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UNIVERSAL REGULAR AUTONOMOUS ASYNCHRONOUS SYSTEMS: FIXED POINTS, EQUIVALENCIES AND DYNAMICAL BIFURCATIONS

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Abstract The asynchronous systems are the non-deterministic models of the asynchronous circuits from the digital electrical engineering. In the autonomous version, such a system is a set of functions $x : \mathbf{R} \to \{0,1\}^n$ called states ($\mathbf{R}$ is the time set). If an autonomous asynchronous system is defined by making use of a so called generator function $\Phi : \{0,1\}^n \to \{0,1\}^n$, then it is called regular. The regular autonomous asynchronous systems compute in real time the iterates of Φ when these are not made, in general, on all the coordinates $\Phi_1, ..., \Phi_n$ simultaneosly. The property of universality means the greatest in the sense of the inclusion.

The purpose of the paper is that of defining and of characterizing the fixed points, the equivalencies and the dynamical bifurcations of the universal regular autonomous asynchronous systems. We use analogies with the dynamical systems theory.

Keywords: asynchronous system, fixed point, dynamical bifurcation.

2000 MSC: 94C10, 93A30, 47H10, 37G99.

1. INTRODUCTION

Switching theory, more precisely: what we mean by switching theory, has been practiced in the 50's and the 60's by many mathematicians, in dialogue with engineers. The last book from this series was published by Moisil in 1969 **??**. After 1970, the theory of modeling the asynchronous circuits from the digital electrical engineering has developed in a manner suggesting that the main interest of the researchers is to keep away their works from publication. In this context, we have started some years ago the construction of a theory of modeling the asynchronous circuits under the name of asynchronous systems theory. A part of this theory, related with the universal regular autonomous asynchronous systems is presented in this paper. The bibliography that we indicate consists in works on dynamical systems (written as usual on real numbers, we use binary numbers here) that create analogies. They are not relevant to the readers that are familiar with the concepts of orbit, nullclin, dynamical bifurcation etc, except for showing the source of inspiration of the construction. The paper is obviously self-contained.

2. PRELIMINARIES

Definition 2.1. *We denote by* $\mathbf{B} = \{0, 1\}$ *the* ***binary Boole algebra****, endowed with the discrete topology and with the usual laws.*

Definition 2.2. *Let be the Boolean function* $\Phi : \mathbf{B}^n \to \mathbf{B}^n, \Phi = (\Phi_1, ..., \Phi_n)$ *and* $\nu \in \mathbf{B}^n, \nu = (\nu_1, ..., \nu_n)$. *We define* $\Phi^\nu : \mathbf{B}^n \to \mathbf{B}^n$ *by* $\forall \mu \in \mathbf{B}^n$,

$$\Phi^\nu(\mu) = (\overline{\nu_1} \cdot \mu_1 \oplus \nu_1 \cdot \Phi_1(\mu), ..., \overline{\nu_n} \cdot \mu_n \oplus \nu_n \cdot \Phi_n(\mu)).$$

Remark 1. Φ^ν *represents the function resulting from* Φ *when this one is not computed, in general, on all the coordinates* $\Phi_i, i = \overline{1, n}$: *if* $\nu_i = 0$, *then* Φ_i *is not computed,* $\Phi_i^\nu(\mu) = \mu_i$ *and if* $\nu_i = 1$, *then* Φ_i *is computed,* $\Phi_i^\nu(\mu) = \Phi_i(\mu)$.

Definition 2.3. *Let be the sequence* $\alpha^0, \alpha^1, ..., \alpha^k, ... \in \mathbf{B}^n$. *The functions* $\Phi^{\alpha^0\alpha^1...\alpha^k} : \mathbf{B}^n \to \mathbf{B}^n$ *are defined iteratively by* $\forall \mu \in \mathbf{B}^n, \forall k \in \mathbf{N}$,

$$\Phi^{\alpha^0\alpha^1...\alpha^k\alpha^{k+1}}(\mu) = \Phi^{\alpha^{k+1}}(\Phi^{\alpha^0\alpha^1...\alpha^k}(\mu)).$$

Definition 2.4. *The sequence* $\alpha^0, \alpha^1, ..., \alpha^k, ... \in \mathbf{B}^n$ *is called* ***progressive*** *if*

$$\forall i \in \{1, ..., n\}, \text{ the set } \{k | k \in \mathbf{N}, \alpha_i^k = 1\} \text{ is infinite.}$$

The set of the progressive sequences is denoted by Π_n.

Remark 2.1. *Let be* $\mu \in \mathbf{B}^n$. *When* $\alpha = \alpha^0, \alpha^1, ..., \alpha^k, ...$ *is progressive, each coordinate* $\Phi_i, i = \overline{1, n}$ *is computed infinitely many times in the sequence* $\Phi^{\alpha^0\alpha^1...\alpha^k}(\mu)$, $k \in \mathbf{N}$.

Definition 2.5. *The* ***initial value****, denoted by* $x(-\infty + 0)$ *or* $\lim\limits_{t \to -\infty} x(t) \in \mathbf{B}^n$ *and the* ***final value****, denoted by* $x(\infty - 0)$ *or* $\lim\limits_{t \to \infty} x(t) \in \mathbf{B}^n$ *of the function*

$x : \mathbf{R} \to \mathbf{B}^n$ *are defined by*

$$\exists t' \in \mathbf{R}, \forall t < t', x(t) = x(-\infty + 0),$$

$$\exists t' \in \mathbf{R}, \forall t > t', x(t) = x(\infty - 0).$$

Definition 2.6. *The function* $x : \mathbf{R} \to \mathbf{B}^n$ *is called (**pseudo**)**periodical with the period*** $T_0 > 0$ *if*

a) $\lim\limits_{t \to \infty} x(t)$ *does not exist and*
b) $\exists t' \in \mathbf{R}, \forall t \geq t', x(t) = x(t + T_0)$.

Definition 2.7. *The **characteristic function*** $\chi_A : \mathbf{R} \to \mathbf{B}$ *of the set* $A \subset \mathbf{R}$ *is defined in the following way:*

$$\chi_A(t) = \begin{cases} 1, \ if \ t \in A \\ 0, otherwise \end{cases}.$$

Notation 2.1. *We denote by Seq the set of the real sequences* $t_0 < t_1 < ... < t_k < ...$ *which are unbounded from above.*

Remark 2.2. *The sequences* $(t_k) \in Seq$ *act as time sets. At this level of generality of the exposure, a double uncertainty exists in the real time iterative computations of the function* $\Phi : \mathbf{B}^n \to \mathbf{B}^n$ *: we do not know precisely neither the coordinates* Φ_i *of* Φ *that are computed, nor when the computation happens. This uncertainty implies the non-determinism of the model and its origin consists in structural fluctuations in the fabrication process, the variations in ambiental temperature and the power supply etc.*

Definition 1. *A **signal** (or* $n-$***signal****) is a function* $x : \mathbf{R} \to \mathbf{B}^n$ *of the form*

$$x(t) = x(-\infty + 0) \cdot \chi_{(-\infty, t_0)}(t) \oplus x(t_0) \cdot \chi_{[t_0, t_1)}(t) \oplus ... \quad (1)$$

$$... \oplus x(t_k) \cdot \chi_{[t_k, t_{k+1})}(t) \oplus ...$$

with $(t_k) \in Seq$*. The set of the signals is denoted by* $S^{(n)}$*.*

Remark 2. *The signals* $x \in S^{(n)}$ *model the electrical signals from the digital electrical engineering. They have by definition initial values and they avoid 'Dirichlet type' properties (called Zeno properties by the engineers) such as*

$$\exists t \in \mathbf{R}, \forall \varepsilon > 0, \exists t' \in (t - \varepsilon, t), \exists t'' \in (t - \varepsilon, t), x(t') \neq x(t''),$$

$$\exists t \in \mathbf{R}, \forall \varepsilon > 0, \exists t' \in (t, t+\varepsilon), \exists t'' \in (t, t+\varepsilon), x(t') \neq x(t'')$$

because these properties cannot characterize the inertial devices.

We can interpret now Definition 2.6 of (pseudo)periodicity in the situation when $x \in S^{(n)}$. If at b) we would have had $\forall t \in \mathbf{R}, x(t) = x(t+T_0)$, then the existence of $x(-\infty+0)$ implies that x is constant. Similarly, if a) would be false, then x would be constant. In other words Definition 2.6 was formulated in a way that makes us work with non-constant functions, a request of non-triviality.

