^\alpha(\mu)$ *are defined by* $\widehat{Or}^\alpha(\mu) = \{\widehat{\Phi}^\alpha(\mu, k) | k \in \mathbf{N}_{\_}\}$, $\widehat{\omega}^\alpha(\mu) = \{\mu' | \{k | k \in \mathbf{N}_{\_}, \widehat{\Phi}^\alpha(\mu, k) = \mu'\}$ *is infinite*$\}$.

Remark 6 *We can prove [2] that* $\widehat{Or}^\alpha(\mu) = \{\mu\} \Longleftrightarrow \Phi(\mu) = \mu$.

2 Invariant set and invariant subset

Definition 7 *Let* $\Phi : \mathbf{B}^n \longrightarrow \mathbf{B}^n$. *The non-empty set* $A \subset \mathbf{B}^n$ *is called* ***invariant*** *if* $\forall \mu \in A, \exists \alpha \in \widehat{\Pi}_n, \widehat{Or}^\alpha(\mu) \subset A$ *holds.*

Remark 8 *If the non-empty sets* $A, B \subset \mathbf{B}^n$ *are invariant, then* $A \cup B$ *is invariant.*

Definition 9 *Let* $\Phi : \mathbf{B}^n \longrightarrow \mathbf{B}^n$ *and the sets* $\emptyset \neq A \subset X \subset \mathbf{B}^n$. *A is said to be an* ***invariant subset*** *of X if* $\begin{cases} \forall \mu \in X, \exists \alpha \in \widehat{\Pi}_n, \widehat{Or}^\alpha(\mu) \subset X, \\ \forall \mu \in A, \exists \alpha \in \widehat{\Pi}_n, \widehat{Or}^\alpha(\mu) \subset A. \end{cases}$

Example 10 *For any* $\mu \in \mathbf{B}^n$ *and any* $\alpha \in \widehat{\Pi}_n$, *the orbit* $X = \widehat{Or}^\alpha(\mu)$ *is an invariant set satisfying* $\forall \mu' \in X, \exists \beta \in \widehat{\Pi}_n, \widehat{Or}^\beta(\mu') \subset X$, *where* $\mu' = \widehat{\Phi}^\alpha(\mu, k')$ *and* $\beta = \widehat{\sigma}^{k'+1}(\alpha)$. *Moreover,* $A = \widehat{Or}^\beta(\mu')$ *is an invariant subset of X.*

Example 11 $A = \widehat{\omega}^\alpha(\mu)$ *is an invariant subset of* $X = \widehat{Or}^\alpha(\mu)$. *Special case:* $\mu' \in X$ *exists with* $\Phi(\mu') = \mu'$ *and* $A = \{\mu'\}$ *(such a* μ' *is called rest position).*

3 Minimal and maximal invariant subset

Definition 12 *Let* $X \subset \mathbf{B}^n$ *and we suppose that* $A \subset X$ *is non-empty and invariant. We say that A is the* ***minimal invariant subset*** *of X if* $\forall Y, (Y \neq \emptyset$ *and* $Y \subset X$ *and* $\forall \mu \in Y, \exists \alpha \in \widehat{\Pi}_n, \widehat{Or}^\alpha(\mu) \subset Y) \Longrightarrow A \subset Y$. *The notation of the minimal invariant subset of X is* $\underline{X}$. *If* $\forall Y, (Y \neq \emptyset$ *and* $Y \subset X$ *and* $\forall \mu \in Y, \exists \alpha \in \widehat{\Pi}_n, \widehat{Or}^\alpha(\mu) \subset Y) \Longrightarrow Y \subset A$, *then A is called the* ***maximal invariant subset*** *of X and the notation is* $\overline{X}$.

Theorem 13 *Let* $X \subset \mathbf{B}^n, X \neq \emptyset$ *and we define* $X' = \{\mu | \mu \in X, \exists \alpha \in \widehat{\Pi}_n, \widehat{Or}^\alpha(\mu) \subset X\}$. *If* $X' \neq \emptyset$, *then* $X' = \overline{X}$. *In addition, if X is invariant, then* $X = \overline{X}$.

