UNIVERSITATEA “AL. I. CUZA” IASI
FACULTATEA DE MATEMATICA

LUCRARILE CONFERINTEI NATIONALE DE GEOMETRIE SI TOPOLOGIE
Suceava, 6-9 octombrie, 1988
Iasi, 1989

BOOLEAN FUNCTIONS: DIFFERENTIABLE MANIFOLDS AND AFFINE SPACES

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The Boolean affine applications are the Boolean differentiable applications. If we accept to define the differentiable manifolds by replacing the differentiable structure with some atlas, then the affine spaces are special cases of differentiable manifolds.

0. Preliminaries

(0.1) **Definition** $\boldsymbol{B}_2 = \{0,1,\oplus,\cdot\}$ is the field of the binary numbers, where $'\oplus'$ is the modulo 2 sum and $'\cdot'$ is the intersection. It has the rough topology.

(0.2) **Definition** The functions $f : L \to L'$, where L, L' are Boolean linear spaces, are called *Boolean applications*. From now on, we shall drop the attribute ‘Boolean’ whenever possible.

(0.3) **Remark** L becomes an affine space together with the function

$$L \times L \ni (x,y) \mapsto \bar{x}(y) = \bar{y}(x) = \overline{xy} \overset{def}{=} x \oplus y \in L$$

where $\bar{x}, \bar{y} : L \to L$ are translations.

(0.4) **Definition** The *affine applications* f are these that satisfy:

$$\forall k \in \boldsymbol{N}, \forall x_1,..., x_{2k+1} \in L,\ f(x_1 \oplus ... \oplus x_{2k+1}) = f(x_1) \oplus ... \oplus f(x_{2k+1})$$

(0.5) **Definition** If $\{e^i \mid i \in I\}$ is a base of L, then the *first order derivatives* of f are:

$$\partial_i f(x) = f(x \oplus e^i) \oplus f(x), i \in I$$

1. Differentiable applications

(1.1) **Theorem** The following statements are equivalent:

a) $\forall x, y \in L, f(x \oplus y) = f(x) \oplus f(y) \oplus f(0)$

b) f is affine

c) $\exists df$ linear, making the following diagram commutative for any $y \in L$:

$$\begin{array}{ccc} L & \xrightarrow{f} & L' \\ \overline{y}\downarrow & & \downarrow\overline{f(y)} \\ L & \xrightarrow{df} & L' \end{array}$$

d) $\forall i, j \in I, \forall x \in L, \partial_{ij}^2 f(x) = 0$

(1.2) **Remark** By taking at (1.1) c) $y = 0,$ we get $\forall x \in L, f(x) = df \cdot x \oplus f(0)$

(1.3) **Definition** An application that satisfies one of the conditions from above is called *differentiable* and df is called its *differential*.

(1.4) **Theorem** There are true

a) If $L \xrightarrow{f} L' \xrightarrow{g} L''$ are differentiable applications, then $g \circ f$ is differentiable and $d(g \circ f) = dg \cdot df$

b) If f is a differentiable bijection, then f^{-1} is differentiable, df is bijective with linear inverse and $d(f^{-1}) = (df)^{-1}$

(1.5) **Remark** If the dimension of L is ≥ 3, there are non differentiable bijections $L \to L$.

(1.6) **Remark** If $f, g : L \to \boldsymbol{B}_2$ are differentiable, $f \cdot g$ is not, generally.

2. Differentiable Manifolds

(2.1) **Remark** The *differentiable manifolds* are triplets (S, L, A), where the set S has the rough topology, L is a linear space with the rough topology and A is an atlas. The maps are bijections $S \to L$ giving differentiable changes of coordinates.

(2.2) **Remark** We mention that the tangent vectors may be defined via the equivalence relation:

$$\forall x, x' \in L, \forall \chi, \chi' \in A, (x, \chi) \approx (x', \chi') \Leftrightarrow x' = d(\chi' \circ \chi^{-1}) \cdot x$$

3. Affine Spaces

(3.1) **Definition** The *affine spaces* are the differentiable manifolds $(S, L, \overline{S})$ for which there is defined a bijection $\overline{} : S \to \overline{S}, x \mapsto \overline{x}$ making commutative $\forall x, y \in S$ the diagram

$$\begin{array}{ccc} S & ===== & S \\ \overline{x}\downarrow & & \downarrow\overline{y} \\ L & \xrightarrow{\overline{\overline{yx}}} & L \end{array}$$

i.e.

$$\forall z \in S, \overline{yx} \oplus \overline{xz} = \overline{yz} \text{ (Chasles)}$$

(3.2) **Remark** There results that the changes of coordinates are translations, $d(\overline{\overline{yx}}) = 1_L$ and the tangent space T may be identified with L.

4. Examples

Examples of differentiable manifolds are given by the linear space L and any set which is homomorphic with it. Examples of affine spaces are L, any set which is homomorphic with it and the linear varieties.

5. Conclusions

Many familiar concepts may be reconstructed in this context. For example, the differentiable vector fields behave ‘well’ under the action of the differentiable applications (see proposition 12, pg 57, in [1]).

Bibliography

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