Notation 2.2. *We denote by P^* the set of the non-empty subsets of a set.*

Definition 2.8. *The **autonomous asynchronous systems** are the non-empty sets $X \in P^*(S^{(n)})$.*

Example 2.1. *We give the following simple example that shows how the autonomous asynchronous systems model the asynchronous circuits. In Figure 1 we have drawn the (logical) gate NOT with the input $u \in S^{(1)}$ and the state*

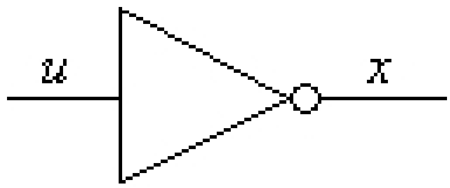

Fig. 1. Circuit with the logical gate NOT.

(the output) $x \in S^{(1)}$. For $\lambda \in \mathbf{B}$ and

$$u(t) = \lambda,$$

the state x represents the computation of the negation of u and it is of the form

$$x(t) = \mu \cdot \chi_{(-\infty,t_0)}(t) \oplus \overline{\lambda} \cdot \chi_{[t_0,t_1)}(t) \oplus \overline{\lambda} \cdot \chi_{[t_1,t_2)}(t) \oplus ... \oplus \overline{\lambda} \cdot \chi_{[t_k,t_{k+1})}(t) \oplus ...$$

$$= \mu \cdot \chi_{(-\infty,t_0)}(t) \oplus \overline{\lambda} \cdot \chi_{[t_0,\infty)}(t),$$

where $\mu \in \mathbf{B}$ is the initial value of x and $(t_k) \in Seq$ is arbitrary. As we can see, x depends on t_0, μ, λ only and it is independent on $t_1, t_2, ...$

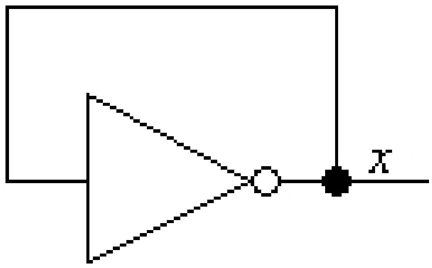

Fig. 2. Circuit with feedback with the logical gate NOT.

In Figure 2, we have

$$x(t) = \mu \cdot \chi_{(-\infty,t_0)}(t) \oplus \overline{\mu} \cdot \chi_{[t_0,t_1)}(t) \oplus \mu \cdot \chi_{[t_1,t_2)}(t) \oplus ...$$

$$\oplus \overline{\mu} \cdot \chi_{[t_{2k},t_{2k+1})}(t) \oplus \mu \cdot \chi_{[t_{2k+1},t_{2k+2})}(t) \oplus ...$$

thus this circuit is modeled by the autonomous asynchronous system

$$X = \{\mu \cdot \chi_{(-\infty,t_0)}(t) \oplus \overline{\mu} \cdot \chi_{[t_0,t_1)}(t) \oplus \mu \cdot \chi_{[t_1,t_2)}(t) \oplus ...$$

$$\oplus \overline{\mu} \cdot \chi_{[t_{2k},t_{2k+1})}(t) \oplus \mu \cdot \chi_{[t_{2k+1},t_{2k+2})}(t) \oplus ... | \mu \in \mathbf{B}, (t_k) \in Seq\} \in P^*(S^{(1)}).$$

Definition 2.9. *The **progressive functions** $\rho : \mathbf{R} \to \mathbf{B}^n$ are by definition the functions*

$$\rho(t) = \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus \alpha^1 \cdot \chi_{\{t_1\}}(t) \oplus ... \oplus \alpha^k \cdot \chi_{\{t_k\}}(t) \oplus ... \qquad (2)$$

where $(t_k) \in Seq$ and $\alpha^0, \alpha^1, ..., \alpha^k, ... \in \Pi_n$. The set of the progressive functions is denoted by P_n.

Definition 2.10. *For $\Phi : \mathbf{B}^n \to \mathbf{B}^n$ and $\rho \in P_n$ like at (2), we define $\Phi^\rho : \mathbf{B}^n \times \mathbf{R} \to \mathbf{B}^n$ by $\forall \mu \in \mathbf{B}^n, \forall t \in \mathbf{R}$,*

$$\Phi^\rho(\mu, t) = \mu \cdot \chi_{(-\infty,t_0)}(t) \oplus \Phi^{\alpha^0}(\mu) \cdot \chi_{[t_0,t_1)}(t) \oplus ... \oplus \Phi^{\alpha^0 ... \alpha^k}(\mu) \cdot \chi_{[t_k,t_{k+1})}(t) \oplus ...$$

Remark 2.3. *The previous equation reminds the iterations of a discrete time real dynamical system. The time is not exactly discrete in it, but some sort of intermediate situation occurs between the discrete and the real time; on the other hand the iterations of Φ do not happen in general on all the coordinates (synchronicity), but on some coordinates only, such that any coordinate Φ_i is* *computed infinitely many times,* $i = \overline{1,n}$ *(asynchronicity) when* $t \in \mathbf{R}$*. This is the meaning of the progress property, giving the so called 'unbounded delay model' of computation of the Boolean functions.*

3. DISCRETE TIME

Notation 3.1. *We denote by*

$$\mathbf{N}_{\_} = \mathbf{N} \cup \{-1\}$$

the discrete time set.

Definition 3.1. *Let be* $\Phi : \mathbf{B}^n \to \mathbf{B}^n$ *and* $\alpha \in \Pi_n, \alpha = \alpha^0, ..., \alpha^k, ...$ *We define the function* $\widehat{\Phi}^\alpha : \mathbf{B}^n \times \mathbf{N}_{\_} \to \mathbf{B}^n$ *by* $\forall(\mu, k) \in \mathbf{B}^n \times \mathbf{N}_{\_}$,

$$\widehat{\Phi}^\alpha(\mu, k) = \begin{cases} \mu, k = -1, \\ \Phi^{\alpha^0 ... \alpha^k}(\mu), k \geq 0 \end{cases}.$$

Notation 3.2. *Let us denote*

$$\widehat{\Pi}_n = \{\alpha | \alpha \in \Pi_n, \forall k \in \mathbf{N}, \alpha^k \neq (0, ..., 0)\}.$$

Definition 3.2. *The equivalence of* $\rho, \rho' \in P_n$ *is defined by:* $\exists(t_k) \in Seq, \exists(t'_k) \in Seq, \exists \alpha \in \widehat{\Pi}_n$ *such that (2) and*

$$\rho'(t) = \alpha^0 \cdot \chi_{\{t'_0\}}(t) \oplus \alpha^1 \cdot \chi_{\{t'_1\}}(t) \oplus ... \oplus \alpha^k \cdot \chi_{\{t'_k\}}(t) \oplus ...$$

are true.

Definition 3.3. *The 'canonical surjection'* $s : P_n \to \widehat{\Pi}_n$ *is by definition the function* $\forall \rho \in P_n$,

$$s(\rho) = \alpha$$

where $\alpha \in \widehat{\Pi}_n$ *is the only sequence such that* $(t_k) \in Seq$ *exists, making the equation (2) true.*

Remark 3.1. *The relation between the continuous and the discrete time is the following: for any* $\mu \in \mathbf{B}^n$ *and any* $\rho \in P_n$*, the sequences* $\alpha \in \widehat{\Pi}_n$ *and* $(t_k) \in Seq$ *exist making the equation (2) true and we have*

$$\Phi^\rho(\mu, t) = \widehat{\Phi}^\alpha(\mu, -1) \cdot \chi_{(-\infty, t_0)}(t) \oplus \widehat{\Phi}^\alpha(\mu, 0) \cdot \chi_{[t_0, t_1)}(t) \oplus ...$$

$$... \oplus \widehat{\Phi}^{\alpha}(\mu, k) \cdot \chi_{[t_k, t_{k+1})}(t) \oplus ...$$

Equivalent progressive functions $\rho, \rho' \in P_n$ (i.e. $s(\rho) = s(\rho')$) give 'equivalent' functions $\Phi^{\rho}(\mu, t), \Phi^{\rho'}(\mu, t)$ in the sense that the computations $\widehat{\Phi}^{\alpha}(\mu, k), k \in \mathbf{N}_-$ are the same $\forall \mu \in \mathbf{B}^n$, but the time flow is piecewise faster or slower in the two situations.