Proof. We prove that X' is invariant first and let an arbitrary $\mu \in X'$. We have from the definition of X' the existence of $\alpha \in \widehat{\Pi}_n$ with $\widehat{Or}^{\alpha}(\mu) \subset X$. We take an arbitrary $\mu' \in \widehat{Or}^{\alpha}(\mu)$, meaning the existence of $k' \in \mathbf{N}_{\_}$ with $\mu' = \widehat{\Phi}^{\alpha}(\mu, k')$. From the fact [3] that $\forall k \in \mathbf{N}_{\_}, \widehat{\Phi}^{\widehat{\sigma}^{k'+1}(\alpha)}(\mu', k) = \widehat{\Phi}^{\widehat{\sigma}^{k'+1}(\alpha)}(\widehat{\Phi}^{\alpha}(\mu, k'), k) = \widehat{\Phi}^{\alpha}(\mu, k + k' + 1)$, we infer $\widehat{Or}^{\widehat{\sigma}^{k'+1}(\alpha)}(\mu') \subset \widehat{Or}^{\alpha}(\mu) \subset X$, thus $\mu' \in X'$. As μ' was arbitrarily chosen, we have obtained that $\widehat{Or}^{\alpha}(\mu) \subset X'$.

We prove the maximality of X'. For this, we consider an arbitrary $A \subset X$, $A \neq \emptyset$ fulfilling $\forall \mu \in A, \exists \alpha \in \widehat{\Pi}_n, \widehat{Or}^{\alpha}(\mu) \subset A$. For any $\mu \in A$, we get $\mu \in X$ and some $\alpha \in \widehat{\Pi}_n$ exists with $\widehat{Or}^{\alpha}(\mu) \subset A \subset X$, wherefrom $\mu \in X'$. We infer that $A \subset X'$.

We suppose now that X is invariant. The inclusion $\overline{X} \subset X$ is obvious and the inclusion $X \subset \overline{X}$ results from the maximality of $\overline{X}$. We obtain $\overline{X} = X$. ■

4 Connected set and disconnected set

Definition 14 *Let $\Phi : \mathbf{B}^n \longrightarrow \mathbf{B}^n$ and the non-empty invariant set $X \subset \mathbf{B}^n$. If $\forall A, (\emptyset \neq A$ and $A \subset X$ and $A \neq X) \Longrightarrow \exists \mu \in A, \forall \alpha \in \widehat{\Pi}_n, \widehat{Or}^{\alpha}(\mu) \setminus A \neq \emptyset$ is true, we say that X is **connected** and if $\exists A, \emptyset \neq A$ and $A \subset X$ and $A \neq X$ and $\forall \mu \in A, \exists \alpha \in \widehat{\Pi}_n, \widehat{Or}^{\alpha}(\mu) \subset A$ holds, we say that the set X is **disconnected** or **separated** and each A like previously is called a **separation** of X.*

Remark 15 *X is connected if it has no proper invariant subset. This is the situation when each proper subset A contains a point μ with the property that no orbit $\widehat{Or}^{\alpha}(\mu)$ is included in A. X is disconnected otherwise, meaning that it has a proper invariant subset A; thus A is not connected with $X \setminus A$.*

Remark 16 *If $X_1, X_2 \subset \mathbf{B}^n$ are invariant and A is a separation of X_1, then A is a separation of $X_1 \cup X_2$. If X_1, X_2 are invariant and $X_2 \setminus X_1 \neq \emptyset$, then X_1 is a separation of $X_1 \cup X_2$.*

Theorem 17 *If $X \subset \mathbf{B}^n, X \neq \emptyset$ is connected, then $\forall \mu \in X, \exists \alpha \in \widehat{\Pi}_n, \widehat{Or}^{\alpha}(\mu) = X$.*

Proof. If some μ exists with $X = \{\mu\}$, then the statement of the Theorem is trivial, under the form: $\Phi(\mu) = \mu$ and $\forall \alpha \in \widehat{\Pi}_n, \widehat{Or}^{\alpha}(\mu) = \{\mu\}$, see Remark 6.

We put now X under the form $X = \{\mu^0, \mu^1, ..., \mu^p\}, p \geq 1$. The invariance of X shows the existence of $\alpha^{(0)}, \alpha^{(1)}, ..., \alpha^{(p-1)} \in \widehat{\Pi}_n$ with $\widehat{Or}^{\alpha^{(i)}}(\mu^i) \subset X, i = \overline{0, p-1}$.

