On the basins of attraction of the regular autonomous asynchronous systems

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Abstract

The Boolean autonomous dynamical systems, also called regular autonomous asynchronous systems are systems whose 'vector field' is a function $\Phi : \{0,1\}^n \to \{0,1\}^n$ and time is discrete or continuous. While the synchronous systems have their coordinate functions $\Phi_1, ..., \Phi_n$ computed at the same time: $\Phi, \Phi \circ \Phi, \Phi \circ \Phi \circ \Phi, ...$ the asynchronous systems have $\Phi_1, ..., \Phi_n$ computed independently on each other. The purpose of the paper is that of studying the basins of attraction of the fixed points, of the orbits and of the ω-limit sets of the regular autonomous asynchronous systems, by continuing the study started in [8]. The bibliography consists in analogies.

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keywords: asynchronous system, $\omega-$limit set, invariance, basin of attraction

1 Introduction

The $\mathbf{R} \to \{0,1\}$ functions model the digital electrical signals and they are not studied in literature. An asynchronous circuit without input, considered as a collection of n signals, should be deterministically modelled by a function $x : \mathbf{R} \to \{0,1\}^n$ called state. Several parameters related with the asynchronous circuit are either unknown, or perhaps variable or simply ignored in modeling: the temperature, the tension of the mains, the delays the occur in the computation of the Boolean functions etc. For this reason, instead of a function x we have in general a set X of functions x, called state space or autonomous system, where each x represents a possibility of modeling the circuit. When X is constructed by making use of a 'vector field' $\Phi : \{0,1\}^n \to \{0,1\}^n$, the system X is

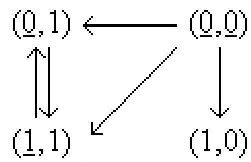

Figure 1: Example of state portrait

called regular. The universal regular autonomous asynchronous systems are the Boolean dynamical systems and they are identified with Φ.

The dynamics of these systems is described by the so called state portraits. We give the example of the function $\Phi : \{0,1\}^2 \to \{0,1\}^2$ that is defined by Table 1, where $\mu = (\mu_1, \mu_2) \in \{0,1\}^2$:

(μ_1, μ_2)	$(\Phi_1(\mu_1, \mu_2), \Phi_2(\mu_1, \mu_2))$
$(0,0)$	$(1,1)$
$(0,1)$	$(1,1)$
$(1,0)$	$(1,0)$
$(1,1)$	$(0,1)$

Table 1

The state portrait of Φ was drawn in Figure 1 where the arrows show the increase of time. The coordinates $\mu_i, i \in \{1,2\}$ are underlined if $\Phi_i(\mu_1, \mu_2) \neq \mu_i$ and they are called unstable, or enabled, or excited. These are the coordinates that are about to change their value. The coordinates μ_i that are not underlined satisfy by definition $\Phi_i(\mu_1, \mu_2) = \mu_i$ and they are called stable, or disabled, or not excited. These are the coordinates that cannot change their value. Three arrows start from the point $(0,0)$ where both coordinates are unstable, showing the fact that $\Phi_1(0,0)$ may be computed first, $\Phi_2(0,0)$ may be computed first or $\Phi_1(0,0), \Phi_2(0,0)$ may be computed simultaneously. Note that the system was identified with the function Φ.

The existence of several possibilities of evolution of the system (three possibilities in $(0,0)$) is the key characteristic of asynchronicity, as opposed to synchronicity where the coordinates $\Phi_i(\mu)$ are always computed simultaneously, $i \in \{1,...,n\}$ for all $\mu \in \{0,1\}^n$ and the system's run is: $\mu, \Phi(\mu), (\Phi \circ \Phi)(\mu), ..., (\Phi \circ ... \circ \Phi)(\mu), ...$

The purpose of the paper is that of defining in the asynchronous systems theory, by following analogies, the basins of attraction of the fixed points and of the orbits from the dynamical systems theory. We shall also define the basins of attraction of the $\omega-$limit sets. The paper continues the study of the basins of attraction that was started in [8] and many introductory issues are taken from that work.

2 Preliminaries

Notation 1 *The set* $\mathbf{B}=\{0,1\}$ *is the binary Boole algebra, endowed with the usual algebraical laws*

$$\begin{array}{cc} & - \\ 0 & 1 \\ 1 & 0 \end{array}, \quad \begin{array}{ccc} \cdot & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}, \quad \begin{array}{ccc} \cup & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{array}, \quad \begin{array}{ccc} \oplus & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}$$

Table 2

and with the discrete topology.

Definition 2 *The sequence* $\alpha:\mathbf{N}\rightarrow\mathbf{B}^n$*, usually denoted by* $\alpha^k, k\in\mathbf{N}$*, is called* ***progressive*** *if the sets*

$$\{k|k\in\mathbf{N},\alpha_i^k=1\}$$

are infinite for all $i\in\{1,...,n\}$*. We denote the set of the progressive sequences by* Π_n*.*

Definition 3 *For the function* $\Phi:\mathbf{B}^n\rightarrow\mathbf{B}^n$ *and* $\nu\in\mathbf{B}^n$ *we define* $\Phi^\nu:\mathbf{B}^n\rightarrow\mathbf{B}^n$ *by* $\forall\mu\in\mathbf{B}^n$,

$$\Phi^\nu(\mu)=(\overline{\nu_1}\cdot\mu_1\oplus\nu_1\cdot\Phi_1(\mu),...,\overline{\nu_n}\cdot\mu_n\oplus\nu_n\cdot\Phi_n(\mu)).$$

Definition 4 *The functions* $\Phi^{\alpha^0...\alpha^k}:\mathbf{B}^n\rightarrow\mathbf{B}^n$ *are defined for* $k\in\mathbf{N}$ *and* $\alpha^0,...,\alpha^k\in\mathbf{B}^n$ *iteratively:* $\forall\mu\in\mathbf{B}^n$,

$$\Phi^{\alpha^0...\alpha^k\alpha^{k+1}}(\mu)=\Phi^{\alpha^{k+1}}(\Phi^{\alpha^0...\alpha^k}(\mu)).$$

Notation 5 *We denote by* $\chi_A:\mathbf{R}\rightarrow\mathbf{B}$ *the characteristic function of the set* $A\subset\mathbf{R}$*:* $\forall t\in\mathbf{R}$,

$$\chi_A(t)=\begin{cases}1, t\in A\\0, t\notin A\end{cases}.$$

Notation 6 *We denote by Seq the set of the sequences* $t_0<t_1<...<t_k<...$ *of real numbers that are unbounded from above.*

Definition 7 *The functions* $\rho:\mathbf{R}\rightarrow\mathbf{B}^n$ *of the form* $\forall t\in\mathbf{R}$,

$$\rho(t)=\alpha^0\cdot\chi_{\{t_0\}}(t)\oplus\alpha^1\cdot\chi_{\{t_1\}}(t)\oplus...\oplus\alpha^k\cdot\chi_{\{t_k\}}(t)\oplus... \quad (1)$$

where $\alpha\in\Pi_n$ *and* $(t_k)\in Seq$ *are called* ***progressive*** *and their set is denoted by* P_n*.*

Definition 8 *The function* $\Phi^{\rho}:\mathbf{B}^n\times\mathbf{R}\rightarrow\mathbf{B}^n$ *that is defined in the following way*

$$\Phi^{\rho}(\mu,t)=\mu\cdot\chi_{(-\infty,t_0)}(t)\oplus\Phi^{\alpha^0}(\mu)\cdot\chi_{[t_0,t_1)}(t)\oplus\Phi^{\alpha^0\alpha^1}(\mu)\cdot\chi_{[t_1,t_2)}(t)\oplus...$$

$$...\oplus\Phi^{\alpha^0...\alpha^k}(\mu)\cdot\chi_{[t_k,t_{k+1})}(t)\oplus...$$

is called ***flow, motion*** *or* ***orbit*** *(of* $\mu\in\mathbf{B}^n$*). We have supposed that* $\rho\in P_n$ *is like at (1).*

Definition 9 *The set*

$$Or_{\rho}(\mu)=\{\Phi^{\rho}(\mu,t)|t\in\mathbf{R}\}$$

is also called ***orbit*** *(of* $\mu\in\mathbf{B}^n$*).*

Remark 10 *The function* Φ^{ν} *shows how an asynchronous iteration of* Φ *is made: for any* $i\in\{1,...,n\}$*, if* $\nu_i=0$ *then* Φ_i *is not computed, since* $\Phi_i^{\nu}(\mu)=\mu_i$ *and if* $\nu_i=1$ *then* Φ_i *is computed, since* $\Phi_i^{\nu}(\mu)=\Phi_i(\mu)$*.*