4. REGULAR AUTONOMOUS ASYNCHRONOUS SYSTEMS

Definition 4.1. *The **universal regular autonomous asynchronous system** $\Xi_{\Phi} \in P^*(S^{(n)})$ that is generated by the function $\Phi : \mathbf{B}^n \to \mathbf{B}^n$ is defined by*

$$\Xi_{\Phi} = \{\Phi^{\rho}(\mu, \cdot) | \mu \in \mathbf{B}^n, \rho \in P_n\}.$$

Definition 4.2. *An autonomous asynchronous system $X \in P^*(S^{(n)})$ is called **regular**, if Φ exists such that $X \subset \Xi_{\Phi}$. In this case Φ is called the **generator function** of X.*

Remark 4.1. ***1.*** *The terminology of 'generator function' is also used in [1], meaning the vector field of a discrete time dynamical system. In [3] the terminology of 'generator' (function) of a dynamical system is mentioned too. Moisil called Φ 'network function' in a non-autonomous, discrete time context; for Moisil, 'network' means 'system' or 'circuit'.*

2. *In the last two definitions, the attribute 'regular' refers to the existence of a generator function Φ and the attribute 'universal' means maximal relative to the inclusion.*

For a regular system, Φ is not unique in general.

Example 4.1. *For any $\mu^0 \in \mathbf{B}^n$ and $\rho^* \in P_n$, the autonomous systems $\{\Phi^{\rho^*}(\mu^0, \cdot)\}$, $\{\Phi^{\rho}(\mu^0, \cdot) | \rho \in P_n\}$, $\{\Phi^{\rho^*}(\mu, \cdot) | \mu \in \mathbf{B}^n\}$ and Ξ_{Φ} are regular.*

For $\Phi = 1_{\mathbf{B}^n}$, the system $\Xi_{1_{\mathbf{B}^n}} = \{\mu | \mu \in \mathbf{B}^n\} = \mathbf{B}^n$ is regular and we have identified the constant function $x \in S^{(n)}, x(t) = \mu$ with the constant $\mu \in \mathbf{B}^n$.

Another example of universal regular autonomous asynchronous system is given by $\Phi = \mu^0$, the constant function, for which $\Xi_{\mu^0} = \{x | x_i = \mu_i \cdot \chi_{(-\infty, t_i)} \oplus \mu_i^0 \cdot \chi_{[t_i, \infty)}, \mu_i \in \mathbf{B}, t_i \in \mathbf{R}, i = \overline{1, n}\}$.

Remark 4.2. *These examples suggest several possibilities of defining the systems* $X \subset \Xi_\Phi$ *which are not universal. For example by putting appropriate supplementary requests on the functions* ρ*, one could rediscover the 'bounded delay model' of computation of the Boolean functions. If* ρ *is fixed, we get the 'fixed delay model' of computation of the Boolean functions.*

5. ORBITS AND STATE PORTRAITS

Definition 5.1. *Let be* $\rho \in P_n$*. Two things are understood by* ***orbit****, or (****state****, or* ***phase****)* ***trajectory of*** Ξ_Φ ***starting at*** $\mu \in \mathbf{B}^n$*:*

a) the function $\Phi^\rho(\mu, \cdot) : \mathbf{R} \to \mathbf{B}^n$;

b) the set $Or_\rho(\mu) = \{\Phi^\rho(\mu, t) | t \in \mathbf{R}\}$ *representing the values of the previous function.*

Sometimes the function from a) is called the ***motion*** *(or the* ***dynamic****) of* μ *through* Φ^ρ.

Definition 5.2. *The equivalent properties*

$$\exists t \in \mathbf{R}, \Phi^\rho(\mu, t) = \mu'$$

and

$$\mu' \in Or_\rho(\mu)$$

are called of ***accessibility****; the points* $\mu' \in Or_\rho(\mu)$ *are said to be* ***accessible****.*

Remark 5.1. *The orbits are the curves in* $\mathbf{B}^n$*, parametrized by* ρ *and* t*. On the other hand* $\rho \in P_n$*,* $t' \in \mathbf{R}$ *imply* $\rho \cdot \chi_{(t',\infty)} \in P_n$ *and we see the truth of the implication*

$$\mu' = \Phi^\rho(\mu, t') \Longrightarrow \forall t \geq t', \Phi^\rho(\mu, t) = \Phi^{\rho \cdot \chi_{(t',\infty)}}(\mu', t).$$

Definition 5.3. *The* ***state*** *(or the* ***phase****)* ***portrait*** *of* Ξ_Φ *is the set of its orbits.*

Example 5.1. *The function* $\Phi : \mathbf{B}^2 \to \mathbf{B}^2$ *is defined by the following table*

(μ_1, μ_2)	$\Phi(\mu_1, \mu_2)$
$(0,0)$	$(0,0)$
$(0,1)$	$(1,0)$
$(1,0)$	$(1,1)$
$(1,1)$	$(1,1)$

The state portrait of Ξ_Φ *is:*

$$\{(0,1)\cdot\chi_{(-\infty,t_0)} \oplus (0,0)\cdot\chi_{[t_0,\infty)} | t_0 \in \mathbf{R}\}\cup$$
$$\cup\{(0,1)\cdot\chi_{(-\infty,t_0)} \oplus (1,0)\cdot\chi_{[t_0,t_1)} \oplus (1,1)\cdot\chi_{[t_1,\infty)} | t_0, t_1 \in \mathbf{R}, t_0 < t_1\}\cup$$
$$\cup\{(0,1)\cdot\chi_{(-\infty,t_0)} \oplus (1,1)\cdot\chi_{[t_0,\infty)} | t_0 \in \mathbf{R}\}\cup$$
$$\cup\{(1,0)\cdot\chi_{(-\infty,t_0)} \oplus (1,1)\cdot\chi_{[t_0,\infty)} | t_0 \in \mathbf{R}\} \cup \{(0,0)\} \cup \{(1,1)\}.$$

This set is drawn in Figure 3,

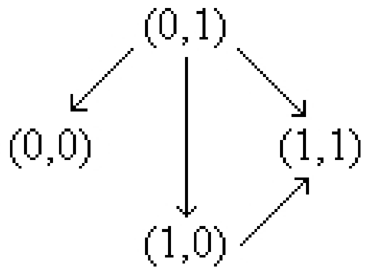

Fig. 3. The state portrait of the system from Example 5.1.

where the arrows show the increase of time. One might want to put arrows from $(0,0)$ *to itself and from* $(1,1)$ *to itself.*

6. NULLCLINS

Definition 6.1. *Let be* $\Phi : \mathbf{B}^n \to \mathbf{B}^n$. *For any* $i \in \{1, ..., n\}$, *the* ***nullclins*** *of* Φ *are the sets*

$$NC_i = \{\mu | \mu \in \mathbf{B}^n, \Phi_i(\mu) = \mu_i\}.$$

If $\mu \in NC_i$, *then the coordinate* i *is said to be* ***not excited****, or* ***not enabled****, or* ***stable*** *and if* $\mu \in \mathbf{B}^n \setminus NC_i$ *then it is called* ***excited****, or* ***enabled****, or* ***unstable****.*

Remark 6.1. *Sometimes, instead of indicating* Φ *by a table like previously, we can replace Figure 3 by Figure 4,*

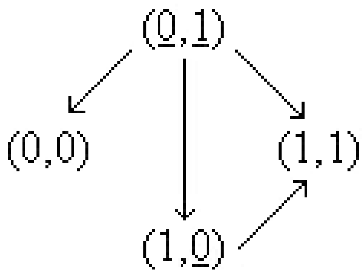

Fig. 4. The state portrait of the system from Example 5.1, version.

where we have underlined the unstable coordinates. For example in Figure 4, $(\underline{0},\underline{1})$ *means that* $\Phi(0,1) = (1,0)$, $(1,\underline{0})$ *means that* $\Phi(1,0) = (1,1)$ *etc.*

In fact Figure 4 results uniquely from Figure 3, one could know by looking at Figure 3 which coordinates should be underlined and which should be not. For example the existence of an arrow from $(0,1)$ *to* $(1,0)$ *shows that in* $(0,1)$ *both coordinates are enabled.*