We take $A = \{\mu^0\}$. The hypothesis of connectedness of X implies $\widehat{Or}^{\alpha^{(0)}}(\mu^0) \setminus \{\mu^0\} \neq \emptyset$. We define $k_1 \geq 1$ be the duration that $\widehat{\Phi}^{\alpha^{(0)}}(\mu^0, k)$ needs to leave $\{\mu^0\}$, and we can suppose without loosing the generality that

$$\begin{cases} \forall k \in \{-1, 0, ..., k_1 - 2\}, \widehat{\Phi}^{\alpha^{(0)}}(\mu^0, k) = \mu^0, \\ \widehat{\Phi}^{\alpha^{(0)}}(\mu^0, k_1 - 1) = \mu^1. \end{cases} \quad (1)$$

We take $A=\{\mu^0,\mu^1\}$. The hypothesis of connectedness of X implies that one of $\widehat{Or}^{\alpha^{(0)}}(\mu^0)\setminus\{\mu^0,\mu^1\}\neq\emptyset$, $\widehat{Or}^{\alpha^{(1)}}(\mu^1)\setminus\{\mu^0,\mu^1\}\neq\emptyset$ is true. We define $k_2\geq 1$ be the duration that $\widehat{\Phi}^{\alpha^{(0)}}(\mu^0,k)$ needs in the first case, and that $\widehat{\Phi}^{\alpha^{(1)}}(\mu^1,k)$ needs in the second case to leave $\{\mu^0,\mu^1\}$. We can suppose without loosing the generality that one of the following statements is true:

$$\begin{cases}\forall k\in\{-1,0,...,k_2+k_1-2\},\widehat{\Phi}^{\alpha^{(0)}}(\mu^0,k)\in\{\mu^0,\mu^1\},\\ \widehat{\Phi}^{\alpha^{(0)}}(\mu^0,k_2+k_1-1)=\mu^2,\end{cases}\tag{2}$$

$$\begin{cases}\forall k\in\{-1,0,...,k_2-2\},\widehat{\Phi}^{\alpha^{(1)}}(\mu^1,k)\in\{\mu^0,\mu^1\},\\ \widehat{\Phi}^{\alpha^{(1)}}(\mu^1,k_2-1)=\mu^2...\end{cases}\tag{3}$$

... We take $A=\{\mu^0,\mu^1,...,\mu^{p-1}\}$. The hypothesis implies that one of $\widehat{Or}^{\alpha^{(0)}}(\mu^0)\setminus\{\mu^0,\mu^1,...,\mu^{p-1}\}\neq\emptyset$, $\widehat{Or}^{\alpha^{(1)}}(\mu^1)\setminus\{\mu^0,\mu^1,...,\mu^{p-1}\}\neq\emptyset$,..., $\widehat{Or}^{\alpha^{(p-1)}}(\mu^{p-1})\setminus\{\mu^0,\mu^1,...,\mu^{p-1}\}\neq\emptyset$ takes place. We define $k_p\geq 1$ be the duration that $\widehat{\Phi}^{\alpha^{(0)}}(\mu^0,k)$ needs in the first case, that $\widehat{\Phi}^{\alpha^{(1)}}(\mu^1,k)$ needs in the second case,..., that $\widehat{\Phi}^{\alpha^{(p-1)}}(\mu^{p-1},k)$ needs in the $p-$th case to leave the set $\{\mu^0,\mu^1,...,\mu^{p-1}\}$, i.e. it needs to take the value μ^p. One of the following properties holds:

$$\begin{cases}\forall k\in\{-1,0,...,k_p+k_{p-1}+...+k_2+k_1-2\},\\ \widehat{\Phi}^{\alpha^{(0)}}(\mu^0,k)\in\{\mu^0,\mu^1,...,\mu^{p-1}\},\\ \widehat{\Phi}^{\alpha^{(0)}}(\mu^0,k_p+k_{p-1}+...+k_2+k_1-1)=\mu^p,\end{cases}\tag{4}$$

$$\begin{cases}\forall k\in\{-1,0,...,k_p+k_{p-1}+...+k_2-2\},\widehat{\Phi}^{\alpha^{(1)}}(\mu^1,k)\in\{\mu^0,\mu^1,...,\mu^{p-1}\},\\ \widehat{\Phi}^{\alpha^{(1)}}(\mu^1,k_p+k_{p-1}+...+k_2-1)=\mu^p,...\end{cases}\tag{5}$$

$$...\begin{cases}\forall k\in\{-1,0,...,k_p-2\},\widehat{\Phi}^{\alpha^{(p-1)}}(\mu^{p-1},k)\in\{\mu^0,\mu^1,...,\mu^{p-1}\},\\ \widehat{\Phi}^{\alpha^{(p-1)}}(\mu^{p-1},k_p-1)=\mu^p.\end{cases}\tag{6}$$