The definition of $\Phi^{\alpha^0...\alpha^k}$ *generalizes this idea to an arbitrary number* $k+1$ *of asynchronous iterations, with the supplementary request that each coordinate* Φ_i *is computed infinitely many times in the sequence* $\mu,\Phi^{\alpha^0}(\mu),\Phi^{\alpha^0\alpha^1}(\mu),...,\Phi^{\alpha^0...\alpha^k}(\mu),...$ *whenever* $\alpha\in\Pi_n$*.*

The sequences $(t_k)\in Seq$ *make the pass from the discrete time* $\mathbf{N}$ *to the continuous time* $\mathbf{R}$ *and each* $\rho\in P_n$ *shows, in addition to* $\alpha\in\Pi_n$*, the time instants* t_k *when* Φ *is computed (asynchronously). Thus* $\Phi^{\rho}(\mu,t),t\in\mathbf{R}$ *is the continuous time computation of the sequence* μ*,* $\Phi^{\alpha^0}(\mu)$*,* $\Phi^{\alpha^0\alpha^1}(\mu)$*, ...,* $\Phi^{\alpha^0...\alpha^k}(\mu)$*, ...*

When α *runs in* Π_n *and* (t_k) *runs in* Seq *we get the 'unbounded delay model' of computation of the Boolean function* Φ*, represented in discrete time by the sequences* $\mu,\Phi^{\alpha^0}(\mu),\Phi^{\alpha^0\alpha^1}(\mu),...,\Phi^{\alpha^0...\alpha^k}(\mu),...$ *and in continuous time by the orbits* $\Phi^{\rho}(\mu,t)$ *respectively. We shall not insist on the non-formalized way that the engineers describe this model; we just mention that the 'unbounded delay model' is a reasonable way of starting the analysis of a circuit in which the delays occurring in the computation of the Boolean functions* Φ *are arbitrary positive numbers. If we restrict suitably the ranges of* α *and* (t_k) *we get the 'bounded delay model' of computation of* Φ *and if both* α*,* (t_k) *are fixed, then we obtain the 'fixed delay model' of computation of* Φ*, determinism.*

Theorem 11 *[8] Let* $\alpha\in\Pi_n,(t_k)\in Seq$ *be arbitrary and the function*

$$\rho(t)=\alpha^0\cdot\chi_{\{t_0\}}(t)\oplus\alpha^1\cdot\chi_{\{t_1\}}(t)\oplus...\oplus\alpha^k\cdot\chi_{\{t_k\}}(t)\oplus...,$$

$\rho\in P_n$*. The following statements are true:*

a) $\{\alpha^k | k \geq k_1\} \in \Pi_n$ *for any* $k_1 \in \mathbf{N}$;
b) $(t_k) \cap (t', \infty) \in Seq$ *for any* $t' \in \mathbf{R}$;
c) $\rho \cdot \chi_{(t',\infty)} \in P_n$ *for any* $t' \in \mathbf{R}$;
d) $\forall \mu \in \mathbf{B}^n, \forall \mu' \in \mathbf{B}^n, \forall t' \in \mathbf{R}$,

$$\Phi^{\rho}(\mu, t') = \mu' \Longrightarrow \forall t \geq t', \Phi^{\rho}(\mu, t) = \Phi^{\rho \cdot \chi_{(t',\infty)}}(\mu', t).$$

Notation 12 *For any* $d \in \mathbf{R}$, *we denote with* $\tau^d : \mathbf{R} \to \mathbf{R}$ *the translation* $\forall t \in \mathbf{R}, \tau^d(t) = t - d$.

Theorem 13 *[8] Let be* $\mu \in \mathbf{B}^n, \rho \in P_n$ *and* $d \in \mathbf{R}$. *The function* $\rho \circ \tau^d$ *is progressive and we have*

$$\Phi^{\rho \circ \tau^d}(\mu, t) = \Phi^{\rho}(\mu, t - d).$$

Definition 14 *The* ***universal regular autonomous asynchronous system*** *that is generated by* $\Phi : \mathbf{B}^n \to \mathbf{B}^n$ *is by definition*

$$\Xi_{\Phi} = \{\Phi^{\rho}(\mu, \cdot) | \mu \in \mathbf{B}^n, \rho \in P_n\};$$

any $x(t) = \Phi^{\rho}(\mu, t)$ *is called* ***state*** *(of* Ξ_{Φ}*),* μ *is called* ***initial value*** *(of* x*), or* ***initial state*** *(of* Ξ_{Φ}*) and* Φ *is called* ***generator function*** *(of* Ξ_{Φ}*).*

Remark 15 *The asynchronous systems are non-deterministic in general, due to the uncertainties that occur in the modeling of the asynchronous circuits. Non-determinism is produced, in the case of* Ξ_{Φ}, *by the fact that the initial state* μ *and the way* ρ *of iterating* Φ *are not known.*

Some notes on the terminology:

- universality means the greatest in the sense of inclusion. Any $X \subset \Xi_{\Phi}$ *is a system, but we shall not study such systems in the present paper;*

- regularity means the existence of a generator function Φ, *i.e. analogies with the dynamical systems theory;*

- autonomy means here that no input exists. We mention the fact that autonomy has another non-equivalent definition also, a system is called autonomous if its input set has exactly one element;

- asynchronicity refers (vaguely) to the fact that the coordinate functions $\Phi_1, ..., \Phi_n$ *are computed independently on each other. Its antonym synchronicity means that the iterates of* Φ *are:* $\Phi, \Phi \circ \Phi, ..., \Phi \circ ... \circ \Phi, ...$ *i.e. in the sequence* $\Phi^{\alpha^0}, \Phi^{\alpha^0 \alpha^1}, ..., \Phi^{\alpha^0 ... \alpha^k}, ...$ *all* α^k *are* $(1, ..., 1), k \in \mathbf{N}$. *That is the discrete time of the dynamical systems.*

Definition 16 *Let $x : \mathbf{R} \to \mathbf{B}^n$ be some function. If*

$$\exists t' \in \mathbf{R}, \forall t \geq t', x(t) = x(t'),$$

*we say that **the limit** $\lim\limits_{t\to\infty} x(t)$ (or the **final value** of x) **exists** and we denote*

$$\lim_{t\to\infty} x(t) = x(t').$$

Theorem 17 *[7],[8] $\forall \mu \in \mathbf{B}^n, \forall \mu' \in \mathbf{B}^n, \forall \rho \in P_n$,*

$$\lim_{t\to\infty} \Phi^\rho(\mu, t) = \mu' \Longrightarrow \Phi(\mu') = \mu',$$

i.e. if the final value of $\Phi^\rho(\mu, \cdot)$ exists, it is a fixed point of Φ.

Theorem 18 *[7],[8] $\forall \mu \in \mathbf{B}^n, \forall \mu' \in \mathbf{B}^n, \forall \rho \in P_n$,*

$$(\Phi(\mu') = \mu' \text{ and } \exists t' \in \mathbf{R}, \Phi^\rho(\mu, t') = \mu') \Longrightarrow \forall t \geq t', \Phi^\rho(\mu, t) = \mu',$$

meaning that if the fixed point μ' of Φ is accessible, then it is the final value of $\Phi^\rho(\mu, \cdot)$.

Corollary 19 *[8] We have $\forall \mu \in \mathbf{B}^n, \forall \rho \in P_n$,*

$$\Phi(\mu) = \mu \Longrightarrow \forall t \in \mathbf{R}, \Phi^\rho(\mu, t) = \mu.$$

3 $\omega-$limit sets

Definition 20 *For $\mu \in \mathbf{B}^n$ and $\rho \in P_n$, the set*

$$\omega_\rho(\mu) = \{\mu' | \mu' \in \mathbf{B}^n, \exists (t_k) \in Seq, \lim_{k\to\infty} \Phi^\rho(\mu, t_k) = \mu'\}$$