7. FIXED POINTS. REST POSITION

Definition 7.1. *A point* $\mu \in \mathbf{B}^n$ *that fulfills* $\Phi(\mu) = \mu$ *is called a* ***fixed point*** *(an* ***equilibrium point****, a* ***critical point****, a* ***singular point****), shortly an* ***equilibrium*** *of* Φ. *A point that is not fixed is called* ***ordinary****.*

Theorem 7.1. *The following statements are equivalent for* $\mu \in \mathbf{B}^n$:

$$\Phi(\mu) = \mu, \tag{3}$$

$$\exists \rho \in P_n, \forall t \in \mathbf{R}, \Phi^{\rho}(\mu, t) = \mu, \tag{4}$$

$$\forall \rho \in P_n, \forall t \in \mathbf{R}, \Phi^{\rho}(\mu, t) = \mu, \tag{5}$$

$$\exists \rho \in P_n, Or_{\rho}(\mu) = \{\mu\}, \tag{6}$$

$$\forall \rho \in P_n, Or_{\rho}(\mu) = \{\mu\}, \tag{7}$$

$$\mu \in NC_1 \cap ... \cap NC_n. \tag{8}$$

Proof. (3)$\Longrightarrow$(4) We take $\rho \in P_n$ in the following way

$$\rho(t) = (1, ..., 1) \cdot \chi_{\{t_0\}}(t) \oplus ... \oplus (1, ..., 1) \cdot \chi_{\{t_k\}}(t) \oplus ...$$

with $(t_k) \in Seq$. For the sequence

$$\forall k \in \mathbf{N}, \alpha^k = (1, ..., 1)$$

from Π_n we can prove by induction on k that

$$\forall k \in \mathbf{N}, \Phi^{\alpha^0 ... \alpha^k}(\mu) = \mu \tag{9}$$

wherefrom

$$\Phi^\rho(\mu, t) = \mu \cdot \chi_{(-\infty, t_0)}(t) \oplus \mu \cdot \chi_{[t_0, t_1)}(t) \oplus ... \oplus \mu \cdot \chi_{[t_k, t_{k+1})}(t) \oplus ... = \mu \tag{10}$$

(4)$\Longrightarrow$(3) From (4) we have the existence of $\alpha \in \Pi_n$ and $(t_k) \in Seq$ with the property that (10) is true, thus (9) is true. We denote

$$I_0 = \{i | i \in \{1, ..., n\}, \alpha_i^0 = 1\},$$

$$I_1 = \{i | i \in \{1, ..., n\}, \alpha_i^1 = 1\},$$

$$...$$

$$I_k = \{i | i \in \{1, ..., n\}, \alpha_i^k = 1\},$$

$$...$$

and we have from (9):

$\forall i \in \{1, .., n\}$,

$$\Phi_i^{\alpha^0}(\mu) = \begin{cases} \Phi_i(\mu), i \in I_0 \\ \mu_i, i \in \{1, ..., n\} \setminus I_0 \end{cases} = \mu_i;$$

$\forall i \in \{1, .., n\}, \Phi_i^{\alpha^0 \alpha^1}(\mu) = \Phi_i^{\alpha^1}(\Phi^{\alpha^0}(\mu)) =$

$$= \Phi_i^{\alpha^1}(\mu) = \begin{cases} \Phi_i(\mu), i \in I_1 \\ \mu_i, i \in \{1, ..., n\} \setminus I_1 \end{cases} = \mu_i;$$

$$...$$

$\forall i \in \{1, .., n\}, \Phi_i^{\alpha^0 \alpha^1 ... \alpha^k}(\mu) = \Phi_i^{\alpha^k}(\Phi^{\alpha^0 ... \alpha^{k-1}}(\mu)) =$

$$= \Phi_i^{\alpha^k}(\mu) = \begin{cases} \Phi_i(\mu), i \in I_k \\ \mu_i, i \in \{1, ..., n\} \setminus I_k \end{cases} = \mu_i;$$

...

with the conclusion that

$$\forall k \in \mathbf{N}, \forall i \in I_0 \cup I_1 \cup ... \cup I_k, \Phi_i(\mu) = \mu_i.$$

As α is progressive, some $k' \in \mathbf{N}$ exists with the property that

$$I_0 \cup I_1 \cup ... \cup I_{k'} = \{1, ..., n\},$$

thus (3) is true.

(3)$\Longrightarrow$(5) Let be

$$\rho(t) = \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus ... \oplus \alpha^k \cdot \chi_{\{t_k\}}(t) \oplus ... \tag{11}$$

with $\alpha^0, ..., \alpha^k, ... \in \Pi_n$ and $(t_k) \in Seq$ arbitrary. It is proved by induction on k the validity of (9) and this implies the truth of (10).

(5)$\Longrightarrow$(3) This is true because (5)$\Longrightarrow$(4) and (4)$\Longrightarrow$(3) are true.

(4)$\Longleftrightarrow$(6) and (5)$\Longleftrightarrow$(7) are obvious.

(3)$\Longleftrightarrow$(8) $\Phi(\mu) = \mu \Longleftrightarrow \Phi_1(\mu) = \mu_1$ and...and $\Phi_n(\mu) = \mu_n \Longleftrightarrow \mu \in NC_1$ and...and $\mu \in NC_n \Longleftrightarrow \mu \in NC_1 \cap ... \cap NC_n$. ∎

Definition 2. *If $\Phi(\mu) = \mu$, then $\forall \rho \in P_n$, the orbit $\Phi^\rho(\mu, t) = \mu$ is called* ***rest position****.*

8. FIXED POINTS VS. FINAL VALUES OF THE ORBITS

Theorem 8.1. *([8], Theorem 49) The following fixed point property is true*

$$\forall \mu \in \mathbf{B}^n, \forall \mu' \in \mathbf{B}^n, \forall \rho \in P_n, \lim_{t \to \infty} \Phi^\rho(\mu, t) = \mu' \Longrightarrow \Phi(\mu') = \mu'.$$

Proof. Let $\mu \in \mathbf{B}^n, \mu' \in \mathbf{B}^n, \rho \in P_n$ be arbitrary and fixed. Some $t' \in \mathbf{R}$ exists such that $\forall t \geq t'$,

$$\mu' = \Phi^\rho(\mu, t) \overset{\text{Remark 5.1}}{=} \Phi^{\rho \cdot \chi_{(t', \infty)}}(\mu', t).$$

Because $\forall t < t'$,

$$\Phi^{\rho \cdot \chi_{(t', \infty)}}(\mu', t) = \Phi^{(0, ..., 0)}(\mu', t) = \mu',$$

from Theorem 7.1, (4)$\Longrightarrow$(3) we have that $\Phi(\mu') = \mu'$. ∎

Remark 3. *Theorem 8.1 shows that the final values of the states of the system* Ξ_Φ *are fixed points of* Φ*.*

Theorem 8.2. *([8], Theorem 50) We have* $\forall \mu \in \mathbf{B}^n, \forall \mu' \in \mathbf{B}^n, \forall \rho \in P_n$,

$$(\Phi(\mu') = \mu' \text{ and } \exists t' \in \mathbf{R}, \Phi^\rho(\mu, t') = \mu') \Longrightarrow \forall t \geq t', \Phi^\rho(\mu, t) = \mu'.$$

Proof. For arbitrary $\mu \in \mathbf{B}^n, \mu' \in \mathbf{B}^n, \rho \in P_n$ we suppose that $\Phi(\mu') = \mu'$ and $\Phi^\rho(\mu, t') = \mu'$. We have $\forall t \geq t'$,

$$\Phi^\rho(\mu, t) \overset{\text{Remark 5.1}}{=} \Phi^{\rho \cdot \chi_{(t',\infty)}}(\mu', t) \overset{\text{Theorem 7.1, (3)}\Longrightarrow\text{(5)}}{=} \mu'.$$

∎

Remark 4. *As resulting from Theorem 8.2, the accessible fixed points are final values of the states of the system* Ξ_Φ*.*

The properties of the fixed points that are expressed by Theorems 7.1, 8.1, 8.2 give a better understanding of Example 5.1.