We define the sequence $\beta\in\widehat{\Pi}_n$ like this:

$$\forall k\in\{0,...,k_1-1\},\beta^k=\alpha^{(0)k},$$

$$\forall k\in\{k_1,...,k_2+k_1-1\},\beta^k=\begin{cases}\alpha^{(0)k},\text{ if (2) is true,}\\ \alpha^{(1)k-k_1},\text{ if (3) is true,...}\end{cases}$$

$$...\forall k\geq k_{p-1}+k_{p-2}+...+k_1,\beta^k=\begin{cases}\alpha^{(0)k},\text{ if (4) is true,}\\ \alpha^{(1)k-k_1},\text{ if (5) is true,}\\ ...\\ \alpha^{(p-1)k-k_{p-1}-k_{p-2}-...-k_1},\text{ if (6) is true.}\end{cases}$$

We notice that $\beta\in\widehat{\Pi}_n$ indeed. We have $\widehat{Or}^{\beta}(\mu^0)\subset X$, due to the way that $\alpha^{(0)},\alpha^{(1)},...,\alpha^{(p-1)}$ were taken. We also infer: $\widehat{\Phi}^{\beta}(\mu^0,-1)=\mu^0,\widehat{\Phi}^{\beta}(\mu^0,k_1-1)=\mu^1,...,\widehat{\Phi}^{\beta}(\mu^0,k_p+k_{p-1}+...+k_1-1)=\mu^p$. We have proved that $\widehat{Or}^{\beta}(\mu^0)=X$. ■

5 Path connected set

Definition 18 *Let $X \subset \mathbf{B}^n$ and $\mu, \mu' \in X$. A **path** in X from μ to μ' is a flow $\widehat{\Phi}^{\alpha}(\mu, \cdot) : \mathbf{N}_{\_} \to \mathbf{B}^n$, where $\alpha \in \widehat{\Pi}_n$, with the property that $k' \in \mathbf{N}_{\_}$ exists such that $\forall k \in \{-1, 0, ..., k'\}, \widehat{\Phi}^{\alpha}(\mu, k) \in X$ and $\widehat{\Phi}^{\alpha}(\mu, k') = \mu'$.*

Definition 19 *We say that X is **path connected** if $\forall \mu \in X, \forall \mu' \in X, \exists \alpha \in \widehat{\Pi}_n, \exists k' \in \mathbf{N}_{\_}, \forall k \in \{-1, 0, ..., k'\}, \widehat{\Phi}^{\alpha}(\mu, k) \in X$ and $\widehat{\Phi}^{\alpha}(\mu, k') = \mu'$.*

Theorem 20 *If X is path connected, then $\forall \mu \in X, \exists \alpha \in \widehat{\Pi}_n, X \subset \widehat{Or}^{\alpha}(\mu)$ holds.*

Proof. As the statement is obvious for $card(X) = 1$, we suppose that $X = \{\mu^0, ..., \mu^p\}, p \geq 1$ and let $\mu \in X$ arbitrary. We can suppose without loosing the generality that $\mu = \mu^0$. The proof is based on the existence of $\alpha^{(i)} \in \widehat{\Pi}_n$ and $k_i \geq 1, i = \overline{1,p}$ with $\Phi^{\alpha^{(i)}}(\mu^{i-1}, k_i - 1) = \mu^i, i = \overline{1,p}$. ■

Theorem 21 *If $X_1, ..., X_k \subset \mathbf{B}^n$ are path connected, $k \geq 2$ and $\forall i \in \{1, ..., k\}$, $\forall j \in \{1, ..., k\}$, $X_i \cap X_j \neq \emptyset$, then $X_1 \cup ... \cup X_k$ is path connected.*

Proof. Let $\mu, \mu' \in X_1 \cup ... \cup X_k$ arbitrary, thus we have $i, j \in \{1, ..., k\}$ with $\mu \in X_i, \mu' \in X_j$. Then $\widetilde{\mu} \in X_i \cap X_j$ and $\alpha, \beta \in \widehat{\Pi}_n, k_1', k_2' \geq 1$ exist such that

$$\forall k'' \in \{-1, 0, ..., k_1' - 1\}, \widehat{\Phi}^{\alpha}(\mu, k'') \in X_i \text{ and } \widehat{\Phi}^{\alpha}(\mu, k_1' - 1) = \widetilde{\mu},$$

$$\forall k'' \in \{-1, 0, ..., k_2' - 1\}, \widehat{\Phi}^{\beta}(\widetilde{\mu}, k'') \in X_j \text{ and } \widehat{\Phi}^{\beta}(\widetilde{\mu}, k_2' - 1) = \mu'.$$