*is called the $\omega-$**limit set** of the orbit $\Phi^\rho(\mu, \cdot)$.*

Theorem 21 *[8] For any $\mu \in \mathbf{B}^n$ and any $\rho \in P_n$, we have:*
a) $\omega_\rho(\mu) \neq \emptyset$;
b) $\forall t' \in \mathbf{R}$, $\omega_\rho(\mu) \subset \{\Phi^\rho(\mu, t) | t \geq t'\} \subset Or_\rho(\mu)$;
c) $\exists t' \in \mathbf{R}, \omega_\rho(\mu) = \{\Phi^\rho(\mu, t) | t \geq t'\}$ and any $t'' \geq t'$ fulfills $\omega_\rho(\mu) = \{\Phi^\rho(\mu, t) | t \geq t''\}$;
d) $\forall t' \in \mathbf{R}, \forall t'' \geq t', \{\Phi^\rho(\mu, t) | t \geq t'\} = \{\Phi^\rho(\mu, t) | t \geq t''\}$ implies $\omega_\rho(\mu) = \{\Phi^\rho(\mu, t) | t \geq t'\}$;
e) we suppose that $\omega_\rho(\mu) = \{\Phi^\rho(\mu, t) | t \geq t'\}, t' \in \mathbf{R}$. Then $\forall \mu' \in \omega_\rho(\mu), \forall t'' \geq t'$, if $\Phi^\rho(\mu, t'') = \mu'$ we get

$$\omega_\rho(\mu) = \{\Phi^{\rho\cdot\chi_{(t'',\infty)}}(\mu', t) | t \geq t''\} = Or_{\rho\cdot\chi_{(t'',\infty)}}(\mu') = \omega_{\rho\cdot\chi_{(t'',\infty)}}(\mu').$$

Remark 22 *If in Theorem 21 e) we take $t'' \in \mathbf{R}$ arbitrarily, the equation*

$$\omega_\rho(\mu) = \omega_{\rho \cdot \chi_{(t'',\infty)}}(\Phi^\rho(\mu, t'')) \tag{2}$$

is still true. Indeed, for sufficiently great t''', the terms in (2) are equal with

$$\{\Phi^\rho(\mu, t) | t \geq t'''\} = \{\Phi^{\rho \cdot \chi_{(t'',\infty)}}(\Phi^\rho(\mu, t''), t) | t \geq t'''\}.$$

Theorem 23 *[8] For arbitrary $\mu \in \mathbf{B}^n$,$\rho \in P_n$ the following statements are true:*

a) $\lim\limits_{t\to\infty} \Phi^\rho(\mu, t)$ exists $\Longleftrightarrow card(\omega_\rho(\mu)) = 1$;

b) if $\exists \mu' \in \mathbf{B}^n, \omega_\rho(\mu) = \{\mu'\}$, then $\lim\limits_{t\to\infty} \Phi^\rho(\mu, t) = \mu'$ and $\Phi(\mu') = \mu'$;

c) if $\exists \mu' \in \mathbf{B}^n, \Phi(\mu') = \mu'$ and $\mu' \in Or_\rho(\mu)$, then $\omega_\rho(\mu) = \{\mu'\}$.

Theorem 24 *[8] Let be $\mu \in \mathbf{B}^n, \rho \in P_n, d \in \mathbf{R}$. We have $\omega_\rho(\mu) = \omega_{\rho \circ \tau^d}(\mu)$.*

4 Invariant sets

Theorem 25 *[8] We consider the function $\Phi : \mathbf{B}^n \to \mathbf{B}^n$ and let be the set $A \in P^*(\mathbf{B}^n)$. For any $\mu \in A$, the following properties are equivalent*

$$\exists \alpha \in \Pi_n, \forall k \in \mathbf{N}, \Phi^{\alpha^0 \ldots \alpha^k}(\mu) \in A, \tag{3}$$

$$\exists \rho \in P_n, \forall t \in \mathbf{R}, \Phi^\rho(\mu, t) \in A, \tag{4}$$

$$\exists \rho \in P_n, Or_\rho(\mu) \subset A \tag{5}$$

and the following properties are also equivalent

$$\forall \alpha \in \Pi_n, \forall k \in \mathbf{N}, \Phi^{\alpha^0 \ldots \alpha^k}(\mu) \in A, \tag{6}$$

$$\forall \rho \in P_n, \forall t \in \mathbf{R}, \Phi^\rho(\mu, t) \in A, \tag{7}$$

$$\forall \rho \in P_n, Or_\rho(\mu) \subset A, \tag{8}$$

$$\forall \lambda \in \mathbf{B}^n, \Phi^\lambda(\mu) \in A. \tag{9}$$

Definition 26 *The set $A \in P^*(\mathbf{B}^n)$ is called a* ***p-invariant*** *(or* ***p-stable****)* ***set*** *of the system Ξ_Φ if it fulfills for any $\mu \in A$ one of (3),..., (5) and it is called an* ***n-invariant*** *(or n-**stable**)* ***set*** *of Ξ_Φ if it fulfills $\forall \mu \in A$ one of (6),..., (9).*

Remark 27 *In the previous terminology, the letter 'p' comes from 'possibly' and the letter 'n' comes from 'necessarily'. Both 'p' and 'n' refer to the quantification of* ρ*. Such kind of p-definitions and n-definitions recalling logic are caused by the fact that we translate 'real' concepts into 'binary' concepts and the former have no* ρ *parameters, thus after translation* ρ *may appear quantified in two ways. The obvious implication is n-invariance* $\Longrightarrow$ *p-invariance.*

Theorem 28 *[8] Let be* $\mu \in \mathbf{B}^n$ *and* $\rho' \in P_n$*.*

a) If $\Phi(\mu) = \mu$*, then* $\{\mu\}$ *is an n-invariant set and the set* Eq *of the fixed points of* Φ *is also n-invariant;*

b) the set $Or_{\rho'}(\mu)$ *is p-invariant and* $\bigcup\limits_{\rho \in P_n} Or_\rho(\mu)$ *is n-invariant;*

c) the set $\omega_{\rho'}(\mu)$ *is p-invariant.*

5 The basin of attraction

Theorem 29 *[8] We consider the set* $A \in P^*(\mathbf{B}^n)$*. For any* $\mu \in \mathbf{B}^n$*, the following statements are equivalent*

$$\exists \alpha \in \Pi_n, \exists k' \in \mathbf{N}, \forall k \geq k', \Phi^{\alpha^0 \ldots \alpha^k}(\mu) \in A, \tag{10}$$

$$\exists \rho \in P_n, \exists t' \in R, \forall t \geq t', \Phi^\rho(\mu, t) \in A, \tag{11}$$

$$\exists \rho \in P_n, \omega_\rho(\mu) \subset A \tag{12}$$

and the following statements are equivalent too

$$\forall \alpha \in \Pi_n, \exists k' \in \mathbf{N}, \forall k \geq k', \Phi^{\alpha^0 \ldots \alpha^k}(\mu) \in A, \tag{13}$$

$$\forall \rho \in P_n, \exists t' \in R, \forall t \geq t', \Phi^\rho(\mu, t) \in A, \tag{14}$$

$$\forall \rho \in P_n, \omega_\rho(\mu) \subset A. \tag{15}$$

Definition 30 *The* ***basin*** *(or* ***kingdom****, or* ***domain****)* ***of p-attraction*** *or the* ***p-stable set*** *of the set* $A \in P^*(\mathbf{B}^n)$ *is given by*

$$\overline{W}(A) = \{\mu | \mu \in \mathbf{B}^n, \exists \rho \in P_n, \omega_\rho(\mu) \subset A\}; \tag{16}$$

the ***basin*** *(or* ***kingdom****, or* ***domain****)* ***of n-attraction*** *or the* ***n-stable set*** *of the set* A *is given by*

$$\underline{W}(A) = \{\mu | \mu \in \mathbf{B}^n, \forall \rho \in P_n, \omega_\rho(\mu) \subset A\}. \tag{17}$$

Remark 31 *Definition 30 makes use of the properties (12) and (15). We can make use also in this Definition of the other equivalent properties from Theorem 29.*

In Definition 30, one or both basins of attraction $\overline{W}(A), \underline{W}(A)$ *may be empty.*

Theorem 32 *[8] We have:*

i) $\overline{W}(\mathbf{B}^n) = \underline{W}(\mathbf{B}^n) = \mathbf{B}^n$;

ii) if $A \subset A'$, *then* $\overline{W}(A) \subset \overline{W}(A')$ *and* $\underline{W}(A) \subset \underline{W}(A')$ *hold.*

Definition 33 *When* $\overline{W}(A) \neq \emptyset$, A *is said to be* ***p-attractive*** *and for any non-empty set* $B \subset \overline{W}(A)$, *we say that* A *is* ***p-attractive*** *for* B *and that* B *is* ***p-attracted*** *by* A; A *is by definition* ***partially p-attractive*** *if* $\overline{W}(A) \notin \{\emptyset, \mathbf{B}^n\}$ *and* ***totally p-attractive*** *whenever* $\overline{W}(A) = \mathbf{B}^n$.

The fact that $\underline{W}(A) \neq \emptyset$ *makes us say that* A *is* ***n-attractive*** *and in this situation for any non-empty* $B \subset \underline{W}(A)$, A *is called* ***n-attractive*** *for* B *and* B *is called to be* ***n-attracted*** *by* A; *we use to say that* A *is* ***partially n-attractive*** *if* $\underline{W}(A) \notin \{\emptyset, \mathbf{B}^n\}$ *and* ***totally n-attractive*** *if* $\underline{W}(A) = \mathbf{B}^n$.