9. TRANSITIVITY

Definition 9.1. *The system* Ξ_Φ *(or the function* Φ*) is* ***transitive****, or* ***minimal*** *if one of the following non-equivalent properties holds true:*

$$\forall \mu \in \mathbf{B}^n, \forall \mu' \in \mathbf{B}^n, \exists \rho \in P_n, \exists t \in \mathbf{R}, \Phi^\rho(\mu, t) = \mu', \tag{12}$$

$$\forall \mu \in \mathbf{B}^n, \forall \mu' \in \mathbf{B}^n, \forall \rho \in P_n, \exists t \in \mathbf{R}, \Phi^\rho(\mu, t) = \mu'. \tag{13}$$

Remark 9.1. *The property of transitivity may be considered one of surjectivity or one of accessibility.*

If Φ *is transitive, then it has no fixed points. For example* $1_{\mathbf{B}^n}$ *is not transitive since all* $\mu \in \mathbf{B}^n$ *are fixed points for this function.*

Example 9.1. *The property (12) of transitivity is exemplified in Figure 5 and the property (13) of transitivity is exemplified in Figure 6.*

10. THE EQUIVALENCE OF THE SYSTEMS

Notation 10.1. *Let* $h : \mathbf{B}^n \to \mathbf{B}^n$ *and* $x : \mathbf{R} \to \mathbf{B}^n$ *be some functions. We denote by* $h(x) : \mathbf{R} \to \mathbf{B}^n$ *the function*

$$\forall t \in \mathbf{R}, h(x)(t) = h(x(t)).$$

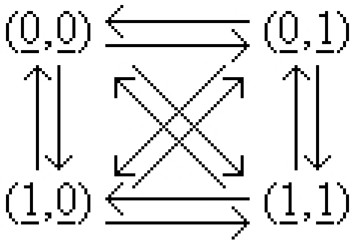

Fig. 5. Transitivity.

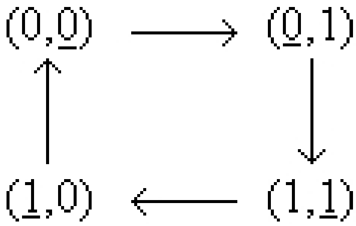

Fig. 6. Transitivity.

Remark 10.1. *If* $h : \mathbf{B}^n \to \mathbf{B}^n$ *and* $x \in S^{(n)}$ *is expressed by*

$$x(t) = x(-\infty + 0) \cdot \chi_{(-\infty,t_0)}(t) \oplus x(t_0) \cdot \chi_{[t_0,t_1)}(t) \oplus ... \oplus x(t_k) \cdot \chi_{[t_k,t_{k+1})}(t) \oplus ...$$

then

$$h(x)(t) = h(x(-\infty + 0)) \cdot \chi_{(-\infty,t_0)}(t) \oplus h(x(t_0)) \cdot \chi_{[t_0,t_1)}(t) \oplus ...$$

$$... \oplus h(x(t_k)) \cdot \chi_{[t_k,t_{k+1})}(t) \oplus ...$$

Notation 10.2. *For* $h : \mathbf{B}^n \to \mathbf{B}^n$ *and* $\alpha = \alpha^0, ..., \alpha^k, ... \in \mathbf{B}^n$*, we denote by* $\widehat{h}(\alpha)$ *the sequence* $h(\alpha^0), ..., h(\alpha^k), ... \in \mathbf{B}^n$.

Notation 10.3. *Let be* $k \geq 2$ *arbitrary and we denote for* $\mu^1, ..., \mu^k \in \mathbf{B}^n$,

$$\mu^1 \cup ... \cup \mu^k = (\mu_1^1 \cup ... \cup \mu_1^k, ..., \mu_n^1 \cup ... \cup \mu_n^k).$$

Notation 10.4. *We denote by* Ω_n *the set of the functions* $h : \mathbf{B}^n \to \mathbf{B}^n$ *that fulfill*

i) h *is bijective;*

ii) $h(0, ..., 0) = (0, ..., 0),\ h(1, ..., 1) = (1, ..., 1);$

iii) $\forall k \geq 2, \forall \mu^1 \in \mathbf{B}^n, ..., \forall \mu^k \in \mathbf{B}^n$,

$$\mu^1 \cup ... \cup \mu^k = (1, ..., 1) \Longleftrightarrow h(\mu^1) \cup ... \cup h(\mu^k) = (1, ..., 1).$$

Theorem 10.1. *a)* Ω_n *is group relative to the composition* $'\circ'$ *of the functions;*

b) $\forall h \in \Omega_n, \forall \alpha \in \Pi_n, \widehat{h}(\alpha) \in \Pi_n$;

c) $\forall h \in \Omega_n, \forall \rho \in P_n, h(\rho) \in P_n$.

Proof. a) We can prove the fact that $1_{\mathbf{B}^n} \in \Omega_n$, $\forall h \in \Omega_n, \forall h' \in \Omega_n, h \circ h' \in \Omega_n$ and $\forall h \in \Omega_n, h^{-1} \in \Omega_n$. For example let be $h \in \Omega_n, k \geq 2$ and $\nu^1, ..., \nu^k \in \mathbf{B}^n$ arbitrary, for which we denote $\mu^1 = h^{-1}(\nu^1), ..., \mu^k = h^{-1}(\nu^k)$. We have:

$$h^{-1}(\nu^1 \cup ... \cup \nu^k) = (1, ..., 1) \Longleftrightarrow \nu^1 \cup ... \cup \nu^k = h(1, ..., 1) = (1, ..., 1)$$

$$\Longleftrightarrow h(\mu^1) \cup ... \cup h(\mu^k) = (1, ..., 1) \Longleftrightarrow \mu^1 \cup ... \cup \mu^k = (1, ..., 1)$$

$$\Longleftrightarrow h^{-1}(\nu^1) \cup ... \cup h^{-1}(\nu^k) = (1, ..., 1),$$

thus h^{-1} fulfills iii) from Notation 10.4.

b) Let $h \in \Omega_n$ and $\alpha = \alpha^0, ..., \alpha^k, ... \in \mathbf{B}^n$ be arbitrary. We denote for $p \geq 1$

$$\{\mu^1, ..., \mu^p\} = \{\mu | \mu \in \mathbf{B}^n, \{k | k \in \mathbf{N}, \alpha^k = \mu\} \text{ is infinite}\}$$

and we remark that

$$\alpha \in \Pi_n \Longleftrightarrow \mu^1, ..., \mu^p, \mu^1, ..., \mu^p, \mu^1, ... \in \Pi_n \Longleftrightarrow$$

$$\Longleftrightarrow \begin{cases} \mu^1 = (1, ..., 1), p = 1 \\ \mu^1 \cup ... \cup \mu^p = (1, ..., 1), p \geq 2 \end{cases},$$

$$\widehat{h}(\alpha) \in \Pi_n \Longleftrightarrow h(\mu^1), ..., h(\mu^p), h(\mu^1), ..., h(\mu^p), h(\mu^1), ... \in \Pi_n \Longleftrightarrow$$

$$\Longleftrightarrow \begin{cases} h(\mu^1) = (1, ..., 1), p = 1 \\ h(\mu^1) \cup ... \cup h(\mu^p) = (1, ..., 1), p \geq 2 \end{cases}.$$

Case $p = 1$,

$$\alpha \in \Pi_n \Longrightarrow \mu^1 = (1, ..., 1) \Longrightarrow h(\mu^1) = (1, ..., 1) \Longrightarrow \widehat{h}(\alpha) \in \Pi_n.$$

Case $p \geq 2$,

$$\alpha \in \Pi_n \Longrightarrow \mu^1 \cup ... \cup \mu^p = (1, ..., 1) \Longrightarrow h(\mu^1) \cup ... \cup h(\mu^p) = (1, ..., 1) \Longrightarrow$$

$$\Longrightarrow \widehat{h}(\alpha) \in \Pi_n.$$

c) Let us take arbitrarily some $h \in \Omega_n$ and a function $\rho \in P_n$,

$$\rho(t) = \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus ... \oplus \alpha^k \cdot \chi_{\{t_k\}}(t) \oplus ...$$

where $\alpha \in \Pi_n$ and $(t_k) \in Seq$. We have

$$h(\rho)(t) = h(\rho(t)) =$$

$$= h((0,...,0) \cdot \chi_{(-\infty,t_0)}(t) \oplus \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus (0,...,0) \cdot \chi_{(t_0,t_1)}(t) \oplus ...$$

$$... \oplus \alpha^k \cdot \chi_{\{t_k\}}(t) \oplus (0,...,0) \cdot \chi_{(t_k,t_{k+1})}(t) \oplus ...)$$

$$= h(0,...,0) \cdot \chi_{(-\infty,t_0)}(t) \oplus h(\alpha^0) \cdot \chi_{\{t_0\}}(t) \oplus h(0,...,0) \cdot \chi_{(t_0,t_1)}(t) \oplus ...$$

$$... \oplus h(\alpha^k) \cdot \chi_{\{t_k\}}(t) \oplus h(0,...,0) \cdot \chi_{(t_k,t_{k+1})}(t) \oplus ...$$

$$= h(\alpha^0) \cdot \chi_{\{t_0\}}(t) \oplus ... \oplus h(\alpha^k) \cdot \chi_{\{t_k\}}(t) \oplus ...$$