We define $\gamma : \mathbf{N} \longrightarrow \mathbf{B}^n$ by $\forall k \in \mathbf{N}, \gamma^k = \begin{cases} \alpha^k, if\ k \in \{0, ..., k_1' - 1\}, \\ \beta^{k-k_1'}, if\ k \geq k_1'. \end{cases}$ We get obviously $\widehat{\sigma}^{k_1'}(\gamma) = \beta$ and $\gamma \in \widehat{\Pi}_n$. We can write: $\forall k'' \in \{-1, 0, ..., k_1' + k_2' - 1\}, \widehat{\Phi}^{\gamma}(\mu, k'') \in X_i \cup X_j \subset X_1 \cup ... \cup X_k$ and on the other hand $\widehat{\Phi}^{\gamma}(\mu, k_1' + k_2' - 1) = \widehat{\Phi}^{\widehat{\sigma}^{k_1'}(\gamma)}(\widehat{\Phi}^{\gamma}(\mu, k_1'), k_2' - 1) = \widehat{\Phi}^{\beta}(\widetilde{\mu}, k_2' - 1) = \mu'$.

■

6 Minimality, connectedness and path connectedness

Theorem 22 *Let $\Phi : \mathbf{B}^n \longrightarrow \mathbf{B}^n$ and $X \subset \mathbf{B}^n$ non-empty. The properties:*

a) X is the minimal invariant subset of $\mathbf{B}^n$;

b) X is connected;

c) X is path connected

fulfill a) $\Longrightarrow$ b) $\Longrightarrow$ c).

Proof. a)$\Longrightarrow$b) We suppose against all reason that b) is false. If X is not invariant there is a contradiction with a), thus we have the existence of A such that $A \neq \emptyset$, $A \subset X$, $A \neq X$ and $\forall \mu \in A, \exists \alpha \in \widehat{\Pi}_n, \widehat{Or}^{\alpha}(\mu) \subset A$. Then a) implies $X \subset A$. We have obtained that $X = A$, contradiction.

b)$\Longrightarrow$c) Let $\mu, \mu' \in X$ arbitrary and fixed. As X is connected, Theorem 17 shows the existence of $\alpha \in \widehat{\Pi}_n$ such that $\widehat{Or}^{\alpha}(\mu) = X$. From $\mu' \in \widehat{Or}^{\alpha}(\mu)$, we have the existence of $k' \in \mathbf{N}_-$ such that $\widehat{\Phi}^{\alpha}(\mu, k') = \mu'$ and the fact that $\forall k \in \{-1, 0, ..., k'\}, \widehat{\Phi}^{\alpha}(\mu, k) \in X$ is obvious. ■

7 Other definitions of invariance

Remark 23 *Let $\Phi : \mathbf{B}^n \longrightarrow \mathbf{B}^n$ and the non-empty set $A \subset \mathbf{B}^n$. The statements*

$$\forall \mu \in A, \exists \alpha \in \widehat{\Pi}_n, \widehat{Or}^{\alpha}(\mu) \subset A, \tag{7}$$

$$\exists \alpha \in \widehat{\Pi}_n, \forall \mu \in A, \widehat{Or}^{\alpha}(\mu) \subset A, \tag{8}$$

$$\forall \alpha \in \widehat{\Pi}_n, \forall \mu \in A, \widehat{Or}^{\alpha}(\mu) \subset A, \tag{9}$$

$$\forall \nu \in \mathbf{B}^n, \Phi^{\nu}(A) \subset A, \tag{10}$$

$$\forall \nu \in \mathbf{B}^n, \Phi^{\nu}(A) = A \tag{11}$$

fulfill (11)$\Longrightarrow$(10)$\Longleftrightarrow$ (9)$\Longrightarrow$(8)$\Longrightarrow$(7) and they are all of invariance of A, resulted by analogy with the (usual) dynamical systems. They define new concepts of connectedness, path connectedness etc that may be analized.

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