Theorem 34 *[8] Let* $A \in P^*(\mathbf{B}^n)$ *be some set. If* A *is p-invariant, then* $A \subset \overline{W}(A)$ *and* A *is also p-attractive; if* A *is n-invariant, then* $A \subset \underline{W}(A)$ *and* A *is also n-attractive.*

Remark 35 *The previous Theorem shows the connection that exists between invariance and attractiveness. If* A *is p-attractive, then* $\overline{W}(A)$ *is the greatest set that is p-attracted by* A *and the point is that this really happens when* A *is p-invariant. The other situation is dual.*

Theorem 36 *[8] Let be* $A \in P^*(\mathbf{B}^n)$. *If* A *is p-attractive, then* $\overline{W}(A)$ *is p-invariant and if* A *is n-attractive, then* $\underline{W}(A)$ *is n-invariant.*

Corollary 37 *[8] If the set* $A \in P^*(\mathbf{B}^n)$ *is p-invariant, then* $\overline{W}(A)$ *is p-invariant and if* A *is n-invariant, then the basin of n-attraction* $\underline{W}(A)$ *is n-invariant.*

6 The basin of attraction of the fixed points

Notation 38 *For any point* $\mu \in \mathbf{B}^n$ *we use the simpler notations* $\overline{W}(\mu)$, $\underline{W}(\mu)$ *instead of* $\overline{W}(\{\mu\})$, $\underline{W}(\{\mu\})$. *Furthermore, if the point* μ *is identified with the* $n-$*tuple* $(\mu_1, ..., \mu_n)$, *it is usual to write* $\overline{W}(\mu_1, ..., \mu_n)$, $\underline{W}(\mu_1, ..., \mu_n)$ *for these sets.*

Remark 39 *This section is dedicated to the special case when in Definition 30 the set* $A \in P^*(\mathbf{B}^n)$ *consists in a point* μ, *in other words*

$$\overline{W}(\mu) = \{\mu' | \mu' \in \mathbf{B}^n, \exists \rho' \in P_n, \omega_{\rho'}(\mu') \subset \{\mu\}\}, \tag{18}$$

$$\underline{W}(\mu) = \{\mu' | \mu' \in \mathbf{B}^n, \forall \rho' \in P_n, \omega_{\rho'}(\mu') \subset \{\mu\}\}. \tag{19}$$

The fact that the point μ *is chosen to be fixed is justified by the*

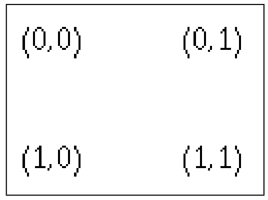

Figure 2: The basins of attraction of the fixed points

Theorem 40 $\overline{W}(\mu) \neq \emptyset \iff \mu$ *is a fixed point of* Φ *and similarly,* $\underline{W}(\mu) \neq \emptyset \iff \mu$ *is a fixed point of* Φ.

Proof. We prove the first statement. If $\mu' \in \overline{W}(\mu)$, then $\rho' \in P_n$ exists such that $\omega_{\rho'}(\mu') \subset \{\mu\}$. In this case $\omega_{\rho'}(\mu')$ is non-empty, thus $\omega_{\rho'}(\mu') = \{\mu\}$ and, from Theorem 23 b), $\Phi(\mu) = \mu$.

Let us suppose now that $\Phi(\mu) = \mu$. For any $\rho' \in P_n$, $Or_{\rho'}(\mu) = \omega_{\rho'}(\mu) = \{\mu\}$ from Corollary 19, thus $\mu \in \overline{W}(\mu)$ and $\overline{W}(\mu) \neq \emptyset$. ■

Remark 41 *In [6] at page 5, the fixed point* $x_0 \in X$ *is called attractive if the neighborhood* $U \subset X$ *and* $t' > 0$ *exist such that*

$$\forall x \in U, \forall t > t', \Phi_t(x) \in U \text{ and } \lim_{t \to \infty} |\Phi_t(x) - x_0| = 0.$$

We give also the point of view from [3] where, at page 110 it is said, in a discrete time context, that the basin of attraction of an attractive fixed point $x_0 \in X$ *is formed by the the set of all the initial points of some sequences of iterates that converge to* x_0.

Example 42 *In Figure 2 we have the property that all the points are fixed points and* $\forall \mu \in \mathbf{B}^2, \forall \rho \in P_2$,

$$\overline{W}(\mu) = \underline{W}(\mu) = \{\mu\}.$$

Any $\mu \in \mathbf{B}^2$ *is partially p-attractive and partially n-attractive.*

Example 43 *The point* $(1,0)$ *is fixed in Figure 3 and*

$$\overline{W}(1,0) = \{(0,0), (1,0)\},$$

$$\underline{W}(1,0) = \{(1,0)\}.$$

The point $(1,0)$ *is partially p-attractive and partially n-attractive.*

Example 44 *We have also the example when in Figure 4:*

$$\overline{W}(1,0) = \underline{W}(1,0) = \mathbf{B}^2,$$

thus the fixed point $(1,0)$ *is totally p-attractive and totally n-attractive.*

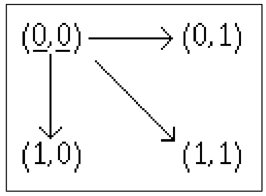

Figure 3: The basins of attraction of the fixed points

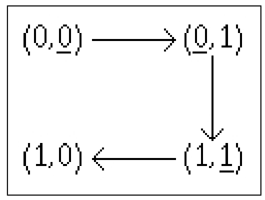

Figure 4: The basins of attraction of the fixed points

Theorem 45 *Let $\mu \in \mathbf{B}^n$ be a fixed point of Φ. The following statements are true:*

a) We have

$$\overline{W}(\mu) = \{\mu' | \mu' \in \mathbf{B}^n, \exists \rho' \in P_n, \lim_{t\to\infty} \Phi^{\rho'}(\mu', t) = \mu\},$$

$$\underline{W}(\mu) = \{\mu' | \mu' \in \mathbf{B}^n, \forall \rho' \in P_n, \lim_{t\to\infty} \Phi^{\rho'}(\mu', t) = \mu\};$$

b) $\{\mu\} \subset \underline{W}(\mu) \subset \overline{W}(\mu)$ thus μ is p-attractive and n-attractive;

c) $\overline{W}(\mu)$ is p-invariant and $\underline{W}(\mu)$ is n-invariant.

Proof. a) If $\mu' \in \overline{W}(\mu)$, then $\rho' \in P_n$ exists such that $\omega_{\rho'}(\mu') = \{\mu\}$. In this situation from Theorem 23 b) we infer that $\lim_{t\to\infty} \Phi^{\rho'}(\mu', t) = \mu$, thus $\overline{W}(\mu) \subset \{\mu' | \mu' \in \mathbf{B}^n, \exists \rho' \in P_n, \lim_{t\to\infty} \Phi^{\rho'}(\mu', t) = \mu\}$.

Conversely, if $\mu' \in \mathbf{B}^n, \rho' \in P_n$ exist such that $\lim_{t\to\infty} \Phi^{\rho'}(\mu', t) = \mu$, then $\omega_{\rho'}(\mu') = \{\mu\}$ from Definition 20 and we get $\{\mu' | \mu' \in \mathbf{B}^n, \exists \rho' \in P_n, \lim_{t\to\infty} \Phi^{\rho'}(\mu', t) = \mu\} \subset \overline{W}(\mu)$.

b) The fact that $\mu \in \underline{W}(\mu)$ is a consequence of the fact that $\forall \rho' \in P_n, \lim_{t\to\infty} \Phi^{\rho'}(\mu, t) = \mu$ (see Corollary 19).

c) μ is p-attractive from b), thus $\overline{W}(\mu)$ is p-invariant (Theorem 36).