Because $\widehat{h}(\alpha) \in \Pi_n$, taking into account b), we conclude that $h(\rho) \in P_n$. ∎

Theorem 10.2. *Let be the functions $\Phi, \Psi : \mathbf{B}^n \to \mathbf{B}^n$ and the bijections $h : \mathbf{B}^n \to \mathbf{B}^n, h' \in \Omega_n$. The following statements are equivalent:*

a) $\forall \nu \in \mathbf{B}^n$, the diagram

$$\begin{array}{ccc} \mathbf{B}^n & \overset{\Phi^\nu}{\to} & \mathbf{B}^n \\ h \downarrow & & \downarrow h \\ \mathbf{B}^n & \overset{\Psi^{h'(\nu)}}{\to} & \mathbf{B}^n \end{array}$$

is commutative;

b) $\forall \mu \in \mathbf{B}^n, \forall \alpha \in \Pi_n, \forall k \in \mathbf{N}_\_$,

$$h(\widehat{\Phi}^\alpha(\mu,k)) = \widehat{\Psi}^{\widehat{h'}(\alpha)}(h(\mu),k);$$

c) $\forall \mu \in \mathbf{B}^n, \forall \rho \in P_n, \forall t \in \mathbf{R}$,

$$h(\Phi^\rho(\mu,t)) = \Psi^{h'(\rho)}(h(\mu),t). \tag{14}$$

Proof. a)$\Longrightarrow$b) It is sufficient to prove that $\forall \mu \in \mathbf{B}^n, \forall \alpha \in \Pi_n, \forall k \in \mathbf{N}$,

$$h(\Phi^{\alpha^0...\alpha^k}(\mu)) = \Psi^{h'(\alpha^0)...h'(\alpha^k)}(h(\mu)) \tag{15}$$

since this is equivalent with b).

We fix arbitrarily some μ and some α and we use the induction on k. For $k=0$ the statement is proved, thus we suppose that it is true for k and we prove it for $k+1$:

$$h(\Phi^{\alpha^0...\alpha^k\alpha^{k+1}}(\mu)) = h(\Phi^{\alpha^{k+1}}(\Phi^{\alpha^0...\alpha^k}(\mu))) = \Psi^{h'(\alpha^{k+1})}(h(\Phi^{\alpha^0...\alpha^k}(\mu))) =$$

$$= \Psi^{h'(\alpha^{k+1})}(\Psi^{h'(\alpha^0)...h'(\alpha^k)}(h(\mu))) = \Psi^{h'(\alpha^0)...h'(\alpha^k)h'(\alpha^{k+1})}(h(\mu)).$$

b)$\Longrightarrow$c) For arbitrary $\mu \in \mathbf{B}^n$ and $\rho \in P_n$,

$$\rho(t) = \rho(t_0) \cdot \chi_{\{t_0\}}(t) \oplus ... \oplus \rho(t_k) \cdot \chi_{\{t_k\}}(t) \oplus ...$$

with $(t_k) \in Seq, \rho(t_0), ..., \rho(t_k), ... \in \Pi_n$, we have that

$$h'(\rho)(t) = h'(\rho(t)) = h'(\rho(t_0)) \cdot \chi_{\{t_0\}}(t) \oplus ... \oplus h'(\rho(t_k)) \cdot \chi_{\{t_k\}}(t) \oplus ... \quad (16)$$

is an element of P_n (see Theorem 10.1 c)) and

$$h(\Phi^{\rho}(\mu, t)) = h(\mu \cdot \chi_{(-\infty,t_0)}(t) \oplus \Phi^{\rho(t_0)}(\mu) \cdot \chi_{[t_0,t_1)}(t) \oplus ...$$

$$... \oplus \Phi^{\rho(t_0)...\rho(t_k)}(\mu) \cdot \chi_{[t_k,t_{k+1})}(t) \oplus ...) =$$

$$= h(\mu) \cdot \chi_{(-\infty,t_0)}(t) \oplus h(\Phi^{\rho(t_0)}(\mu)) \cdot \chi_{[t_0,t_1)}(t) \oplus ...$$

$$... \oplus h(\Phi^{\rho(t_0)...\rho(t_k)}(\mu)) \cdot \chi_{[t_k,t_{k+1})}(t) \oplus ... =$$

$$\overset{(15)}{=} h(\mu) \cdot \chi_{(-\infty,t_0)}(t) \oplus \Psi^{h'(\rho(t_0))}(h(\mu)) \cdot \chi_{[t_0,t_1)}(t) \oplus ...$$

$$... \oplus \Psi^{h'(\rho(t_0))...h'(\rho(t_k))}(h(\mu)) \cdot \chi_{[t_k,t_{k+1})}(t) \oplus ... \overset{(16)}{=} \Psi^{h'(\rho)}(h(\mu), t).$$

c)$\Longrightarrow$a) Let $\nu, \mu \in \mathbf{B}^n$ be arbitrary and fixed and we consider $\rho \in P_n$,

$$\rho(t) = \nu \cdot \chi_{\{t_0\}}(t) \oplus \rho(t_1) \cdot \chi_{\{t_1\}}(t) \oplus ... \oplus \rho(t_k) \cdot \chi_{\{t_k\}}(t) \oplus ...$$

with $(t_k) \in Seq$ fixed too. We have

$$h(\Phi^{\rho}(\mu, t)) = h(\mu \cdot \chi_{(-\infty,t_0)}(t) \oplus \Phi^{\nu}(\mu) \cdot \chi_{[t_0,t_1)}(t) \oplus \Phi^{\nu\rho(t_1)}(\mu) \cdot \chi_{[t_1,t_2)}(t) \oplus ...) =$$

$$= h(\mu) \cdot \chi_{(-\infty,t_0)}(t) \oplus h(\Phi^{\nu}(\mu)) \cdot \chi_{[t_0,t_1)}(t) \oplus h(\Phi^{\nu\rho(t_1)}(\mu)) \cdot \chi_{[t_1,t_2)}(t) \oplus ...$$

But

$$h'(\rho)(t) = h'(\rho(t)) = h'(\nu) \cdot \chi_{\{t_0\}}(t) \oplus h'(\rho(t_1)) \cdot \chi_{\{t_1\}}(t) \oplus ...,$$

$$\Psi^{h'(\rho)}(h(\mu),t) =$$

$$= h(\mu)\cdot\chi_{(-\infty,t_0)}(t)\oplus\Psi^{h'(\nu)}\cdot\chi_{[t_0,t_1)}(t)\oplus\Psi^{h'(\nu)h'(\rho(t_1))}\cdot\chi_{[t_1,t_2)}(t)\oplus ...$$

and from (14), for $t\in[t_0,t_1)$, we obtain

$$h(\Phi^{\nu}(\mu)) = \Psi^{h'(\nu)}(h(\mu)).$$

∎

Definition 10.1. *We consider the functions* $\Phi,\Psi:\mathbf{B}^n\to\mathbf{B}^n$*. If two bijections* $h:\mathbf{B}^n\to\mathbf{B}^n, h'\in\Omega_n$ *exist such that one of the equivalent properties a), b), c) from Theorem 10.2 is satisfied, then* Ξ_Φ,Ξ_Ψ *are called* ***equivalent*** *and* Φ,Ψ *are called* ***conjugated****. In this case we denote* $\Phi\overset{(h,h')}{\to}\Psi$.