■

7 The basin of attraction of the orbits and of the $\omega-$limit sets

Definition 46 *Let be $\Phi : \mathbf{B}^n \to \mathbf{B}^n$, $\mu \in \mathbf{B}^n$ and $\rho \in P_n$. We define the* ***basins*** *(or* ***kingdoms****, or* ***domains****)* ***of p-attraction*** *of $\Phi^\rho(\mu,\cdot), \omega_\rho(\mu)$ by*

$$\overline{W}[\Phi^\rho(\mu,\cdot)] = \{\mu'|\mu' \in \mathbf{B}^n, \exists \rho' \in P_n, \exists t' \in \mathbf{R}, \forall t \geq t', \quad (20)$$
$$\Phi^{\rho'}(\mu',t) = \Phi^\rho(\mu,t)\},$$
$$\overline{W}[\omega_\rho(\mu)] = \{\mu'|\mu' \in \mathbf{B}^n, \exists \rho' \in P_n, \omega_{\rho'}(\mu') = \omega_\rho(\mu)\} \quad (21)$$

and the ***basins*** *(or* ***kingdoms****, or* ***domains****)* ***of n-attraction*** *of $\Phi^\rho(\mu,\cdot)$, $\omega_\rho(\mu)$ respectively by*

$$\underline{W}[\Phi^\rho(\mu,\cdot)] = \{\mu'|\mu' \in \mathbf{B}^n, \forall \rho' \in P_n, \exists t' \in \mathbf{R}, \forall t \geq t', \quad (22)$$
$$\Phi^{\rho'}(\mu',t) = \Phi^\rho(\mu,t)\},$$
$$\underline{W}[\omega_\rho(\mu)] = \{\mu'|\mu' \in \mathbf{B}^n, \forall \rho' \in P_n, \omega_{\rho'}(\mu') = \omega_\rho(\mu)\}. \quad (23)$$

Remark 47 *The attractiveness of the orbits and of the $\omega-$limit sets is defined in the spirit of Definition 33 and it is their property of making one of the previous basins of attraction non-empty.*

We mention [2], page 133, where M is a differentiable manifold together with a distance d on M and a discrete time dynamical system is generated by the C^r-diffeomorphism $\Phi : M \to M$. The orbit through $x_0 \in M$ is called attractive if

$$\exists \delta > 0, \forall x \in B(x_0,\delta), \lim_{n\to\infty} d(\Phi_n(x), \Phi_n(x_0)) = 0, \quad (24)$$

where $B(x_0,\delta)$ is the notation for the open ball of center x_0 and radius δ. In the same work [2], page 133 the orbit through $x_0 \in M$ is called stable if

$$\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0, \forall x \in B(x_0,\delta), \forall n \in \mathbf{N}, d(\Phi_n(x), \Phi_n(x_0)) < \varepsilon. \quad (25)$$

The translation of (24), (25) in our framework gives the statements

$$\exists \mu' \in \mathbf{B}^n, \exists \rho' \in P_n, \omega_{\rho'}(\mu') = \omega_\rho(\mu), \quad (26)$$
$$\exists \mu' \in \mathbf{B}^n, \forall \rho' \in P_n, \omega_{\rho'}(\mu') = \omega_\rho(\mu), \quad (27)$$
$$\exists \mu' \in \mathbf{B}^n, \exists \rho' \in P_n, \forall t \in \mathbf{R}, \Phi^{\rho'}(\mu',t) = \Phi^\rho(\mu,t), \quad (28)$$
$$\exists \mu' \in \mathbf{B}^n, \forall \rho' \in P_n, \forall t \in \mathbf{R}, \Phi^{\rho'}(\mu',t) = \Phi^\rho(\mu,t); \quad (29)$$

we note that (26), (27) are equivalent with $\overline{W}[\omega_\rho(\mu)] \neq \emptyset$, $\underline{W}[\omega_\rho(\mu)] \neq \emptyset$ attractiveness, while (28), (29) are stronger than the attractiveness

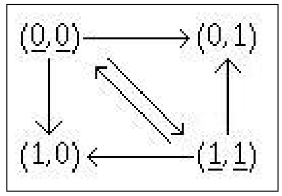

Figure 5: The basins of attraction of the orbits and of the ω-limit sets

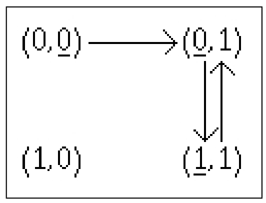

Figure 6: The basins of attraction of the orbits and of the ω-limit sets

properties $\overline{W}[\Phi^{\rho}(\mu,\cdot)]\neq\emptyset$, $\underline{W}[\Phi^{\rho}(\mu,\cdot)]\neq\emptyset$. *On the other hand, if we take in (26) and (28)* $\mu'=\mu,\rho'=\rho$ *we get that these two properties are always true, see Theorem 51 to follow, items a), b).*

Note that the stability of the sets A from the dynamical systems theory is interpreted as invariance [8], while the stability of the orbits from the dynamical systems theory is interpreted to be stronger than attractiveness.

Example 48 *In Figure 5 for*

$$\rho(t)=(1,1)\cdot\chi_{\{0\}}(t)\oplus(1,1)\cdot\chi_{\{1\}}(t)\oplus(1,1)\cdot\chi_{\{2\}}(t)\oplus...$$

we get

$$\overline{W}[\Phi^{\rho}((0,0),\cdot)]=\overline{W}[\omega_{\rho}((0,0))]=\{(0,0),(1,1)\},$$
$$\underline{W}[\Phi^{\rho}((0,0),\cdot)]=\underline{W}[\omega_{\rho}((0,0))]=\emptyset.$$

Example 49 *We take in Figure 6*

$$\rho(t)=(1,1)\cdot\chi_{\{0\}}(t)\oplus(1,1)\cdot\chi_{\{1\}}(t)\oplus(1,1)\cdot\chi_{\{2\}}(t)\oplus...$$

and we obtain

$$\overline{W}[\Phi^{\rho}((0,1),\cdot)]=\overline{W}[\omega_{\rho}((0,1))]=\underline{W}[\omega_{\rho}((0,1))]=\{(0,0),(0,1),(1,1)\},$$
$$\underline{W}[\Phi^{\rho}((0,1),\cdot)]=\emptyset.$$

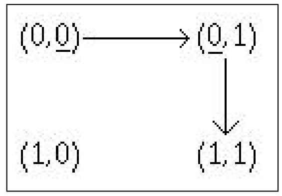

Figure 7: The basins of attraction of the orbits and of the ω-limit sets

Example 50 *In Figure 7 for*

$$\rho(t) = (0,1) \cdot \chi_{\{0\}}(t) \oplus (1,1) \cdot \chi_{\{1\}}(t) \oplus (0,1) \cdot \chi_{\{2\}}(t) \oplus (1,1) \cdot \chi_{\{3\}}(t) \oplus ...$$

we see that

$$\overline{W}[\Phi^{\rho}((0,0),\cdot)] = \overline{W}[\omega_{\rho}((0,0))] = \underline{W}[\Phi^{\rho}((0,0),\cdot)]$$

$$= \underline{W}[\omega_{\rho}((0,0))] = \{(0,0),(0,1),(1,1)\}.$$

Theorem 51 *We consider the point $\mu \in \mathbf{B}^n$ and the function $\rho \in P_n$.*

a) $\overline{W}[\Phi^{\rho}(\mu,\cdot)] = \overline{W}[\omega_{\rho}(\mu)]$;

b) we have $Or_{\rho}(\mu) \subset \overline{W}[\Phi^{\rho}(\mu,\cdot)]$, thus $\overline{W}[\Phi^{\rho}(\mu,\cdot)]$ is non-empty;

c) $\underline{W}[\Phi^{\rho}(\mu,\cdot)] \subset \underline{W}[\omega_{\rho}(\mu)]$;

d) $\underline{W}[\Phi^{\rho}(\mu,\cdot)] \neq \emptyset \iff card(\omega_{\rho}(\mu)) = 1$, thus $card(\omega_{\rho}(\mu)) = 1$ implies that $\underline{W}[\Phi^{\rho}(\mu,\cdot)]$ and $\underline{W}[\omega_{\rho}(\mu)]$ are non-empty;

e) if $\exists \mu' \in \mathbf{B}^n, \omega_{\rho}(\mu) = \{\mu'\}$, then $\underline{W}[\Phi^{\rho}(\mu,\cdot)] = \underline{W}[\omega_{\rho}(\mu)] = \underline{W}(\mu')$.

Proof. a) Let $\mu' \in \overline{W}[\Phi^{\rho}(\mu,\cdot)]$ be arbitrary, for which $\rho' \in P_n, t' \in \mathbf{R}$ exist such that

$$\forall t \geq t', \Phi^{\rho'}(\mu',t) = \Phi^{\rho}(\mu,t), \tag{30}$$

$$\omega_{\rho'}(\mu') = \{\Phi^{\rho'}(\mu',t) | t \geq t'\}, \tag{31}$$

$$\omega_{\rho}(\mu) = \{\Phi^{\rho}(\mu,t) | t \geq t'\}. \tag{32}$$

(30) is fulfilled from the definition (20) of $\overline{W}[\Phi^{\rho}(\mu,\cdot)]$ and, by taking t' sufficiently great, (31), (32) are fulfilled too (Theorem 21). As $\omega_{\rho'}(\mu') = \omega_{\rho}(\mu)$ we get $\mu' \in \overline{W}[\omega_{\rho}(\mu)]$ and because μ' was arbitrary, we infer that $\overline{W}[\Phi^{\rho}(\mu,\cdot)] \subset \overline{W}[\omega_{\rho}(\mu)]$.

Conversely, let $\mu' \in \overline{W}[\omega_{\rho}(\mu)]$ be arbitrary, thus

$$\exists \rho' \in P_n, \omega_{\rho'}(\mu') = \omega_{\rho}(\mu)$$

and let $\mu'' \in \omega_{\rho'}(\mu') = \omega_{\rho}(\mu)$ be some point,

$$\Phi^{\rho'}(\mu',t_1) = \Phi^{\rho}(\mu,t_2) = \mu'', \tag{33}$$

$t_1, t_2 \in \mathbf{R}$. The function

$$\rho''(t) = \rho'(t - t_2 + t_1) \cdot \chi_{(-\infty,t_2]}(t) \oplus \rho(t) \cdot \chi_{(t_2,\infty)}(t) \tag{34}$$

is progressive and fulfills

$$\Phi^{\rho''}(\mu', t_2) \overset{(34)}{=} \Phi^{\rho' \circ \tau^{t_2 - t_1}}(\mu', t_2) \overset{Theorem\ 13}{=} \Phi^{\rho'}(\mu', t_1) \overset{(33)}{=} \Phi^{\rho}(\mu, t_2) \overset{(33)}{=} \mu'',$$

$$\forall t > t_2, \Phi^{\rho''}(\mu', t) \overset{Theorem\ 11\ d)}{=} \Phi^{\rho'' \cdot \chi_{(t_2,\infty)}}(\mu'', t)$$
$$\overset{(34)}{=} \Phi^{\rho \cdot \chi_{(t_2,\infty)}}(\mu'', t) \overset{Theorem\ 11\ d)}{=} \Phi^{\rho}(\mu, t)$$

in other words $\mu' \in \overline{W}[\Phi^{\rho}(\mu, \cdot)]$. The fact that μ' was arbitrary gives the conclusion that $\overline{W}[\omega_{\rho}(\mu)] \subset \overline{W}[\Phi^{\rho}(\mu, \cdot)]$.

b) Let $\mu' \in Or_{\rho}(\mu)$ be arbitrary, thus $\exists t' \in \mathbf{R}$ with $\mu' = \Phi^{\rho}(\mu, t')$. We get

$$\forall t \geq t', \Phi^{\rho \cdot \chi_{(t',\infty)}}(\mu', t) \overset{Theorem\ 11\ d)}{=} \Phi^{\rho}(\mu, t).$$

We have shown that $\mu' \in \overline{W}[\Phi^{\rho}(\mu, \cdot)]$ and as μ' was arbitrarily chosen, we infer $Or_{\rho}(\mu) \subset \overline{W}[\Phi^{\rho}(\mu, \cdot)]$.

c) Let $\mu' \in \underline{W}[\Phi^{\rho}(\mu, \cdot)]$ and $\rho' \in P_n$ be arbitrary. Some sufficiently great $t' \in \mathbf{R}$ exists such that

$$\forall t \geq t', \Phi^{\rho'}(\mu', t) = \Phi^{\rho}(\mu, t)$$

and we have

$$\omega_{\rho'}(\mu') = \{\Phi^{\rho'}(\mu', t) | t \geq t'\} = \{\Phi^{\rho}(\mu, t) | t \geq t'\} = \omega_{\rho}(\mu),$$

i.e. $\mu' \in \underline{W}[\omega_{\rho}(\mu)]$.

d) $\Longrightarrow$ Let be ρ given by

$$\rho(t) = \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus ... \oplus \alpha^k \cdot \chi_{\{t_k\}}(t) \oplus ...,$$

$\alpha \in \Pi_n, (t_k) \in Seq$ and we define

$$\rho'(t) = \alpha^0 \cdot \chi_{\{t'_0\}}(t) \oplus ... \oplus \alpha^k \cdot \chi_{\{t'_k\}}(t) \oplus ...,$$

where

$$t'_k = \frac{t_k + t_{k+1}}{2}, k \in \mathbf{N} \tag{35}$$

belongs to Seq. We call point of discontinuity of $\Phi^{\rho}(\mu, \cdot)$ a point $\xi \in \mathbf{R}$ with the property that $\mu', \mu'' \in \mathbf{B}^n$ and $\varepsilon > 0$ exist such that

$$\forall t \in (\xi - \varepsilon, \xi), \Phi^{\rho}(\mu, t) = \mu',$$

$$\forall t \in [\xi, \xi + \varepsilon), \Phi^{\rho}(\mu, t) = \mu'',$$
$$\mu' \neq \mu''.$$

The hypothesis states that $\widetilde{\mu} \in \underline{W}[\Phi^{\rho}(\mu, \cdot)]$ exists fulfilling the property

$$\exists t' \in \mathbf{R}, \forall t \geq t', \Phi^{\rho'}(\widetilde{\mu}, t) = \Phi^{\rho}(\mu, t). \tag{36}$$

Let us suppose against all reason that $card(\omega_{\rho'}(\widetilde{\mu})) = card(\omega_{\rho}(\mu)) > 1$ and

$$\omega_{\rho'}(\widetilde{\mu}) = \{\Phi^{\rho'}(\widetilde{\mu}, t) | t \geq t''\} = \{\Phi^{\rho}(\mu, t) | t \geq t''\} = \omega_{\rho}(\mu),$$

$t'' \geq t'$. Then equation (36) is contradictory, since the discontinuity points of $\Phi^{\rho'}(\widetilde{\mu}, \cdot)_{|[t',\infty)}$[1] and $\Phi^{\rho}(\mu, \cdot)_{|[t',\infty)}$ are included in the disjoint sets $[t', \infty) \cap (t'_k)$ and $[t', \infty) \cap (t_k)$. The conclusion is that $card(\omega_{\rho'}(\widetilde{\mu})) = card(\omega_{\rho}(\mu)) = 1$ and in (36) the disjoint sets $[t', \infty) \cap (t'_k)$ and $[t', \infty) \cap (t_k)$ contain no discontinuity points.

$\Longleftarrow$ We presume that $\mu' \in \mathbf{B}^n$ exists with $\omega_{\rho}(\mu) = \{\mu'\}$ and then for an arbitrary $\rho' \in P_n$ we get $\forall t \in \mathbf{R}, \Phi^{\rho'}(\mu', t) = \mu'$. As $\lim\limits_{t \to \infty} \Phi^{\rho}(\mu, t) = \mu'$, we conclude that $\exists t' \in \mathbf{R}$ such that $\forall t \geq t'$,

$$\Phi^{\rho}(\mu, t) = \Phi^{\rho'}(\mu', t) = \mu',$$

thus $\mu' \in \underline{W}[\Phi^{\rho}(\mu, \cdot)]$.

e) The fact that $\omega_{\rho}(\mu) = \{\mu'\}$ shows that μ' is a fixed point of Φ (Theorem 23 b)), thus $\underline{W}(\mu') \neq \emptyset$. $\underline{W}[\Phi^{\rho}(\mu, \cdot)]$, $\underline{W}[\omega_{\rho}(\mu)]$ and $\underline{W}(\mu')$ are all equal with the set

$$\{\mu'' | \mu'' \in \mathbf{B}^n, \forall \rho' \in P_n, \lim_{t \to \infty} \Phi^{\rho'}(\mu'', t) = \mu'\}.$$

■

Theorem 52 *For any $\mu \in \mathbf{B}^n$ and any $\rho \in P_n$,*

a) the basin of p-attraction $\overline{W}[\Phi^{\rho}(\mu, \cdot)]$ is p-invariant;

b) if $\Phi^{\rho}(\mu, \cdot)$ is n-attractive, then the basin of n-attraction $\underline{W}[\Phi^{\rho}(\mu, \cdot)]$ is n-invariant;

c) if $\omega_{\rho}(\mu)$ is n-attractive, then $\underline{W}[\omega_{\rho}(\mu)]$ is n-invariant.