Remark 10.2. *The equivalence of the universal regular autonomous asynchronous systems is indeed an equivalence and it should be understood as a change of coordinates. Thus* Φ *and* Ψ *are indistinguishable.*

Example 10.1. $\Phi,\Psi:\mathbf{B}^2\to\mathbf{B}^2$ *are given by, see Figure 7*

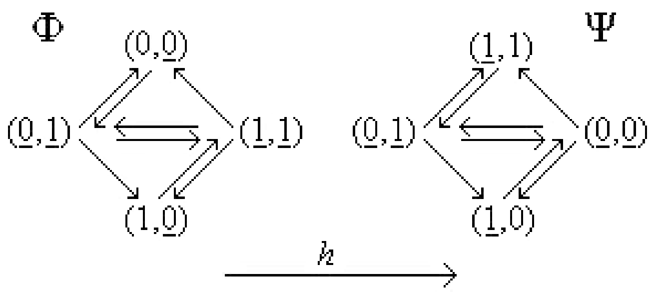

Fig. 7. Equivalent systems.

$$\forall(\mu_1,\mu_2)\in\mathbf{B}^2,\Phi(\mu_1,\mu_2)=(\mu_1\oplus\mu_2,\overline{\mu_2}),$$

$$\forall(\mu_1,\mu_2)\in\mathbf{B}^2,\Psi(\mu_1,\mu_2)=(\overline{\mu_1},\overline{\mu_1}\cdot\overline{\mu_2}\cup\mu_1\cdot\mu_2)$$

and the bijection $h:\mathbf{B}^2\to\mathbf{B}^2$ *is*

$$\forall(\mu_1,\mu_2)\in\mathbf{B}^2,h(\mu_1,\mu_2)=(\overline{\mu_2},\overline{\mu_1}).$$

The diagram

$$\begin{array}{ccc} \mathbf{B}^2 & \stackrel{\Phi^\nu}{\rightarrow} & \mathbf{B}^2 \\ h\downarrow & & \downarrow h \\ \mathbf{B}^2 & \stackrel{\Psi^{\nu'}}{\rightarrow} & \mathbf{B}^2 \end{array}$$

commutes for $\nu = \nu' = (0,0)$ *and on the other hand for* $\nu = \nu' = (1,1)$ *we have the assignments*

$$\begin{array}{ccc} (0,0) & \stackrel{\Phi}{\rightarrow} & (0,1) \\ h\downarrow & & \downarrow h \\ (1,1) & \stackrel{\Psi}{\rightarrow} & (0,1) \end{array}, \quad \begin{array}{ccc} (0,1) & \stackrel{\Phi}{\rightarrow} & (1,0) \\ h\downarrow & & \downarrow h \\ (0,1) & \stackrel{\Psi}{\rightarrow} & (1,0) \end{array}, \quad \begin{array}{ccc} (1,0) & \stackrel{\Phi}{\rightarrow} & (1,1) \\ h\downarrow & & \downarrow h \\ (1,0) & \stackrel{\Psi}{\rightarrow} & (0,0) \end{array}, \quad \begin{array}{ccc} (1,1) & \stackrel{\Phi}{\rightarrow} & (0,0) \\ h\downarrow & & \downarrow h \\ (0,0) & \stackrel{\Psi}{\rightarrow} & (1,1) \end{array}.$$

We denote $\pi_i : \mathbf{B}^2 \rightarrow \mathbf{B}, \forall (\mu_1, \mu_2) \in \mathbf{B}^2$,

$$\pi_i(\mu_1, \mu_2) = \mu_i, i = \overline{1,2}.$$

For $\nu = (0,1), \nu' = (1,0)$ *we have*

$$\begin{array}{ccc} (0,0) & \stackrel{(\pi_1,\Phi_2)}{\rightarrow} & (0,1) \\ h\downarrow & & \downarrow h \\ (1,1) & \stackrel{(\Psi_1,\pi_2)}{\rightarrow} & (0,1) \end{array}, \quad \begin{array}{ccc} (0,1) & \stackrel{(\pi_1,\Phi_2)}{\rightarrow} & (0,0) \\ h\downarrow & & \downarrow h \\ (0,1) & \stackrel{(\Psi_1,\pi_2)}{\rightarrow} & (1,1) \end{array},$$

$$\begin{array}{ccc} (1,0) & \stackrel{(\pi_1,\Phi_2)}{\rightarrow} & (1,1) \\ h\downarrow & & \downarrow h \\ (1,0) & \stackrel{(\Psi_1,\pi_2)}{\rightarrow} & (0,0) \end{array}, \quad \begin{array}{ccc} (1,1) & \stackrel{(\pi_1,\Phi_2)}{\rightarrow} & (1,0) \\ h\downarrow & & \downarrow h \\ (0,0) & \stackrel{(\Psi_1,\pi_2)}{\rightarrow} & (1,0) \end{array},$$

and for $\nu = (1,0), \nu' = (0,1)$ *the assignments are*

$$\begin{array}{ccc} (0,0) & \stackrel{(\Phi_1,\pi_2)}{\rightarrow} & (0,0) \\ h\downarrow & & \downarrow h \\ (1,1) & \stackrel{(\pi_1,\Psi_2)}{\rightarrow} & (1,1) \end{array}, \quad \begin{array}{ccc} (0,1) & \stackrel{(\Phi_1,\pi_2)}{\rightarrow} & (1,1) \\ h\downarrow & & \downarrow h \\ (0,1) & \stackrel{(\pi_1,\Psi_2)}{\rightarrow} & (0,0) \end{array},$$

$$\begin{array}{ccc} (1,0) & \stackrel{(\Phi_1,\pi_2)}{\rightarrow} & (1,0) \\ h\downarrow & & \downarrow h \\ (1,0) & \stackrel{(\pi_1,\Psi_2)}{\rightarrow} & (1,0) \end{array}, \quad \begin{array}{ccc} (1,1) & \stackrel{(\Phi_1,\pi_2)}{\rightarrow} & (0,1) \\ h\downarrow & & \downarrow h \\ (0,0) & \stackrel{(\pi_1,\Psi_2)}{\rightarrow} & (0,1) \end{array},$$

respectively. Φ *and* Ψ *are conjugated.*

Example 10.2. *The functions $h, h' : \mathbf{B}^2 \to \mathbf{B}^2$ are given in the following table*

(μ_1,μ_2)	$h(\mu_1,\mu_2)$	$h'(\mu_1,\mu_2)$
$(0,0)$	$(0,1)$	$(0,0)$
$(0,1)$	$(1,1)$	$(1,0)$
$(1,0)$	$(0,0)$	$(0,1)$
$(1,1)$	$(1,0)$	$(1,1)$

and the state portraits of the two systems are given in Figure 8. Ξ_Φ and Ξ_Ψ are equivalent.

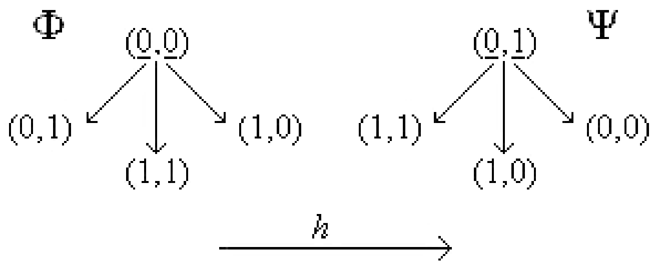

Fig. 8. Equivalent systems.

Theorem 10.3. *If Φ and Ψ are conjugated, then the following possibilities exist:*

a) $\Phi = \Psi = 1_{\mathbf{B}^n}$;

b) $\Phi \neq 1_{\mathbf{B}^n}$ *and* $\Psi \neq 1_{\mathbf{B}^n}$.