Proof. a) From $Or_{\rho}(\mu) \neq \emptyset$ and $Or_{\rho}(\mu) \subset \overline{W}[\Phi^{\rho}(\mu, \cdot)]$, see Theorem 51 b), we have that $\overline{W}[\Phi^{\rho}(\mu, \cdot)] \neq \emptyset$. Let $\mu' \in \overline{W}[\Phi^{\rho}(\mu, \cdot)]$ be arbitrary, meaning that

$$\exists \rho' \in P_n, \exists t' \in \mathbf{R}, \forall t \geq t', \Phi^{\rho'}(\mu', t) = \Phi^{\rho}(\mu, t)$$

[1] $\Phi^{\rho'}(\widetilde{\mu}, \cdot)_{|[t',\infty)}$ is the notation for the restriction of $\Phi^{\rho'}(\widetilde{\mu}, \cdot) : \mathbf{R} \to \mathbf{B}^n$ to $[t', \infty)$.

and we prove the inclusion $Or_{\rho'}(\mu') \subset \overline{W}[\Phi^\rho(\mu,\cdot)]$. Indeed, the points μ'' of the orbit $Or_{\rho'}(\mu')$ are of the form

$$\exists t_1 \in \mathbf{R}, \mu'' = \Phi^{\rho'}(\mu', t_1)$$

and they fulfill

$$\forall t \geq t_1, \Phi^{\rho'}(\mu', t) \overset{Theorem\ 11\ d)}{=} \Phi^{\rho' \cdot \chi_{(t_1,\infty)}}(\mu'', t),$$

thus

$$\forall t \geq \max\{t', t_1\}, \Phi^\rho(\mu, t) = \Phi^{\rho'}(\mu', t) = \Phi^{\rho' \cdot \chi_{(t_1,\infty)}}(\mu'', t).$$

We infer that $\mu'' \in \overline{W}[\Phi^\rho(\mu,\cdot)]$.

b) We must prove any of the following three equivalent statements:

$$\forall \mu' \in \underline{W}[\Phi^\rho(\mu,\cdot)], \forall \rho' \in P_n, Or_{\rho'}(\mu') \subset \underline{W}[\Phi^\rho(\mu,\cdot)], \tag{37}$$

$$\forall \mu' \in \underline{W}[\Phi^\rho(\mu,\cdot)], \forall \rho' \in P_n, \forall t_1 \in \mathbf{R}, \Phi^{\rho'}(\mu', t_1) \in \underline{W}[\Phi^\rho(\mu,\cdot)], \tag{38}$$

$$\forall \mu' \in \mathbf{B}^n, \forall \rho'' \in P_n, \exists t' \in \mathbf{R}, \forall t \geq t', \Phi^{\rho''}(\mu', t) = \Phi^\rho(\mu, t) \Longrightarrow \tag{39}$$

$$\Longrightarrow \forall \rho' \in P_n, \forall t_1 \in \mathbf{R}, \forall \rho''' \in P_n, \exists t'' \in \mathbf{R},$$

$$\forall t \geq t'', \Phi^{\rho'''}(\Phi^{\rho'}(\mu', t_1), t) = \Phi^\rho(\mu, t).$$

For this, let $\mu' \in \mathbf{B}^n$ be arbitrary, fixed, making the following property true:

$$\forall \rho'' \in P_n, \exists t' \in \mathbf{R}, \forall t \geq t', \Phi^{\rho''}(\mu', t) = \Phi^\rho(\mu, t) \tag{40}$$

and we take $\rho' \in P_n, t_1 \in \mathbf{R}, \rho''' \in P_n$ arbitrarily. The truth of (40) for

$$\rho'' = \rho' \cdot \chi_{(-\infty,t_1]} \oplus \rho''' \cdot \chi_{(t_1,\infty)} \tag{41}$$

shows the existence of $t' \in \mathbf{R}$ that we choose $> t_1$ such that $\forall t \geq t'$,

$$\Phi^\rho(\mu, t) \overset{(40)}{=} \Phi^{\rho''}(\mu', t) \overset{Theorem\ 11\ d)}{=} \Phi^{\rho'' \cdot \chi_{(t_1,\infty)}}(\Phi^{\rho''}(\mu', t_1), t) =$$

$$\overset{(41)}{=} \Phi^{\rho''' \cdot \chi_{(t_1,\infty)}}(\Phi^{\rho''}(\mu', t_1), t) \overset{(41)}{=} \Phi^{\rho''' \cdot \chi_{(t_1,\infty)}}(\Phi^{\rho'}(\mu', t_1), t)$$

$$\overset{Theorem\ 11\ d)}{=} \Phi^{\rho'''}(\Phi^{\rho'}(\mu', t_1), t).$$

(39) holds.

c) We must prove any of the equivalent statements:

$$\forall \mu' \in \underline{W}[\omega_\rho(\mu)], \forall \rho' \in P_n, Or_{\rho'}(\mu') \subset \underline{W}[\omega_\rho(\mu)], \tag{42}$$

$$\forall \mu' \in \underline{W}[\omega_\rho(\mu)], \forall \rho' \in P_n, \forall t_1 \in \mathbf{R}, \Phi^{\rho'}(\mu', t_1) \in \underline{W}[\omega_\rho(\mu)], \tag{43}$$

$$\forall \mu' \in \mathbf{B}^n, \forall \rho'' \in P_n, \omega_{\rho''}(\mu') = \omega_\rho(\mu) \Longrightarrow \tag{44}$$
$$\Longrightarrow \forall \rho' \in P_n, \forall t_1 \in \mathbf{R}, \forall \rho''' \in P_n, \omega_{\rho'''}(\Phi^{\rho'}(\mu', t_1)) = \omega_\rho(\mu).$$

Let $\mu' \in \mathbf{B}^n$ be arbitrary, fixed, that fulfills

$$\forall \rho'' \in P_n, \omega_{\rho''}(\mu') = \omega_\rho(\mu) \tag{45}$$

and we take some arbitrary $\rho' \in P_n, t_1 \in \mathbf{R}, \rho''' \in P_n$. A number t_2 exists with the property $\forall t \leq t_2, \rho'''(t) = 0$. We define $\widetilde{\rho} \in P_n$ by

$$\widetilde{\rho}(t) = \rho'''(t - t_1 + t_2)$$

and we note that (as far as $\forall t \leq t_1, t - t_1 + t_2 \leq t_2$) we have $\widetilde{\rho} \cdot \chi_{(t_1,\infty)}(t) = \rho'''(t - t_1 + t_2) = (\rho''' \circ \tau^{t_1 - t_2})(t)$, thus

$$\omega_{\widetilde{\rho} \cdot \chi_{(t_1,\infty)}}(\Phi^{\rho'}(\mu', t_1)) = \omega_{\rho''' \circ \tau^{t_1 - t_2}}(\Phi^{\rho'}(\mu', t_1)) = \omega_{\rho'''}(\Phi^{\rho'}(\mu', t_1)), \tag{46}$$

see Theorem 24. The truth of (45) for

$$\rho'' = \rho' \cdot \chi_{(-\infty, t_1]} \oplus \widetilde{\rho} \cdot \chi_{(t_1,\infty)} \tag{47}$$

shows that

$$\omega_\rho(\mu) \overset{(45)}{=} \omega_{\rho''}(\mu') \overset{(47)}{=} \omega_{\rho' \cdot \chi_{(-\infty,t_1]} \oplus \widetilde{\rho} \cdot \chi_{(t_1,\infty)}}(\mu') =$$
$$\overset{(2)}{=} \omega_{\widetilde{\rho} \cdot \chi_{(t_1,\infty)}}(\Phi^{\rho'}(\mu', t_1)) \overset{(46)}{=} \omega_{\rho'''}(\Phi^{\rho'}(\mu', t_1)).$$

The statement (44) was proved. ■

Theorem 53 *Let $\mu \in \mathbf{B}^n$ be a fixed point of Φ and $\rho \in P_n$. We have*

$$\overline{W}(\mu) = \overline{W}[\Phi^\rho(\mu, \cdot)] = \overline{W}[\omega_\rho(\mu)],$$
$$\underline{W}(\mu) = \underline{W}[\Phi^\rho(\mu, \cdot)] = \underline{W}[\omega_\rho(\mu)].$$

Proof. Because

$$\forall t \in \mathbf{R}, \Phi^\rho(\mu, t) = \mu,$$
$$\omega_\rho(\mu) = \{\mu\}$$

we get for any $\mu' \in \mathbf{B}^n$ the equivalence of the statements

$$\exists \rho' \in P_n, \omega_{\rho'}(\mu') \subset \{\mu\} \text{ (equation (12))},$$
$$\exists \rho' \in P_n, \exists t' \in \mathbf{R}, \forall t \geq t', \Phi^{\rho'}(\mu', t) = \mu \text{ (see (20))},$$
$$\exists \rho' \in P_n, \omega_{\rho'}(\mu') = \{\mu\} \text{ (see (21))}$$

meaning that $\mu' \in \overline{W}(\mu), \mu' \in \overline{W}[\Phi^\rho(\mu, \cdot)], \mu' \in \overline{W}[\omega_\rho(\mu)]$, thus $\overline{W}(\mu) = \overline{W}[\Phi^\rho(\mu, \cdot)] = \overline{W}[\omega_\rho(\mu)]$. ■

Theorem 54 *Let be* $\mu \in \mathbf{B}^n$ *and* $\rho \in P_n$. *The following statements hold:*

a) $\overline{W}[\Phi^{\rho}(\mu,\cdot)] = \overline{W}(Or_{\rho}(\mu))$;

b) $\underline{W}[\Phi^{\rho}(\mu,\cdot)] \subset \underline{W}(Or_{\rho}(\mu))$;

c) $\overline{W}[\omega_{\rho}(\mu)] = \overline{W}(\omega_{\rho}(\mu))$;

d) $\underline{W}[\omega_{\rho}(\mu)] \subset \underline{W}(\omega_{\rho}(\mu))$.

Proof. a) We prove that $\overline{W}[\Phi^{\rho}(\mu,\cdot)] \subset \overline{W}(Or_{\rho}(\mu))$ and let $\mu' \in \overline{W}[\Phi^{\rho}(\mu,\cdot)]$ be arbitrary, thus $\mu' \in \overline{W}[\omega_{\rho}(\mu)]$ (Theorem 51 a)). We get $\exists \rho' \in P_n, \omega_{\rho'}(\mu') = \omega_{\rho}(\mu) \subset Or_{\rho}(\mu)$ and finally $\mu' \in \overline{W}(Or_{\rho}(\mu))$.

We prove now that $\overline{W}(Or_{\rho}(\mu)) \subset \overline{W}[\Phi^{\rho}(\mu,\cdot)]$. We presume that $\mu' \in \overline{W}(Or_{\rho}(\mu))$, i.e. $\exists \rho' \in P_n, \omega_{\rho'}(\mu') \subset Or_{\rho}(\mu)$. Let $\mu'' \in \omega_{\rho'}(\mu')$ be arbitrary. $t_1 \in \mathbf{R}, t_2 \in \mathbf{R}$ exist then such that

$$\Phi^{\rho'}(\mu', t_1) = \Phi^{\rho}(\mu, t_2) = \mu'' \tag{48}$$

and we define

$$\rho''(t) = \rho'(t - t_2 + t_1) \cdot \chi_{(-\infty, t_2]}(t) \oplus \rho(t) \cdot \chi_{(t_2,\infty)}(t). \tag{49}$$

We note that $\rho'' \in P_n$. We have

$$\forall t \leq t_2, \Phi^{\rho''}(\mu', t) \overset{(49)}{=} \Phi^{\rho' \circ \tau^{t_2 - t_1}}(\mu', t) \overset{Theorem\ 13}{=} \Phi^{\rho'}(\mu', t - t_2 + t_1), \tag{50}$$

$$\Phi^{\rho''}(\mu', t_2) \overset{(50)}{=} \Phi^{\rho'}(\mu', t_1) \overset{(48)}{=} \mu'', \tag{51}$$

thus $\forall t > t_2$,

$$\Phi^{\rho''}(\mu', t) \overset{(51)}{=} \Phi^{\rho'' \cdot \chi_{(t_2,\infty)}}(\mu'', t) \overset{(49)}{=} \Phi^{\rho \cdot \chi_{(t_2,\infty)}}(\mu'', t) = \Phi^{\rho}(\mu, t).$$

We have proved the fact that $\mu' \in \overline{W}[\Phi^{\rho}(\mu,\cdot)]$, thus $\overline{W}(Or_{\rho}(\mu)) \subset \overline{W}[\Phi^{\rho}(\mu,\cdot)]$.

b) Let $\mu' \in \underline{W}[\Phi^{\rho}(\mu,\cdot)]$ be arbitrary, in other words $\exists \mu'' \in \mathbf{B}^n$ such that $\omega_{\rho}(\mu) = \{\mu''\}$ (Theorem 51 d)). We infer that $\mu'' \in Or_{\rho}(\mu)$ and

$$\underline{W}[\Phi^{\rho}(\mu,\cdot)] \overset{Theorem\ 51\ e)}{=} \underline{W}(\mu'') \overset{Theorem\ 32\ ii)}{\subset} \underline{W}(Or_{\rho}(\mu)).$$

c) For any $\mu' \in \overline{W}[\omega_{\rho}(\mu)]$ we have $\exists \rho' \in P_n, \omega_{\rho'}(\mu') = \omega_{\rho}(\mu)$, thus $\exists \rho' \in P_n, \omega_{\rho'}(\mu') \subset \omega_{\rho}(\mu)$ proving that $\mu' \in \overline{W}(\omega_{\rho}(\mu))$ and the conclusion is that

$$\overline{W}[\omega_{\rho}(\mu)] \subset \overline{W}(\omega_{\rho}(\mu)). \tag{52}$$

In order to show that the inclusion (52) takes place under the form of an equality, we presume against all reason that $\mu' \in \overline{W}(\omega_{\rho}(\mu)) \setminus \overline{W}[\omega_{\rho}(\mu)]$ exists, wherefrom we get

$$\exists \rho' \in P_n, \omega_{\rho'}(\mu') \subset \omega_{\rho}(\mu), \tag{53}$$

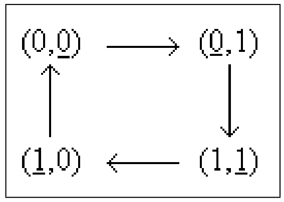

Figure 8: Showing that the inclusion $\underline{W}[\Phi^{\rho}(\mu,\cdot)] \subset \underline{W}(Or_{\rho}(\mu))$ is not equality

$$\forall \rho'' \in P_n, \omega_{\rho''}(\mu') \neq \omega_{\rho}(\mu). \tag{54}$$

From (53), $t_1 \in \mathbf{R}, t_2 \in \mathbf{R}, \mu'' \in \mathbf{B}^n$ exist such that

$$\Phi^{\rho'}(\mu', t_1) = \Phi^{\rho}(\mu, t_2) = \mu'' \tag{55}$$

and we define

$$\rho''(t) = \rho'(t - t_2 + t_1) \cdot \chi_{(-\infty, t_2]}(t) \oplus \rho(t) \cdot \chi_{(t_2, \infty)}(t). \tag{56}$$

Obviously $\rho'' \in P_n$. We have

$$\forall t \leq t_2, \Phi^{\rho''}(\mu', t) \overset{(56)}{=} \Phi^{\rho' \circ \tau^{t_2 - t_1}}(\mu', t) \overset{Theorem\ 13}{=} \Phi^{\rho'}(\mu', t - t_2 + t_1), \tag{57}$$

$$\Phi^{\rho''}(\mu', t_2) \overset{(57)}{=} \Phi^{\rho'}(\mu', t_1) \overset{(55)}{=} \mu'', \tag{58}$$

thus $\forall t > t_2$,

$$\Phi^{\rho''}(\mu', t) \overset{(58)}{=} \Phi^{\rho'' \cdot \chi_{(t_2, \infty)}}(\mu'', t) \overset{(56)}{=} \Phi^{\rho \cdot \chi_{(t_2, \infty)}}(\mu'', t) \overset{Theorem\ 11\ d)}{=} \Phi^{\rho}(\mu, t).$$

We have obtained $\omega_{\rho''}(\mu') = \omega_{\rho}(\mu)$, contradiction with (54).

d) We take an arbitrary $\mu' \in \underline{W}[\omega_{\rho}(\mu)]$. The truth of

$$\forall \rho' \in P_n, \omega_{\rho'}(\mu') = \omega_{\rho}(\mu)$$

implies that

$$\forall \rho' \in P_n, \omega_{\rho'}(\mu') \subset \omega_{\rho}(\mu)$$

is true thus $\mu' \in \underline{W}(\omega_{\rho}(\mu))$. ■

Example 55 *In Figure 8 for any $\rho \in P_2$ we have that $\underline{W}[\Phi^{\rho}((0,0),\cdot)] = \emptyset$, $Or_{\rho}(0,0) = \mathbf{B}^2$, $\underline{W}(\mathbf{B}^2) = \mathbf{B}^2$ and the inclusion from Theorem 54 b) is not equality.*

Example 56 *In Figure 9 we take $\mu = (0,0), \omega_{\rho}(\mu) = \mathbf{B}^2$ so that $\underline{W}(\omega_{\rho}(\mu)) = \underline{W}(\mathbf{B}^2) = \mathbf{B}^2$. On the other hand we can see that $\underline{W}[\omega_{\rho}(\mu)] = \emptyset$, showing that the inclusion from Theorem 54 d) is not equality.*

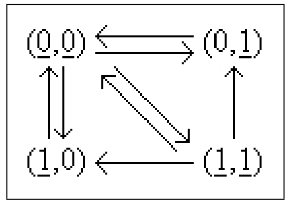

Figure 9: Showing that the inclusion $\underline{W}[\omega_\rho(\mu)] \subset \underline{W}(\omega_\rho(\mu))$ is not equality

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