Proof. We presume that $\Phi \overset{(h,h')}{\to} \Psi$. In the equation

$$\forall \nu \in \mathbf{B}^n, \forall \mu \in \mathbf{B}^n, h(\Phi^\nu(\mu)) = \Psi^{h'(\nu)}(h(\mu))$$

we put $\Psi = 1_{\mathbf{B}^n}$ and we have

$$\forall \nu \in \mathbf{B}^n, \forall \mu \in \mathbf{B}^n, h(\Phi^\nu(\mu)) = h(\mu)$$

thus $\forall \nu \in \mathbf{B}^n, \Phi^\nu = 1_{\mathbf{B}^n}$ and finally $\Phi = 1_{\mathbf{B}^n}$. ∎

Theorem 10.4. *We suppose that Ξ_Φ and Ξ_Ψ are equivalent and let be h, h' such that $\Phi \overset{(h,h')}{\to} \Psi$.*

a) If μ is a fixed point of Φ, then $h(\mu)$ is a fixed point of Ψ.

b) For any $\mu \in \mathbf{B}^n$ *and any* $\rho \in P_n$*, if* $\Phi^\rho(\mu, t)$ *is periodical with the period* T_0*, then* $\Psi^{h'(\rho)}(h(\mu), t)$ *is periodical with the period* T_0*.*

c) If Ξ_Φ *is transitive, then* Ξ_Ψ *is transitive.*

Proof. a) We suppose that $\Phi(\mu) = \mu$. The commutativity of the diagram

$$\begin{array}{ccc} \mathbf{B}^n & \overset{\Phi^\nu}{\rightarrow} & \mathbf{B}^n \\ h \downarrow & & \downarrow h \\ \mathbf{B}^n & \overset{\Psi^{h'(\nu)}}{\rightarrow} & \mathbf{B}^n \end{array}$$

for $\nu = (1, ..., 1)$ gives

$$h(\mu) = h(\Phi(\mu)) = h(\Phi^{(1,\ldots,1)}(\mu)) = \Psi^{h'(1,\ldots,1)}(h(\mu)) =$$

$$= \Psi^{(1,\ldots,1)}(h(\mu)) = \Psi(h(\mu)).$$

b) Let be $\mu \in \mathbf{B}^n$ and $\rho \in P_n$. The hypothesis states that $\exists t' \in \mathbf{R}, \forall t \geq t'$,

$$\Phi^\rho(\mu, t) = \Phi^\rho(\mu, t + T_0)$$

and in this situation

$$\Psi^{h'(\rho)}(h(\mu), t) = h(\Phi^\rho(\mu, t)) = h(\Phi^\rho(\mu, t + T_0)) = \Psi^{h'(\rho)}(h(\mu), t + T_0).$$

c) Let $\mu, \mu' \in \mathbf{B}^n$ be arbitrary and fixed. The hypothesis (12) states that

$$\exists \rho \in P_n, \exists t \in \mathbf{R}, \Phi^\rho(h^{-1}(\mu), t) = h^{-1}(\mu'),$$

wherefrom

$$\Psi^{h'(\rho)}(\mu, t) = \Psi^{h'(\rho)}(h(h^{-1}(\mu)), t) = h(\Phi^\rho(h^{-1}(\mu), t) = h(h^{-1}(\mu')) = \mu'.$$

The situation with (13) is similar. ∎

11. DYNAMICAL BIFURCATIONS

Definition 11.1. *We consider the case when the generator function* $\Phi : \mathbf{B}^n \times \mathbf{B}^m \rightarrow \mathbf{B}^n$*,* $\mathbf{B}^n \times \mathbf{B}^m \ni (\mu, \lambda) \rightarrow \Phi(\mu, \lambda) \in \mathbf{B}^n$ *of* $\Xi_{\Phi(\cdot,\lambda)}$ *depends on the parameter* $\lambda \in \mathbf{B}^m$*. We fix* λ *and let be* $\lambda' \in \mathbf{B}^m$*. If* $\Phi(\cdot, \lambda)$ *and* $\Phi(\cdot, \lambda')$ *are conjugated, then* $\Phi(\cdot, \lambda')$ *is called an **admissible** (or **allowable**) **perturbation of*** $\Phi(\cdot, \lambda)$*.*

Remark 11.1. *Intuitively speaking (Ott, [2]) a dynamical bifurcation is a qualitative change in the dynamic of the system* $\Xi_{\Phi(\cdot,\lambda)}$ *that occurs at the variation of the parameter* λ*.*

Definition 11.2. *If for any parameters* $\lambda, \lambda' \in \mathbf{B}^m$ *the systems* $\Xi_{\Phi(\cdot,\lambda)}$ *and* $\Xi_{\Phi(\cdot,\lambda')}$ *are equivalent, then* Φ *is called* ***structurally stable****; the existence of* λ, λ' *such that* $\Xi_{\Phi(\cdot,\lambda)}$ *and* $\Xi_{\Phi(\cdot,\lambda')}$ *are not equivalent is called a* ***dynamical bifurcation****.*

Equivalently, let us fix an arbitrary $\lambda \in \mathbf{B}^m$*. If* $\forall \lambda' \in \mathbf{B}^m$*,* $\Phi(\cdot, \lambda')$ *is an admissible perturbation of* $\Phi(\cdot, \lambda)$*, then* Φ *is said to be* ***structurally stable****, otherwise we say that* Φ *has a* ***dynamical bifurcation****.*

Remark 11.2. *If* $\forall \lambda \in \mathbf{B}^m, \forall \lambda' \in \mathbf{B}^m$ *the bijections* $h : \mathbf{B}^n \to \mathbf{B}^n, h' \in \Omega_n$ *exist such that* $\forall \nu \in \mathbf{B}^n$*, the diagram*

$$\begin{array}{ccc} \mathbf{B}^n & \overset{\Phi^{\nu}(\cdot,\lambda)}{\rightarrow} & \mathbf{B}^n \\ h \downarrow & & \downarrow h \\ \mathbf{B}^n & \overset{\Phi^{h'(\nu)}(\cdot,\lambda')}{\rightarrow} & \mathbf{B}^n \end{array}$$

commutes, then Φ *is structurally stable, otherwise we have a dynamical bifurcation.*

Example 11.1. *In Figure 9* $(n = 2, m = 1)$*,*

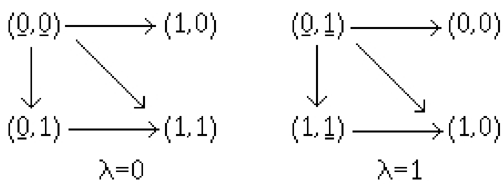

Fig. 9. Structural stability.

Φ *is structurally stable and the bijections* h, h' *are defined accordingly to the following table:*

(μ_1, μ_2)	$h(\mu_1, \mu_2)$	$h'(\mu_1, \mu_2)$
$(0, 0)$	$(0, 1)$	$(0, 0)$
$(0, 1)$	$(1, 1)$	$(1, 0)$
$(1, 0)$	$(0, 0)$	$(0, 1)$
$(1, 1)$	$(1, 0)$	$(1, 1)$

Example 11.2. *In Figure 10 ($n = 2, m = 1$),*

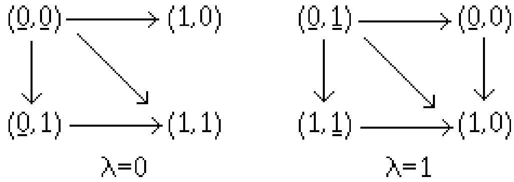

Fig. 10. Dynamical bifurcation.

Φ *has a dynamical bifurcation.*

Definition 11.3. *The **bifurcation diagram** is a partition of the set of systems $\{\Xi_{\Phi(\cdot,\lambda)}|\lambda \in \mathbf{B}^m\}$ in classes of equivalence given by the equivalence of the systems, together with representative state portraits for each class of equivalence.*

Example 11.3. *Figure 10 is a bifurcation diagram.*

Definition 11.4. *The **bifurcation diagram** ([2], page 5) is the graph that gives the position of the fixed points depending on a parameter, such that a bifurcation exists.*

Remark 11.3. *Such a(n informal) definition works for calling Figure 10 a bifurcation diagram, since there fixed points exist. However for Figure 11*

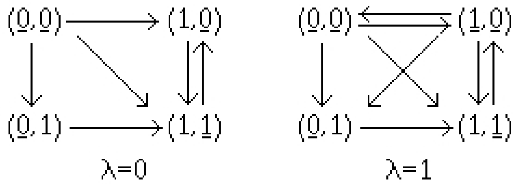

Fig. 11. Dynamical bifurcation.

this definition does not work, because a bifurcation exists there, but no fixed points.

Definition 11.5. *Let be $\Phi, \Psi : \mathbf{B}^n \times \mathbf{B}^m \to \mathbf{B}^n$. The families of systems $(\Xi_{\Phi(\cdot,\lambda)})_{\lambda \in \mathbf{B}^m}$ and $(\Xi_{\Psi(\cdot,\lambda)})_{\lambda \in \mathbf{B}^m}$ are called **equivalent** if there exists a bijection $h'' : \mathbf{B}^m \to \mathbf{B}^m$ such that $\forall \lambda \in \mathbf{B}^m, \Xi_{\Phi(\cdot,\lambda)}$ and $\Xi_{\Psi(\cdot,h''(\lambda))}$ are equivalent in the sense of Definition 10.1.*

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